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ON THE FRAGMENTAL STRUCTURES

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Abstract. In this work we study the fragment structures over a ring extension R of a ring R_0. The defining conditions of the fragments with the partial actions on the descending chains of R_0-modules measure how far they are from being R-modules. The category of R-fragments lies between the categories of R_0-modules and of R-modules. Inspite of R-fragments, in a general setting, are far from being R-modules; they behave, in some ways, the same as R-modules. We prove some imprtant results for finitely spanned fragments and some of their related properties.

1. Introduction

Let R be a ring with unity 1. Let $R_0 \subseteq R_1 \subseteq \ldots \subseteq R$ be a positive filtration of R; with $1 \in R_0$. An abelian group $(M, +)$ with the descending chain

$$M = M_0 \supseteq M_1 \supseteq M_2 \supseteq \ldots \supseteq M_n \supseteq \ldots$$

of subgroups of M is called a left R-fragment with respect to the mappings $\varphi_{i,j} : R_i \times M_j \to M_{j-i}$ (for all i, j; with $j \geq i$) if the following conditions are satisfied:

1. $\varphi_{i,j}|_{R_q \times M_r} = \varphi_{q,r}$; for all $q \leq i \leq j \leq r$.
2. $\varphi_{i,j}(\alpha, m+n) = \varphi_{i,j}(\alpha, m) + \varphi_{i,j}(\alpha, n)$; for all $\alpha) \in) R_i, m, n \in M_j$, and $i \leq j$.
3. $\varphi_{i,j}(\alpha+\beta, m) = \varphi_{i,j}(\alpha, m) + \varphi_{i,j}(\alpha, m)$; for all $\alpha, \beta \in R_i, m \in M_j$, and $i \leq j$.
4. $\varphi_{i,i}(\alpha, \varphi_{i,(i+j)}(\beta, m)) = \varphi_{(i+j),(i+j)}(\alpha\beta, m)$; for all $\alpha \in R_i, \beta \in R_j$, and $m \in M_{i+j}$.
5. $\varphi_{i,j}(1, m) = m$; for all $m \in Mj$.

A right R-fragment can be defined in a similar fasion. For all $\alpha \in R_i$ and $m \in M_j$, we write αm for $\varphi_{i,j}(\alpha, m)$. We shall usually say simply that M, rather than $M = M_0 \supseteq M_1 \supseteq M_2 \supseteq \ldots \supseteq M_n \supseteq \ldots$ with respect to $\varphi_{i,j} : R_i \times M_j \to M_{j-i}$ (for all i, j; with $j \geq i$), is a left R-fragment. This allows some ambiguity, for a given chain of abelian groups may admit more than one left R-fragment structure. We clear this out by fixing a chain of subgroups of M, $M_0 \supseteq M_1 \supseteq M_2 \supseteq \ldots \supseteq M_n \supseteq \ldots$ and certain maps $\varphi_{i,j} : R_i \times M_j \to M_{j-i}$.

Observe that, any left R-module M can be considered as a left R-fragment with respect to the chain $M = M_0 \supseteq M_1 \supseteq M_2 \supseteq \ldots \supseteq M_n \supseteq \ldots$, where $M_i = M$ for all i, and $R_0 \subseteq R_1 \subseteq \ldots \subseteq R$ is any positive filtration of R, with rm taken to be

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the left multiplication of elements of R by elements of M as a left R-module; for all $r \in R_i$, $m \in M_j = M$. Observe also that, each abelian subgroup M_i, in the chain of a left R-fragment M, is a left R_0-module.

Let M be a left R-fragment, we denote $\bigcap_{i\in I} M_i$ by $B(M)$ and call it the body of M. If the filtration of R is exhaustive; i.e. $R = \bigcup_{i\in I} R_i$, then the left actions of the elements of R_i on the elements of M_j (for all i, j) induce an R-module structure on $B(M)$. It is clear that $B(M)$ is the largest abelian subgroup of M on which the left actions of R_i on M_j form a left R-module structure.

In this paper, R will be a filtered ring with a positive exhaustive filteration $\{R_i\}_{i\in I}$; i.e. $R_0 \subseteq R_1 \subseteq \ldots \subseteq R$, and $R = \bigcup_{i\in I} R_i$.

Let M be a left R-fragment. We simply say X is subset of M, for any subset X of M_0; and we say x is an element of M for any element x of M_0. Let $x \in M$ be an arbitrary element, then either $x \in M_n$ and $x \notin M_{n+1}$ for some positive integer n, or $x \in B(M)$. In the first case x is called of depth n (denoted by $d_M(x) = n$), while in the second case we say x has an infinite depth (i.e. $d_M(x) = \infty$). It is clear that $d_M(0) = \infty$.

Let $N = N_0 \supseteq N_1 \supseteq N_2 \supseteq \ldots \supseteq N_n \supseteq \ldots$ be a chain of subgroups of a left R-fragment M. N is called a subfragment of M if $N_i \subseteq M_i$ $(i = 1, 2, \ldots)$, and for all i, j with $j \geq i$, $rx \in N_{j-i}$ for all $r \in R_i$ and $x \in N_j$. If $N_i = 0$ for all i, then N is called the zero subfragment, and is denoted by 0.

2. Nontrivial subfragments

A subfragment N of a left R-fragment M is called a nontrivial subfragment in case of if $R_i x \subseteq N_0$, for $x \in N_0$ with $d_M(x) \geq i$, then $x \in N_i$. N is called an improper (otherwise it is called proper) subfragment of M if, for some $j \geq 0$, $N_0 = M_j$, for each $i \geq 1$ there exists $k(i) \geq 1$ such that $N_i = M_{j+k(i)}$.

Lemma 1. *Let M be a left R-fragment such that $M_i = M_{i+1}$ for all $i = 0, 1, \ldots$. If N is a nontrivial subfragment such that N_0 is an R-submodule of M, then $N_i = N_{i+1}$ for all i.*

Corollary 2. *Let M be a left R-fragment such that the filteration of R is given by $R_0 = R_1 = \ldots = R$, and $M_i = M_{i+1}$ for all $i = 0, 1, \ldots$; i. e. M is a left R-module. Then the chain of any nontrivial subfragment N of M must be of the form $N_0 = N_1 = N_2 = \ldots = N_n = \ldots$ for some R-submodule N_0 of M.*

Lemma 3. *Let N be a subfragment of a left R-fragment M such that $N_0 = M_0$. If N is nontrivial, then $N_i = M_i$ for all i ; i.e. $N = M$.*

Proof. If $x \in M_k$, then $R_k x M_0 = N_0$. This yields $x \in N_k$, and hence $M_k = N_k$ for all k. □

Let M be a left R-fragment, and $\{N^i\}_{i\in I}$ be a family of subfragments of M. Then the chain

$$\bigcap_{i\in I} N_0^i \supseteq \bigcap_{i\in I} N_1^i \supseteq \ldots \supseteq \bigcap_{i\in I} N_n^i \supseteq \ldots$$

of subgroups of M forms a subfragment. This subfragment of M is called the fragment intersection of the family $\{N^i\}_{i\in I}$, and is denoted by $\bigcap_{i\in I} N^i$. Observe that

$$\bigcap_{i\in I} N^i := \bigcap_{i\in I} N_0^i.$$

Let $\{N^i\}_{i\in I}$ be a chain of subfragments of M, and consider the following chain

$$\bigcap_{i\in I} N_0^i \supseteq \bigcap_{i\in I} N_1^i \supseteq \ldots \supseteq \bigcap_{i\in I} N_k^i \supseteq \ldots$$

of subgroups of M. It is clear that it forms a chain of a subfragment of M. This subfragment is called the fragment union of the chain $\{N^i\}_{i\in I}$.

Lemma 4. *The intersection of nontrivial subfragments of a left R-fragment is nontrivial.*

Proof. Let $\{N^i\}_{i\in I}$ be a family of nontrivial subfragments of a left R-fragment M, and let $x \in \bigcap_{i\in I} N_0^i$. Then the nontriviality of each N^i yeiles $x \in \bigcap_{i\in I} N_j^i$, whenever $R_j x \subseteq \bigcap_{i\in I} N_0^i$ and $d_M(x) \geq j$. □

3. Subfragment spanned by a subset

Let M be a left R-fragment, and X be a subset of M. Define the subset $L(X)$ of M by:

$$L(X) = \{r_n(r_{n-1}((r_0x))) : x \in X, \text{ for some } n\, r_i \in R_i, \text{ and } i \geq d_M(r_{i-1}(\ldots(r_0x)); i = 0, 1, \ldots, n\}.$$

A subfragment N of M is said to be strictly containing X if $L(X) \subseteq N$.

Lemma 5. *Let $\{X_i\}_{i\in I}$ be a family of subsets of a left R-fragment M. Then:*

1) *$X_i \subseteq L(X_i)$, and if $X_i \subseteq X_j$, then $L(X_i) \subseteq L(X_j)$.*
2) *$L(\bigcup_{i\in I} X_i) = \bigcup_{i\in I} L(X_i)$.*

Proof is clear.

Lemma 6. *Let X be a subset of a left R-fragment M, then the intersection of all subfragments, which are strictly containing X, is strictly containing X.*

Proof is clear.

Let Γ be the set of all nontrivial subfragments of M which are strictly containing a subset X. By Lemma 4, and Lemma 6, the intersection $\bigcap \Gamma$ is again a nontrivial subfragment of M, which is strictly containing X. We call it the subfragment spanned by X, and is denoted by $\prec X \succ$. If X is the empty set, then by convention $\prec X \succ = 0$.

Lemma 7. *Let M be a left R-fragment, and X be a subset of M. Define*

$$\begin{aligned} N_0 &= \left\{ \sum_{i=1}^{n} t_i : t_i \in L(X), n \text{ is a positive integer} \right\}, \\ N_k &= \{y \in N_0 : d_M(y) \geq k, \text{ and } R_k y \subseteq N_0\}; \quad k = 1, 2, \ldots \,. \end{aligned}$$

Then

$$N := N_0 \supseteq N_1 \supseteq N_2 \supseteq N_n \supseteq \dots$$

is a subfragment of M, and the subfragment spanned by X is just N.

Proof. It is clear that, from the definition of N_k, that

$$N := N_0 \supseteq N_1 \supseteq N_2 \supseteq N_n \supseteq \dots$$

is a descending chain of abelian subgroups of M. Now let $y \in N_k$ and $r_i \in R_i$, where $i \leq k$. Since $d_M(y) \geq k$, we have that $r_i y \in M_{k-i}$; and hence $d_M(r_i y) \geq k - i$. We have

$$R_{k-i}(r_i y) \subseteq R_{k-1}(R_i y) \subseteq R_k y \subseteq N_0;$$

and thus $r_i y \subseteq N_{k-i}$. This shows that N is a subfragment of M. It is clear, from the definition of N_0, that $L(X) \subseteq N_0$; i.e. N is strictly containig X. It is also clear, from the definition of N_i, that N is nontrivial. Therefore $\prec X \succ \subseteq N$. To show that $N \subseteq X$, let K be any nontrivial subfragment which is strictly containing X. Since $L(X) \subseteq K$, we have that N_0 is contained in K_0. Now let $y \in N_m$, for some positive integer m. Hence $R_m y \subseteq N_0 \subseteq K_0$, and thus the nontriviality of K yeilds $y \in K_m$. Therefore $N \subseteq \prec X \succ$. □

If X is a subset of a left R-fragment M such that $\prec X \succ = M$, then X is said to span M, and X is called a spanning set for M. M with a finite spanning set is said to be finitely spanned. M with a single element spanning set is a cyclic fragment.

Corollary 8. *If X is a spanning set for a left R-fragment M, then the chain of M is given by :*

$$M_k = \left\{ \sum_{i=1}^{n} y_i : y_i \in L(X), d_M(\sum_{i=1}^{n} y_i) \geq k, n \text{ is a positive integer} \right\}; \quad k = 0, 1, \dots$$

Corollary 9. *If X is a spanning set for a left R-fragment M, with $d_M(x) = \infty$ for all $x \in X$, then $M_i = M_{i+1}$ for all i; i.e. the fragment structure forms a left R-module structure on M.*

A nontrivial subfragment K of a left R-fragment M is said to be a strict subfragment if K is strictly containing K_0. By Lemma 5, we have that $L(X) \subseteq K_0$ for any subset X of K_0, and that $L(K_0) = K_0$.

Lemma 10. *The following are equivalent for a nontrivial subfragment K of M:*

1) *K is strict,*
2) *$\prec X \succ$ is contained in K, for any subset X of K,*
3) *$K_i = M_i \cap K_0$, for all $i = 0, 1, 2 \dots$,*
4) *$d_M(x) = d_K(x)$, for all $x \in K$.*

Proof. 1) $\Rightarrow$ 2) is clear.

2) $\Rightarrow$ 3) It is clear that, for all i, $K_i \subseteq M_i \cap K_0$. Now let $x \in M_i \cap K_0$. Then by 2), $R_i x \subseteq \prec x \succ \subseteq K_0$. But since K is nontrivial, we have that $x \in K_i$. Hence $\mathrm{K}_i = M_i \cap K_0$.

3) $\Rightarrow$ 4) It is clear that $d_K(x) \leq d_M(x)$ for all $x \in K$. Now let $d_M(x) = n$, $x \in K$, then $x \in K_0 \cap M_n = K_n$, and $x \notin K_{n+1}$ (due to $K_{n+1} \subseteq M_{n+1}$); i.e. $d_K(x) = n$. It is clear that $d_K(x) = \infty$ whenever $d_M(x) = \infty$.

4) $\Rightarrow$ 1) Let X be a subset of K. By 4), it follows that $L(X) \subseteq K$; i.e. K is strictly containing X for all subsets X of K. Hence we have 1). □

Remarks. 1) From Corollary 8, it is clear that a left R-module M is finitely generated if and only if it is finitely spanned as a natural left R-fragment.

2) If M is a left R-fragment, and if the factor R_0-module M_n/M_{n+i} is finitely generated for all $n = 0, 1, \ldots$, then M need not be a finitely spanned fragment. In fact $M_n/M_{n+i} = 0$ is finitely generated for every R-module M, and for all $n = 0, 1, \ldots$.

3) The intersection of an arbitrary family of strict subfragments of a left R--fragment is strict.

Let M be a left R-fragment, and let $\{N^i\}_{i=1}^m$ be a family of subfragments of M, where the chain of N^i is given by

$$N_0^i \supseteq N_1^i \supseteq N_2^i \supseteq \ldots \supseteq N_n^i \supseteq \ldots;$$

$i = 1, 2, \ldots, m$. Then the following chain

$$\sum_{i=1}^m N_0^i \supseteq \sum_{i=1}^m N_1^i \supseteq \ldots \supseteq \sum_{i=1}^m N_n^i \supseteq \ldots$$

forms a subfragment of M (denoted by $\sum_{i=1}^m N^i$) called the fragment sum of the family $\{N^i\}_{i=1}^m$. If each N^i is a strict subfragment, then we define the fragment sum $\sum_{i=1}^m N^i$ to be the subfragment of M spanned by the subset $\bigcup_{i=1}^m N_0^i$. It is clear, by Lemma 5, that

$$L\left(\bigcup_{i=1}^m N_0^i\right) = \bigcup_{i=1}^m L\left(N_0^i\right) = \bigcup_{i=1}^m N_0^i.$$

Hence $\left(\sum_{i=1}^m N^i\right)_0 = \prec \bigcup_{i=1}^m N_0^i \succ_0 = \sum_{i=1}^m N_0^i$, whenever each of N^i is strict.

Remark. If $\{N^i\}_{i\in I}$ is an arbittrary collection of strict subfragments of a left R-fragment M, then motivated by the finite sum of strict subfragments, we define the fragment sum $\sum_{i\in I} \mathrm{N}^i$of $\{N^i\}_{i\in I}$ to be the subfragment of M spanned by $\bigcup_{i\in I} N_0^i$.

Lemma 11. *Let $\{N^i\}_{i\in I}$ be an arbittrary collection of strict subfragments of a left R-fragment M. Then*

$$K := \bigcup \left\{ \sum_{i\in F} Ni : F \textit{ is a finite subset of } I \right\},$$

with the chain

$$K_0 \supseteq K_1 \supseteq K_2 \supseteq \ldots \supseteq K_n \supseteq \ldots,$$

where

$$K_n = \bigcup \left\{ (\sum_{i\in F} N^i)_n : F \textit{ is a finite subset of } I \right\},$$

is a strict subfragment of M.

Proof. It is clear that K with the given chain is a subfragment of M. Let $x \in K_0$ such that $R_j x \subseteq K_0$, and $j \leq d_M(x)$. Then $x \in (\sum_{i\in F} N^i)_0$ for some finite subset F of I. Since $\sum_{i\in F} N^i$ is a strict subfragment of M, we have by Lemma 10 that $d_{N(F)}(x) = d_M(x) = m \geq j$, where $N(F) := \sum_{i\in F} N^i$; i.e. $x \in (\sum_{i\in F} N^i)_m$. It follows that

$$R_j x \subseteq (\sum_{i\in F} N^i)_{m-j} \subseteq (\sum_{i\in F} N^i)_0$$

(due to $\sum_{i\in F} N^i$ a subfragment of M). Since $\sum_{i\in F} N^i$ is nontrivial, we have that

$$x \in (\sum_{i\in F} N^i)_i \subseteq K_i.$$

This shows that K is nontrivial. Since

$$d_M(x) = d_{N(F)}(x) \leq d_K(x) \leq d_M(x),$$

whenever $x \in \sum_{i\in F} N^i := N(F)$, and $x \in K_0$, it follows that

$$d_M(x) = d_K(x)$$

for all $x \in K_0$. Therefore K is strict. □

Theorem 12. *Let* M *be a left* R*-fragment, and* $\{N^i\}_{i\in I}$ *be a family of strict subfragments of* M. *Then*

$$\sum_{i\in I} N^i = \bigcup \left\{ \sum_{i\in F} N^i : F \textit{ is a finite subset of } I \right\}.$$

Proof. By Lemma 5,

$$L(\bigcup_{i\in I} N_0^i) = \bigcup_{i\in I} L(N_0^i) = \bigcup_{i\in I} N_0^i;$$

and hence

$$\prec \bigcup_{i\in I} N_0^i \succ_0 = (\sum_{i\in I} N^i)_0 = \sum_{i\in I} N_0^i = \bigcup \left\{ \sum_{i\in F} N_0^i : F \text{ is a finite subset of } I \right\}.$$

In Lemma 11, we have seen that $K := \bigcup\{\sum_{i\in F} N^i : F$ is a finite subset of $I\}$, with the given chain, is a strict subfragment of M. From the definition of the fragment sum of the family $\{N^i\}_{i\in I}$, we deduce that $\sum_{i\in I} N^i := N$ is also a strict subfragment of M. Since $K_0 = N_0$, and $d_N(x) = d_M(x) = d_K(x)$ for all $x \in N_0 (= K_0)$, it follows that $K = N$. □

Theorem 13. *Let M be a left R-fragment. Then M is finitely spanned if and only if for every family $\{N^i\}_{i\in I}$ of strict subfragments of M,*

$$\sum_{i\in I} N^i = M \text{ implies } \sum_{i\in F} N^i = M,$$

for some finite subset F of I.

Proof. Let X be a finite spanning set for M, and let $\{N^i\}_{i\in I}$ be a family of strict subfragments of M, with $\sum_{i\in I} N^i = M$. It follows that $X \subseteq \sum_{i\in F} N_0^i$ for some finite subset F of I. Hence $M = \prec X \succ \subseteq \sum_{i\in F} N^i$, and therefore $M = \sum_{i\in F} N^i$.

It is clear that $M = \sum_{x\in M_0} \prec x \succ$. Thus, by assumption, $M = \sum_{x\in X} \prec x \succ = X$ for some finite subset X of M_0 ; i.e. M is finitely spanned. □

Theorem 14. *Let M be a left R-fragment. If for each i, M_i is finitely generated as an R_0-module, then every improper subfragment of M is finitely spanned.*

Proof. Without loss of generality we may show that M is finitely spanned. Let $\mathrm{M}_0 = \sum_{i=1}^{n} \mathrm{R}_0 \mathrm{x}_i$. We claim that M as an R-fragment is spanned by X, where $X = \{x_1, x_2, \ldots, x_n\}$. To this end, let $m \in M_k$. Then $m = \sum_{j=1}^{s} r_j y_j$, where $\{y_1, y_2, \ldots, y_s\}$ is a generating set for M_k as an R_0-module, and $r_j \in R_0$. But for each $i = 1, 2, \ldots, s$, we have

$$y_i = \sum_{j=1}^{n} \beta_{ij} x_j,$$

where $\beta_{ij} \in R_0$. Hence

$$m \in \left\{ \sum_{i=1}^{n} y_i : y_i \in L(X), d_M(\sum_{i=1}^{n} y_i) \geq k \right\};$$

and therefore

$$M_k = \left\{ \sum_{i=1}^{n} y_i : y_i \in L(X), d_M(\sum_{i=1}^{n} y_i) \geq k \right\}; \qquad k = 0, 1, \ldots.$$

Therefore we have our claim. □

4. Factor fragments

Just as for modules, there is a factor fragment of a left R-fragment with respect to each of strict subfragments. Let M be a left R-fragment and let N be a strict subfragment. Then it is easy to see that the chain of abelian factor groups

$$M_0/N_0 \supseteq (M_1 + N_0)/N_0 \supseteq \ldots \supseteq (M_n + N_0)/N_0 \supseteq \ldots$$

is a left R-fragment relative to the left multiplication defined via

$$r(x + N_0) = rx + N_0$$

for all $x \in M_n$, $x \in M_n$, and $r \in R_m$, where $m \leq n$.

The resulting fragment (denoted by M/N), is called the left R-factor fragment of M relative to N. Since N is a strict subfragment, then the given left multiplication is well defined. In fact if $f_n : M_n/(N_0 \cap M_n) \to (M_n + N_0)/N_0$ are the natural abelian group isomorphisms ($n = 0, 1, 2, \ldots$), where $f_0 = 1$, then the sequence of monomorphisms

$$M_0/N_0 \overset{g_1}{\leftarrow} M_1/N_1 \overset{g_2}{\leftarrow} M_2/N_2 \overset{g_3}{\leftarrow} \ldots M_n/N_n \overset{g_{n+1}}{\leftarrow} \ldots,$$

where $g_n = (f_{n-1}^{-1}|L_n)f_n$ and $L_n := (M_n + N_0)/N_0$, gives rise to the chain

$$M_0/N_0 \supseteq g_1(M_1/N_1) \supseteq g_1g_2(M_2/N_2) \supseteq \ldots \supseteq g_1g_2\ldots g_n(M_n/N_n) \supseteq \ldots.$$

Hence we may consider the chain of the factor R-fragment of M relative to N as:

$$M_0/N_0 \supseteq M_1/N_1 \supseteq M_2/N_2 \supseteq \ldots \supseteq M_n/N_n \supseteq \ldots,$$

where the left multiplication is give by:

$$r(m + N_n) = rm + N_{n-i},$$

for all $m + N_n \in M_n/N_n$ and $r \in R_i$ where $i \geq n$.

Lemma 15. *Let M be a left R-fragment spanned by a subset X, and let N be a strict subfragment of M. Then M/N is spanned by $\overline{X} = \{x + N_0 : x \in X\}$.*

Proof. From the definition of the left multiplication of R on M/N, we have that

$$r_n(r_{n-1}((r_0x))) + N_0 = r_n(r_{n-1}(\ldots(r_0(x + N_0))),$$

for all $x \in X$, n is a non negative integer, and $r_i \in R_i$; where $i \leq d_M(r_{i-1}(\ldots(r_0x))$, and $i = 0, 1, \ldots, n$. Hence, for all $k = 0, 1, \ldots$, we have that

$$(M/N)_k = (M_k + N_0)/N_0 = \left[\left\{\sum_{i=1}^{n} y_i : y_i \in L(X), d_M(\sum_{i=1}^{n} y_i) \geq k\right\} + N_0\right]/N_0$$
$$= \left\{\sum_{i=1}^{n}(y_i + N_0) : y_i \in L(X), d_M(\sum_{i=1}^{n} y_i) \geq k\right\} = \prec \overline{X} \succ_k .$$

□

Corollary 16. *Let M be a finitely spanned left R-fragment, and let N be a strict subfragment of M. Then M/N is finitely spanned.*

Theorem 17. *Let M be a left R-fragment, and let N be a strict subfragment of M. If M/N and N are finitely spanned, then M is finitely spanned.*

Proof. Let $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be a spanning set of M/N, and $\{\beta_1, \beta_2, \ldots, \beta_m\}$ be a spanning set of N. We claim that M is spanned by

$$X := \{x_1, x_2, \ldots, x_n, \beta_1, \beta_2, \ldots, \beta_m\},$$

where $x_i \in M_0$ such that $x_i + N_0 = \alpha_i$ ($i = 1, 2, \ldots, n$). Now let $m \in M_k$, it follows that $m + N_0 = \sum_{i=1}^{s} t_i, t_i \in L(\{\alpha_1, \alpha_2, \ldots, \alpha_n\})$, and $d_{M/N}(\sum_{i=1}^{s} t_i) \geq k$. Hence $m + N_0 = (\sum_{i=1}^{s} b_i) + N_0$, where $b_i \in L(\{x_1, x_2, \ldots, x_n\})$, and $d_M(\sum_{i=1}^{s} b_i) \geq k$.

Since N is a strict subfragment of M, we have that $d_N(m - \sum_{i=1}^{s} b_i) \geq k$. Hence $m - \sum_{i=1}^{s} b_i$, as an element of N_k, can be written as $m - \sum_{i=1}^{s} b_i = \sum_{i=1}^{\lambda} c_i$, where $c_i \in L(\{\beta_1, \beta_2, \ldots, \beta_m\})$. Therefore

$$m = \sum_{i=1}^{s} b_i + \sum_{i=1}^{\lambda} c_i \in \prec \{\{x_1, x_2, \ldots, x_n, \beta_1, \beta_2, \ldots, \beta_m\}\} \succ_k;$$

i.e.

$$M_k = \left\{ \sum_{i=1}^{n} y_i : y_i \in L(X), d_M(\sum_{i=1}^{n} y_i) \geq k \right\}, \qquad k = 1, 2, \ldots.$$

Thus we have our claim. □

5. Homomorphisms of Fragments

Let M and N be two left R-fragments. By a fragment homomorphism $f : M \to N$ we mean a function $f : M_0 \to N_0$ such that the following two conditions are satisfied:

1. $d_M(m) \leq d_N(f(m))$, for all $m \in M_0$,
2. $f(rm_1 + sm_2) = rf(m_1) + sf(m_2)$, for all $m_1, m_2 \in M_0$ and $r \in R_i, s \in R_j$, with $i \leq d_M(m_1)$, $j \leq d_M(m_2)$.

Let M and N be two left R-fragments, and let $f : M \to N$ be a fragment homomorphism. Then the chain

$$f(M_0) \supseteq f(M_1) \supseteq f(M_2) \supseteq \ldots \supseteq f(M_n) \supseteq \ldots$$

forms a subfragment of N. This subfragment is called the fragment image of f and is denoted by $\mathrm{Im} f$. If K is a subfragment of the fragment N, then the chain

$$f^{-1}(K_0) \supseteq f^{-1}(K_1) \cap M_1 \supseteq f^{-1}(K_2) \cap M_2 \supseteq \ldots \supseteq f^{-1}(K_n) \cap M_n \supseteq \ldots$$

is a subfragment of M. It is called the fragment inverse image of K under f, and is denoted by $f^{-1}(K)$. The kernel of f (denoted by $\mathrm{Ker} f$) is given by $f^{-1}(0)$, where 0 is the zero subfragment of N. f is called an epimorphism in case of the fragment image $\mathrm{Im} f$ and N, as fragments, are equal. It is called a monomorphism in case of the subfragment $\mathrm{Ker} f$ of M is the zero fragment. A homomorphism which is monomorphism and epimorphism is called an isomorphism.

Proposition 18. *Let M and N be two left R-fragments, and let $f : M \to N$ be a fragment homomorphism. If K is a strict subfragment of N, then $f^{-1}(K)$ is a strict subfragment of M.*

Proof. It is clear that

$$\begin{aligned} f^{-1}(K_0) \cap M_i &= f^{-1}(K_0) \cap f^{-1}(N_i) \cap M_i \\ &= f^{-1}(K_0 \cap N_i) \cap M_i = f^{-1}(K_i) \cap M_i = f^{-1}(K_i) \end{aligned}$$

(due to K strict), for each $i = 0, 1, 2, \ldots$; i.e. $f^{-1}(K)$ is a strict. □

Corollary 19. *Let M and N be two left R-fragments, and let $f : M \to N$ be a fragment homomorphism. Then $\ker f$ is a strict subfragment.*

Proposition 20. *Let M and N be two left R-fragments, and let $f : M \to N$ be a fragment homomorphism. If M is spanned by a subset X, then $\mathrm{Im} f$ is spanned by $f(X)$. Moreover if $g : M \to N$ is also a fragment homomorphism, then $f = g$ if and only if $f(x) = g(x)$ for all $x \in X$.*

Proof. It is clear that $f(r_n(r_{n-1}\ldots((r_0x)))) = r_n(r_{n-1}\ldots((r_0f(x))))$, for each $r_n(r_{n-1}\ldots((r_0x))) \in L(X)$. It follows that

$$\begin{aligned} f(M_k) &= f\left(\left\{\sum_{i=1}^{n} y_i : y_i \in L(X), d_M(\sum_{i=1}^{n} y_i) \geq k\right\}\right) \\ &= \left\{\sum_{i=1}^{n} f(y_i) : y_i \in L(X), d_M(\sum_{i=1}^{n} y_i) \geq k\right\} \\ &= \left\{\sum_{i=1}^{n} t_i : t_i \in L(f(X)), d_{f(M)}(\sum_{i=1}^{n} t_i) \geq k\right\}, \end{aligned}$$

for each $k = 0, 1, 2, \ldots$. This shows that $\mathrm{Im} f = \prec f(X) \succ$.

Now let $f(x) = g(x)$ for all $x \in X$. It is easy to check that the chain

$$K_0 \supseteq K_1 \supseteq K_2 \supseteq K_n \supseteq \ldots, \quad \text{where} \quad K_i = \{m \in M_i : f(m) = g(m)\},$$

is a strict subfragment of M. Since X is contained in K, we have that $K = M$; and hence $f = g$.

The converse is obvious. □

Corollary 21. *Homomorphic image of a finitely spanned fragment is finitely spanned.*

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