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A NOTE ON CLOSED GRAPH THEOREMS

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Abstract. We give a common generalisation of the closed graph theorems of De Wilde and of Popa.

1. Introduction

In the theory of locally convex spaces, M. De Wilde's notion of webs is the abstraction of all that is essential in order to prove very general closed graph theorems. Here we concentrate on his theorem in [**1**] for linear mappings with bornologically closed graph that have an ultrabornological space as domain and a webbed locally convex space as codomain. Henceforth, we shall refer to this theorem as 'De Wilde's closed graph theorem'.

As far as the category of bounded linear mappings between separated convex bornological spaces is concerned, there exists a corresponding bornological notion of so-called nets that enabled N. Popa in [**5**] to prove a bornological version of the closed graph theorem which we name 'Popa's closed graph theorem'.

Although M. De Wilde's theorem in the locally convex and N. Popa's theorem in the convex bornological setting are conceptually similar (see also [**3**]), they do not directly relate to each other.

In this manuscript I present a bornological closed graph theorem that generalises the one of N. Popa and even implies the one of M. De Wilde. First of all, I give a suitable definition of bornological webs on separated convex bornological spaces with excellent stability properties and prove the aforementioned general bornological closed graph theorem. It turns out that there is a connection between topological webs in the sense of M. De Wilde and bornological webs on locally convex spaces. This is the keystone of this paper and the reason why I could reveal the bornological nature of De Wilde's theorem.

2. De Wilde's closed graph theorem

The following definition goes back to M. De Wilde and may be found in [**4**, 5.2].

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Definition 2.1 (Topological webs)**.** Let E be a Hausdorff locally convex space. A mapping $W : \bigcup_{k\in\mathbb{N}} \mathbb{N}^k \to \mathcal{P}(E)$ is called a topological web if all of the conditions below hold. We set

$$W_{s,k} := W(s(0), \dots, s(k)), \text{ where } s : \mathbb{N} \to \mathbb{N}.$$

(TW1): The image of W consists of absolutely convex sets.

(TW2): $W(\emptyset) = E$;

(TW3): Given a finite sequence $(n_0, \dots n_k)$, every point in $W(n_0, \dots, n_k)$ is absorbed by

$$\bigcup_{n\in\mathbb{N}} W(n_0, \dots, n_k, n).$$

Note that in particular $\bigcup_{n\in\mathbb{N}} W(n)$ is absorbent in $W(\emptyset) = E$.

(TW4): For every finite sequence $(n_0, \dots, n_k, n_{k+1})$ one has

$$2\,W(n_0, \dots, n_k, n_{k+1}) \subseteq W(n_0, \dots, n_k).$$

We say that W is completing if the following condition is satisfied:

(TW5): For every $s : \mathbb{N} \to \mathbb{N}$ and for every choice of $y_k \in W_{s,k}$, the series $\sum_k y_k$ converges topologically in E.

A separated locally convex space E that carries a topological web is called webbed locally convex space.

Next, we state M. De Wilde's intriguing closed graph theorem which may be found in [**4**, 13.3.4(a)].

Theorem 2.2 (De Wilde's closed graph theorem)**.** *If E is ultrabornological and F is a webbed locally convex space, then every linear mapping $f : E \to F$ which has bornologically closed graph with respect to those convex bornologies on E and F that are generated by all bounded Banach disks in E and in F, respectively, is continuous even if regarded as a mapping into the ultrabornologification F_{uborn} of F.*

3. Popa's closed graph theorem

Now let us turn to a bornological version of the closed graph theorem that goes back to N. Popa (see [**5**]) and may also be found in [**2**, 4.4.3].

First of all, we give a definition of nets in separated convex bornological spaces. Only the terminology differs from [**2**, 4.4.3].

Definition 3.1. Let $(F, \mathcal{B})$ be a separated convex bornological space. A mapping $N : \bigcup_{k\in\mathbb{N}} \mathbb{N}^k \to \mathcal{P}(F)$ is called a net which is compatible with $\mathcal{B}$ if the conditions below hold.

(N1): The image of N consists of disks.

(N2): $N(\emptyset) = F$;

(N3): For every finite sequence $(n_0, \dots n_k)$ we have
$$N(n_0, \dots n_k) = \bigcup_{n\in\mathbb{N}} N(n_0, \dots, n_k, n).$$

(N4): For every $s : \mathbb{N} \to \mathbb{N}$ there is a $b(s) : \mathbb{N} \to \mathbb{R}_{>0}$ such that for all $x_k \in N_{s,k}$ and $a_k \in [0, b(s)_k]$ the series $\sum_k a_k x_k$ converges bornologically in $(F, \mathcal{B})$ and $\sum_{k\geq n} a_k x_k \in N_{s,n}$ for every $n \in \mathbb{N}$.

(N5): For every sequence $(\lambda_k)_{k\in\mathbb{N}}$ of positive reals and $s : \mathbb{N} \to \mathbb{N}$ the set $\bigcap_{k\in\mathbb{N}} \lambda_k N_{s,k}$ belongs to $\mathcal{B}$.

Given a mapping $b : \mathbb{N}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}_{>0}$, we say that b satisfies (N4) if, for all $s : \mathbb{N} \to \mathbb{N}$, $b(s)$ is suitable for s in the sense of (N4).

Here is the already existent bornological version of the closed graph theorem:

Theorem 3.2 (Popa's bornological closed graph theorem)**.** *Let E and F be separated convex bornological spaces such that E is complete and F has a net which is compatible with its bornology. Then every linear mapping $f : E \to F$ with bornologically closed graph is bounded.*

4. The generalised bornological closed graph theorem

In the rest of this paper, I establish my theory and its connection to the theorems (2.2) and (3.2). So let me first introduce a suitable notion of webs on separated bornological spaces.

Definition 4.1 (Bornological webs)**.** Let F be a separated convex bornological space. A pair (V, b) consisting of mappings $V : \bigcup_{k\in\mathbb{N}} \mathbb{N}^k \to \mathcal{P}(F)$ and $b : \mathbb{N}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}_{>0}$ is called a bornological web if all of the conditions below hold. As in (2.1), we use the abbreviations $V_{s,k}$.

(BW1): The image of V consists of disks.

(BW2): $V(\emptyset) = F$;

(BW3): Given a finite sequence $(n_0, \dots, n_k)$, every point in $V(n_0, \dots, n_k)$ is absorbed by $\bigcup_{n\in\mathbb{N}} V(n_0, \dots, n_k, n)$.

The relationship between V and the given bornology is established by the following property:

(BW4): For every $s : \mathbb{N} \to \mathbb{N}$ the series $\sum_k b(s)_k x_k$ converges bornologically in F, whenever we choose $x_k \in V_{s,k}$.

We define the following sets, which of course depend on b:
$$\forall s : \mathbb{N} \to \mathbb{N},\ \forall n \in \mathbb{N} :\ \widetilde{V_{s,n}} := \langle V_{s,n} \cup \{ \sum_{k\geq n+1} b(s)_k x_k |\ \forall k \geq n+1 : x_k \in V_{s,k}\}\rangle_{\mathrm{ac}},$$
where $\langle A\rangle_{\mathrm{ac}}$ is the absolutely convex hull of A. Furthermore, let $\mathcal{B}_{(V,b)}$ denote the convex linear bornology on F which is generated by all sets of the form $\bigcap_{k\in\mathbb{N}} \lambda_k \widetilde{V_{s,k}}$, where the $(\lambda_k)_{k\in\mathbb{N}}$ are arbitrary real-valued sequences and $s : \mathbb{N} \to \mathbb{N}$.

A separated convex bornological space F which is endowed with a bornological web will be called a webbed convex bornological space. A separated locally convex space E will be called bornologically webbed space if E equipped with its von Neumann bornology is a webbed convex bornological space.

The following theorem states that for a separated convex bornological space it is a weaker condition to be webbed than to carry a net which is compatible with the given bornology. So the concept of webbed convex bornological spaces is a generalisation of convex bornological spaces with nets.

Theorem 4.2. *Let $(F, \mathcal{B})$ be a separated convex bornological space. If N is a net on F which is compatible with $\mathcal{B}$, then, for every $b : \mathbb{N}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}_{>0}$ satisfying (N4) of N, (N, b) is a bornological web on F such that in addition $\mathcal{B}_{(N,b)} \subseteq \mathcal{B}$ holds.*

Proof. (BW1)–(BW3) are direct consequences of (N1)–(N3), respectively.
(BW4) is clear.
$\mathcal{B}_{(N,b)} \subseteq \mathcal{B}$: Let $\bigcap_{k\in\mathbb{N}} \lambda_k \widetilde{N_{s,k}}$ be a typical generator of $\mathcal{B}_{(N,b)}$. Then

$$\sum_{k\geq n+1} b(s)_k x_k \in N_{s,n+1} \overset{\text{(N3)}}{\subseteq} N_{s,n}.$$

Hence $\widetilde{N_{s,n}} = N_{s,n}$ and $\bigcap_{k\in\mathbb{N}} \lambda_k \widetilde{N_{s,k}} = \bigcap_{k\in\mathbb{N}} \lambda_k N_{s,k} \overset{\text{(N5)}}{\in} \mathcal{B}$, at least for sequences of positive reals. For arbitrary real-valued sequences everything remains true, since the $N_{s,k}$ are circled and $\{0\}$ is bounded. $\square$

Even more, the following general bornological closed graph theorem holds:

Theorem 4.3 (General bornological closed graph theorem)**.** *Let E and F be separated convex bornological spaces, where E is complete and F is endowed with a bornological web (V, b). Then every linear mapping $f : E \to F$ with bornologically closed graph is bounded with respect to the given convex bornology on E and $\mathcal{B}_{(V,b)}$ on F.*

Proof. Step 1. First of all, note that $E = \varinjlim_B E_B$, where B runs through all bounded Banach disks, i.e. E can be described as the inductive limit of a family of Banach spaces. In order to get that f is bounded we only have to show that the compositions of f with the coprojections $p_B : E_B \to E$ are bounded. Since the graph of $f \circ p_B$ is bornologically closed, it suffices to show the theorem for Banach spaces E.

Step 2. Let therefore E be a Banach space, F be endowed with a bornological web (V, b), and $f : E \to F$ be a linear mapping with bornologically closed graph.

By (BW2) and (BW3) $F = \bigcup_{k,l\in\mathbb{N}} l\, V(k)$, hence $E = \bigcup_{k,l\in\mathbb{N}} l f^{-1}(V(k))$. Since E is Baire, there must exist an $n_0 \in \mathbb{N}$ such that $f^{-1}(V(n_0))$ is not meagre. By (BW3) $V(n_0) \subseteq \bigcup_{k,l\in\mathbb{N}} l\, V(n_0, k)$.

$$\Rightarrow f^{-1}(V(n_0)) \subseteq \bigcup_{k,l\in\mathbb{N}} l f^{-1}(V(n_0, k)).$$

Since $f^{-1}(V(n_0))$ is not meagre, there is an $n_1 \in \mathbb{N}$ such that $f^{-1}(V(n_0, n_1))$ is not meagre. Thus we recursively find a sequence $n : \mathbb{N} \to \mathbb{N}$ such that the sets $f^{-1}(V(n_0, \dots, n_k))$ are not meagre, for all $k \in \mathbb{N}$. Set $V_k := V_{n,k}$ and $b_k := b(n)(k)$, for all $k \in \mathbb{N}$.

Since (V, b) satisfies (BW4), the series $\sum_k b_k x_k$ converges bornologically in F, whenever we choose $x_k \in V_k$.

For the next step let B denote the unit disk in E. If we can show that $f(B)$ is absorbed by $\widetilde{V_k}$, or equivalently, that B is absorbed by $f^{-1}(\widetilde{V_k})$, for all $k \in \mathbb{N}$, then $f(B) \in \mathcal{B}_{(V,b)}$, and we are done.

Step 3. Define $A_k := b_k f^{-1}(V_k)$, for all $k \in \mathbb{N}$. Since A_k is not meagre and consequently not nowhere dense, the interior of $\overline{A_k}$ is not empty. Hence there exist $\overline{y_k} \in \overline{A_k}$ and $\lambda_k < \frac{1}{k+1} : \overline{y_k} + 2\lambda_k B \subseteq \overline{A_k}$. As $\overline{y_k} \in \overline{A_k}$, there is a $y_k \in A_k$ such that $y_k \in \overline{y_k} + \lambda_k B$.

$$\Rightarrow y_k + \lambda_k B = (y_k - \overline{y_k}) + (\overline{y_k} + \lambda_k B) \subseteq \overline{y_k} + 2\lambda_k B \subseteq \overline{A_k}.$$

We obtain $\lambda_k B \subseteq \overline{A_k} - y_k \subseteq 2\overline{A_k}$.

Step 4. Next we show, for fixed n, that $\overline{f^{-1}(\widetilde{V_n})} \subseteq 3 f^{-1}(\widetilde{V_n})$, which completes the proof, since then we have

$$\lambda_n B \subseteq 2\overline{A_n} = 2b_n \overline{f^{-1}(V_n)} \subseteq 2b_n \overline{f^{-1}(\widetilde{V_n})} \subseteq 6b_n f^{-1}(\widetilde{V_n}).$$

So let $x \in \overline{f^{-1}(\widetilde{V_n})}$. Then there is a $u_n \in f^{-1}(\widetilde{V_n})$ with $x - u_n \in \lambda_{n+1} B$.

$$\Rightarrow x - u_n + y_{n+1} \subseteq \lambda_{n+1} B + y_{n+1} \subseteq \overline{A_{n+1}}$$

(see step 3). Now there is a $u_{n+1} \in A_{n+1}$ with $(x - u_n + y_{n+1}) - u_{n+1} \subseteq \lambda_{n+2} B$. Inductively we find $u_k \in A_k$, $k > n$, such that we have $x - \sum_{k=n}^{l} u_k + \sum_{k=n+1}^{l} y_k \in \lambda_{l+1} B$, for $l > n$. Hence, $x - \sum_{k=n}^{l} u_k + \sum_{k=n+1}^{l} y_k$ converges to 0, since $\lambda_n \to 0$. Define $v_k := f(u_k)$ and $z_k := f(y_k)$. Then $v_n \in \widetilde{V_n}$, $z_n \in b_n V_n$, and $\forall k > n : v_k, z_k \in b_k V_k$. It follows from (BW4) that $\sum_k v_k$ and $\sum_k z_k$ converge bornologically in F. Besides,

$$y := \sum_{k \geq n} v_k - \sum_{k \geq n+1} z_k = v_n + \sum_{k \geq n+1} v_k - \sum_{k \geq n+1} z_k \in \widetilde{V_n} + \widetilde{V_n} - \widetilde{V_n} \subseteq 3\widetilde{V_n},$$

since the $\widetilde{V_n}$ are absolutely convex. Since f has bornologically closed graph, we infer $f(0) = f(x) - y$, i.e. $f(x) = y$. □

Theorem 4.4. *Popa's closed graph theorem* (3.2) *is a consequence of the preceding theorem* (4.3)*.*

Proof. Let E and $(F, \mathcal{B})$ be separated convex bornological spaces, where E is complete and $(F, \mathcal{B})$ carries a net N which is compatible with $\mathcal{B}$. Let further $f : E \to F$ be a bornologically closed linear mapping. By (4.2) there is a sequence $b : \mathbb{N} \to \mathbb{N}$ such that (N, b) is a bornological web on F. Now we apply (4.3) and conclude that f is bounded with respect to the given convex bornology on E and $\mathcal{B}_{(N,b)}$ on F. Since $\mathcal{B}_{(N,b)}$ is contained in $\mathcal{B}$ by (4.2), we are done. □

In order to see the relation between topological and bornological webs on locally convex spaces, we shall need the following

Lemma 4.5. *Let E be a Hausdorff locally convex space which is endowed with a topological web W. Then the mapping $V : \bigcup_{k\in\mathbb{N}} \mathbb{N}^k \to \mathcal{P}(E)$, defined by*

$$V(n_0,\dots,n_k) := \frac{1}{2^k} W(n_0,\dots,n_k),$$

is again a topological web on E.

Proof. (TW1)–(TW3) are clear.

(TW4): $2\,V(n_0,\dots,n_k,n_{k+1}) = \frac{1}{2^k} W(n_0,\dots,n_k,n_{k+1}) \subseteq \frac{1}{2^k} W(n_0,\dots,n_k)$
$$= V(n_0,\dots,n_k)$$

(TW5) is true since the sets $W(n_0,\dots,n_k)$ are in particular circled, and therefore $V(n_0,\dots,n_k) = \frac{1}{2^k} W(n_0,\dots,n_k) \subseteq W(n_0,\dots,n_k)$ holds. □

Apart from what follows, the preceding lemma may also be used to correct the proof of [**4**, 13.3.3]:

Theorem 4.6. *If E is a webbed locally convex space, then so is its ultra-bornologification E_{uborn}.*

There a mapping $\widehat{W}$ on a webbed locally convex space E is defined as follows:

$$\widehat{W}_{\varphi,n} := W_{\varphi,2n-1},$$

for all $\varphi \in \mathbb{N}^{\mathbb{N}}$ and $n \in \mathbb{N}$, where $\mathbb{N} := \{1,2,\dots\}$ and W is a topological web on E. It is claimed then that the first three axioms (W1)–(W3), which correspond to (TW1)–(TW3), are trivially satisfied. But this is not true, since $\widehat{W}$ is not well-defined. Indeed, $\widehat{W_{\varphi,n}}$ depends on more entries than just $\varphi(0),\dots,\varphi(n)$.

Now one may proceed as in the proof of [**4**, 13.3.3] by taking V as in (4.5) instead of $\widehat{W}$.

Notation 4.7. *For locally convex spaces let $\mathcal{B}_{\text{Banach}}$ denote the convex linear bornology which has the bounded Banach disks as basis.*

Theorem 4.8 (Bornological via topological webs)**.** *Let E be a locally convex space. If E is webbed, then $(E,\mathcal{B}_{\text{Banach}})$ is a webbed convex bornological space with a bornological web (V,b) that may be chosen in such a way that $\mathcal{B}_{(V,b)}$ is finer than the von Neumann bornology of E.*

Proof. Let W be a topological web on E. By (4.5) we find another topological web on E which is given by $V(n_0,\dots,n_k) := \frac{1}{2^k} W(n_0,\dots,n_k)$. For $s : \mathbb{N} \to \mathbb{N}$ define $b(s) : \mathbb{N} \to \mathbb{R}_{>0}$ to be constant with value 1. We claim that (V,b) is a bornological web for $(E,\mathcal{B}_{\text{Banach}})$:

(BW1)–(BW3) are clear.

(BW4): Let $s : \mathbb{N} \to \mathbb{N}$ be given and choose $y_k \in V_{s,k}$, for all $k \in \mathbb{N}$. We then find $x_k \in W_{s,k}$ with $x_k = 2^k y_k$, for all $k \in \mathbb{N}$. Because W satisfies (TW5), $\sum_k \lambda_k x_k$ converges for every choice of $\lambda_k \in \mathbb{D}$. Consequently we obtain a linear mapping

$f^* : l^1 \to E$ defined by $(\lambda_k)_{k\in\mathbb{N}} \mapsto \sum_{k\in\mathbb{N}} \lambda_k x_k$. Let B denote the unit ball in l^1, then $f^*(B) \subseteq K$, where K is the bipolar $\{x_k : k \in \mathbb{N}\}^{\circ}{}_{\circ}$.

Observe that f^* is the adjoint of $f : E^* \to c_0$, $f(u) := (u(x_k))_{k\in\mathbb{N}}$. The mapping f is well-defined, since $(x_k)_{k\in\mathbb{N}}$ is a null sequence. The mapping f^* is $(\sigma(l^1, c_0), \sigma(E, E^*))$-continuous. Because B is the polar of the unit ball in c_0, it is $\sigma(l^1, c_0)$-compact, and hence $f^*(B)$ is a $\sigma(E, E^*)$-compact disk. Since $x_k = f^*(e^k)$, for all $k \in \mathbb{N}$, it follows that $f^*(B) = K$. So K is a $\sigma(E, E^*)$-compact disk, hence in particular a bounded Banach disk (see [**4**, 8.4.4 (b)]). Clearly, $\sum_k y_k$ converges in E_K.

In order to prove the last assertion, note first that (V, b) is also a bornological web for F when equipped with its von Neumann bornology, since $\mathcal{B}_{\mathrm{Banach}}$ is contained therein (see (4.11.4)). Next, let $(\lambda_k)_{k\in\mathbb{N}}$ be a real-valued sequence. For every choice of $x_k \in V_{s,k}$, $k \in \mathbb{N}$, the resulting limit $\sum_{k\geq n+1} x_k$ belongs to $\overline{V_{s,n}}$, as follows from property (TW4) of V. Hence $\widetilde{V_{s,n}} \subseteq \overline{V_{s,n}}$, but $\bigcap_{k\in\mathbb{N}} \lambda_k \overline{V_{s,k}}$ is bounded, since for any given closed and absolutely convex 0-neighbourhood U there is an index $n \in \mathbb{N}$ such that $\overline{V_{s,n}} \subseteq U$ (see [**4**, 5.2.1]). Hence, $\lambda_n \overline{V_{s,n}}$, and consequently $\bigcap_{k\in\mathbb{N}} \lambda_k \overline{V_{s,k}}$, is absorbed by U. We thus see that the elements of $\mathcal{B}_{(V,b)}$ are bounded with respect to the canonical bornology of E. □

Corollary 4.9. *Let E be a webbed locally convex space. Then $(E_{\mathrm{uborn}}, \mathcal{B}_{\mathrm{Banach}})$ carries a bornological web (V, b) such that the corresponding convex bornology $\mathcal{B}_{(V,b)}$ is contained in the von Neumann bornology of E_{uborn}.*

Proof. By (4.6) the ultrabornologification E_{uborn} of a webbed locally convex space E is also webbed. Now the preceding theorem applied to E_{uborn} yields the assertion. □

Theorem 4.10. *De Wilde's closed graph theorem* (2.2) *is a consequence of* (4.3).

Proof. We consider $(E, \mathcal{B}_{\mathrm{Banach}})$ as domain space of f and $(F_{\mathrm{uborn}}, \mathcal{B}_{\mathrm{Banach}})$ as codomain of f. Note that the bounded Banach disks of F and its ultrabornologification coincide and that, by (4.9), F_{uborn} carries a bornological web (V, b) such that $\mathcal{B}_{(V,b)}$ is finer than the canonical bornology of F_{uborn}.

By assumption f has bornologically closed graph with respect to $(E, \mathcal{B}_{\mathrm{Banach}})$ and $(F_{\mathrm{uborn}}, \mathcal{B}_{\mathrm{Banach}})$. Hence we may apply (4.3) in order to see that f is $(\mathcal{B}_{\mathrm{Banach}}, \mathcal{B}_{(V,b)})$-bounded, which in turn implies that f is bounded with respect to $\mathcal{B}_{\mathrm{Banach}}$ on E and the von Neumann bornology on F_{uborn}.

Since E is ultrabornological, we finally get that $f : E \to F_{\mathrm{uborn}}$ is continuous. □

Thus one sees that De Wilde's closed graph theorem actually is a statement in the category of bounded linear mappings between separated convex bornological spaces.

Finally, we state some stability properties for webbed convex bornological spaces.

Theorem 4.11 (Stability of bornological webs)**.**

(1): *Let E be a Fréchet space, then E is bornologically webbed.*

(2): *Let E be a diagram in the category CBS of bounded linear mappings between separated convex bornological spaces which is given by a monotonic sequence $E_0 \subseteq E_1 \subseteq \ldots \subseteq E_n \subseteq \ldots$ of webbed convex bornological spaces together with the canonical bounded injections $\mathrm{inj}_n : E_n \to E_{n+1}$. Then the colimit $E_\infty := \varinjlim_{n\in\mathbb{N}} E_n = \bigcup_{n\in\mathbb{N}} E_n$ exists in CBS and is webbed again.*

(3): *Every complete separated convex bornological space with a countable basis is webbed.*

(4): *In case $\mathcal{B}, \mathcal{B}'$ are separated convex linear bornologies on a vector space E, the following is true: If $\mathcal{B} \subseteq \mathcal{B}'$ and $(E, \mathcal{B})$ is webbed, then so is $(E, \mathcal{B}')$.*

(5): *Every bornologically closed subspace $F \subseteq E$ of a webbed convex bornological space is webbed.*

(6): *Countable products of webbed convex bornological spaces are webbed.*

(7): *Countable projective limits of webbed convex bornological spaces are webbed.*

(8): *Countable coproducts of webbed convex bornological spaces are webbed.*

Proof. (1): Being a Fréchet space, E carries a topological web and consequently is bornologically webbed. For details see (4.8).

(2): First of all, we show that E_∞ is separated:
The final convex bornology of an inductive limit in the category CBS is already described by all subsets of sets of the form $f_i(B)$, where $f_i : E_i \to E$ and B is bounded in E_i. If in addition the mappings f_i are injective, we observe that when a set C is bounded in the final convex bornology, then $f_i^{-1}(C) \subseteq f_i^{-1}(f_i(B)) = B$ is bounded in E_i for some i.

So if all E_i are separated convex bornological spaces, then the same is true for the inductive limit.

For each $n \in \mathbb{N}$ let $(V^{(n)}, b^{(n)})$ be the given bornological web on E_n. Now define a web $(V^{(\infty)}, b^{(\infty)})$ on E_∞ as follows:

$$\forall n \in \mathbb{N} : \quad V^{(\infty)}(n) := E_n \quad \text{and}$$
$$\forall\ (n_0, \ldots n_k) \in \mathbb{N}^{k+1},\ k > 0 : \quad V^{(\infty)}(n_0, \ldots n_k) := V^{(n_0)}(n_1, \ldots, n_k)\,.$$

For $s : \mathbb{N} \to \mathbb{N}$ set $\widetilde{s} := (s(1), s(2), \ldots)$ and $b^{(\infty)}(s) := c$, where $c(0) := 1$ and $c(k) := b^{(s(0))}(\widetilde{s})(k-1)$, for $k \geq 1$. Indeed, $(V^{(\infty)}, b^{(\infty)})$ is a bornological web:

(BW1) – (BW3) are trivial.

(BW4): Let $s : \mathbb{N} \to \mathbb{N}$ be given and choose $y_k \in V^{(\infty)}(s(0), \ldots, s(k))$, $k \in \mathbb{N}$. Then the series $\sum_{k\geq 0} b^{(\infty)}(s)_k y_k = y_0 + \sum_{k\geq 1} b^{(s(0))}(\widetilde{s})_{k-1} y_k$ converges bornologically in $E_{s(0)}$.

Note that a set which is bounded in some E_n belongs to the final convex bornology of E_∞.

(3): Let E be a complete separated convex bornological space with a countable basis. Then there exists a monotonic basis $\mathcal{B}$ of the same bornology which consists of Banach disks, and hence $E = \bigcup_{B\in\mathcal{B}} E_B = \varinjlim_{B\in\mathcal{B}} E_B$. Since the E_B are Banach spaces and consequently bornologically webbed by (1), the bornological inductive limit is again webbed.

(4): Indeed, one may use the same bornological web.

(5): For a given bornological web (V, b) on E define a bornological web (V_F, b_F) on F by $V_F(n_0, \ldots, n_k) := F \cap V(n_0, \ldots, n_k)$ and $b_F := b$. Then (BW1) – (BW3) are clear, and (BW4) follows from the assumption that F is bornologically closed in E.

(6): First of all, notice that products of separated convex bornological spaces are separated.

Let (E_n) be a sequence of webbed convex bornological spaces with corresponding webs $(V^{(n)}, b^{(n)})$.

There are bijections $f_n : \mathbb{N}^{n+1} \xrightarrow{\cong} \mathbb{N}$, for all $n \in \mathbb{N}$. Given $s(k) \in \mathbb{N}$, there exists $(a_0^{(k)}, \ldots, a_k^{(k)}) \in \mathbb{N}^{k+1}$ such that $s(k) = f_k(a_0^{(k)}, \ldots, a_k^{(k)})$.

For $n \in \mathbb{N}$ we define mappings $s_n : \mathbb{N} \to \mathbb{N}$ by $s_n(k) := a_n^{(n+k)}$, and we set

$$\forall k \in \mathbb{N} : \quad V_{s,k} := V^{(0)}_{s_0,k} \times V^{(1)}_{s_1,k-1} \times \ldots \times V^{(k)}_{s_k,0} \times \prod_{i>k} E_i \,.$$

If $t : \mathbb{N} \to \mathbb{N}$ is such that $t(i) = s(i)$, $0 \le i \le k$, then clearly $V_{s,k} = V_{t,k}$, since the f_n are bijections. We then get a well-defined mapping $V : \bigcup_{k\in\mathbb{N}} \mathbb{N}^k \to \mathcal{P}(\prod_{n\in\mathbb{N}} E_n)$ by setting $V(\emptyset) = \prod_{n\in\mathbb{N}} E_n$ and $V(n_0, \ldots, n_k) := V_{s,k}$, where $s : \mathbb{N} \to \mathbb{N}$ and $s(i) = n_i$, $0 \le i \le k$.

Let further $b : \mathbb{N}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}_{>0}$ be defined as follows:

$$\forall s : \mathbb{N} \to \mathbb{N}, \forall k \in \mathbb{N} : \quad b(s)(k) := \min\{b^{(i)}(s_i)(k-i) : 0 \le i \le k\}$$

Indeed, (V, b) is a bornological web on $\prod_{n\in\mathbb{N}} E_n$:

(BW1): Clearly products of absolutely convex sets are absolutely convex.

(BW2) is true by definition.

(BW3): First, fix $x = (x_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} E_n$. Since $V^{(0)}$ satisfies (BW2) and (BW3), there is a $\lambda > 0$ and a $u : \mathbb{N} \to \mathbb{N}$ with $x_0 \in \lambda V^{(0)}_{u,0}$. Now set $a_0^{(k)} := u(k)$ and $a_i^{(k)} := 1$, for $k \ge i \ge 1$. Then we obtain a mapping $s : \mathbb{N} \to \mathbb{N}$ via $s(n) := f_n(a_0^{(n)}, \ldots, a_n^{(n)})$. By definition $s_0 = u$. It is obvious that x is absorbed by the set $V_{s,0} = V^{(0)}_{s_0,0} \times \prod_{i>0} E_i$.

Second, let $x = (x_k)_{k\in\mathbb{N}} \in V_{s,n} = V^{(0)}_{s_0,n} \times V^{(1)}_{s_1,n-1} \times \ldots \times V^{(n)}_{s_n,0} \times E_{n+1} \times \prod_{i>n+1} E_i$ be given. Since the $V^{(i)}$ are topological webs, there are $\lambda > 0$ and mappings $t^i : \mathbb{N} \to \mathbb{N}$ such that $\forall\, 0 \le i \le n+1 : x_i \in \lambda V^{(i)}_{t^i,n-i+1}$ and $\forall\, 0 \le i \le n$, $\forall\, 0 \le k \le n-i : t^i(k) = s_i(k)$. For $i > n+1$ choose $t^i : \mathbb{N} \to \mathbb{N}$ arbitrarily. Define $t : \mathbb{N} \to \mathbb{N}$ by $t(n) := f_n(t^0(n), t^1(n-1), \ldots, t^n(0))$. Then t_i coincides with t^i, for $0 \le i \le n+1$. Hence $x \in \lambda \left(V^{(0)}_{t_0,n+1} \times V^{(1)}_{t_1,n} \times \ldots \times V^{(n+1)}_{t_{n+1},0} \times \prod_{i>n+1} E_i\right) = V_{t,n+1}$. Since $t(k) = f_k(t^0(k), t^1(k-1), \ldots, t^k(0)) = f_k(s^0(k), s^1(k-1), \ldots, s^k(0)) = s(k)$ for $0 \le k \le n$, V satisfies (BW3).

(BW4): Since for countable products of separated convex bornological spaces the concept of bornological convergence coincides with the componentwise bornological convergence, it suffices to show that for a given sequence $s : \mathbb{N} \to \mathbb{N}$ and $x_k \in V_{s,k}$ the n-th projection of the series $\sum_k b(s)_k x_k$ converges bornologically in E_n, for all $n \in \mathbb{N}$. Indeed, we have

$$\begin{aligned}\mathrm{pr}_n\left(\sum_{k=0}^{l} b(s)_k x_k\right) &= \sum_{k=0}^{n-1} b(s)_k x_k^{(n)} + \sum_{k=n}^{l} b^{(n)}(s_n)(k-n) \cdot \frac{b(s)_k}{b^{(n)}(s_n)(k-n)}\, x_k^{(n)} \\ &= \sum_{k=0}^{n-1} b(s)_k x_k^{(n)} + \sum_{k=n}^{l} b^{(n)}(s_n)(k-n)\, y_{k-n}^{(n)},\end{aligned}$$

for all $l \geq n$. Observe that by the definition of $b(s)$

$$\forall k \geq n : \left|\frac{b(s)_k}{b^{(n)}(s_n)(k-n)}\right| \leq 1$$

and that the sets $V_{s_n,k-n}^{(n)}$ are circled. Hence $y_{k-n}^{(n)} := \frac{b(s)_k}{b^{(n)}(s_n)(k-n)}\, x_k^{(n)}$ belongs to $V_{s_n,k-n}^{(n)}$. Now the series $\sum_{k\geq n} b^{(n)}(s_n)(k-n)\, y_{k-n}^{(n)}$ converges bornologically, since $(E_n, (V^{(n)}, b^{(n)}))$ is a webbed convex bornological space, and we are done.

(7): The projective limit of a diagram $\mathcal{F}$ is the bornologically closed linear subspace

$$\{(x_\alpha) \in \prod \mathcal{F}(\alpha) : \mathcal{F}(f)(x_\alpha) = x_\beta \text{ for all } f : \alpha \to \beta\}.$$

Now the assertion is a consequence of (6) and (5).

(8): Let $(E_n)_{n\in\mathbb{N}}$ be a sequence of webbed convex bornological spaces. Then the direct sum may be obtained in the following way:

$$\bigoplus_{n\in\mathbb{N}} E_n = \varinjlim_{n\in\mathbb{N}} \prod_{i=0}^{n} E_i\,,$$

where equality holds in the category CBS. Now (4) and (2) imply (5). $\square$

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