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RESTRICTED CONGRUENCE REGULARITY OF ALGEBRAS

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Abstract. An algebra $\mathcal{A}$ is called restricted regular with respect to a subset M of its base set if every single class of some congruence Θ on $\mathcal{A}$ determines $\Theta|M$. A variety is called restricted regular with respect to a unary term t if every of its members (A, F) is restricted regular with respect to $t(A)$. The well-known results on (weak) regularity are generalized to the "restricted case".

Definition 1. Let $\mathcal{A} = (A, F)$ be an algebra and $M \subseteq A$. $\mathcal{A}$ is called **restricted regular with respect to** M if for all $a \in A$ and $\Theta, \Phi \in \mathrm{Con}\mathcal{A}$ the following holds: If $[a]\Theta = [a]\Phi$ then $\Theta|M = \Phi|M$. For an arbitrary set M let ω_M denote the least equivalence relation on M.

Remark 1. *The case $M = A$ yields the concept of a regular algebra (see* [**2**],[**5**] *and* [**7**]*). Every algebra is restricted regular with respect to every at most one-element subset of its base set.*

Lemma 1. *If $\mathcal{A} = (A, F)$ is an algebra, $M \subseteq A$, $\mathcal{A}$ is restricted regular with respect to M and Θ is a congruence on $\mathcal{A}$ having a singleton class then $\Theta|M = \omega_M$.*

Proof. If $[a]\Theta = \{a\}$ for some $a \in A$ then $[a]\Theta = [a]\omega_A$ and hence $\Theta|M = \omega_A|M = \omega_M$. □

Definition 2. Let $\mathcal{V}$ be a variety and t a unary term of $\mathcal{V}$. $\mathcal{V}$ is called **restricted regular with respect to** t if every member (A, F) of $\mathcal{V}$ is restricted regular with respect to $t(A)$.

Remark 2. *The case $t(x) = x$ yields the concept of a regular variety (see* [**2**],[**5**] *and* [**7**]*). Every variety $\mathcal{V}$ is restricted regular with respect to every constant unary term of $\mathcal{V}$.*

Theorem 1. *Let $\mathcal{V}$ be a variety and t a unary term of $\mathcal{V}$. Then $\mathcal{V}$ is restricted regular with respect to t if and only if for each member $\mathcal{A} = (A, F)$ of $\mathcal{V}$ and each congruence Θ on $\mathcal{A}$ having a singleton class it holds $\Theta|t(A) = \omega_{t(A)}$.*

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Proof. If $\mathcal{V}$ satisfies the condition of the theorem, $\mathcal{A} = (A, F) \in \mathcal{V}$, $a \in A$, $\Theta, \Phi \in \mathrm{Con}\mathcal{A}$ and $[a]\Theta = [a]\Phi$ then

$$|[[a](\Theta \cap \Phi)](\Theta/(\Theta \cap \Phi))| = |[[a](\Theta \cap \Phi)](\Phi/(\Theta \cap \Phi))| = 1$$

and hence

$$(\Theta/(\Theta \cap \Phi))|t(A/(\Theta \cap \Phi)) = (\Phi/(\Theta \cap \Phi))|t(A/(\Theta \cap \Phi))$$

which implies $\Theta|t(A) = \Phi|t(A)$. The rest of the proof follows from Lemma 1. □

Theorem 2. *For an algebra $\mathcal{A} = (A, F)$, $a, b \in A$ and $M \subseteq A^2$ the following conditions are equivalent:*

(i) *$(a, b) \in \Theta(M)$.*

(ii) *There exists a positive integer n, $a_0, \dots, a_n \in A$, $(b_1, c_1), \dots, (b_n, c_n) \in M$ and unary polynomial functions $p_1, \dots, p_n$ over $\mathcal{A}$ with $a_0 = a$, $\{a_{i-1}, a_i\} = \{p_i(b_i), p_i(c_i)\}$ for $i = 1, \dots, n$ and $a_n = b$.*

(iii) *There exists a positive integer n, $(b_1, c_1), \dots, (b_n, c_n) \in M$ and binary polynomial functions $q_1, \dots, q_n$ over $\mathcal{A}$ with $q_1(b_1, c_1) = a$, $q_i(c_i, b_i) = q_{i+1}(b_{i+1}, c_{i+1})$ for $i = 1, \dots, n-1$ and $q_n(c_n, b_n) = b$.*

Proof. (i) ⇒ (ii):
See [**4**] (p. 55) or [**6**] (Theorem 4.19).
(ii) ⇒ (iii):
For $i = 1, \dots, n$ define $q_i(x, y) := p_i(x)$ if $p_i(b_i) = a_{i-1}$ and $q_i(x, y) := p_i(y)$ otherwise. Then $q_1, \dots, q_n$ are binary polynomial functions over $\mathcal{A}$ and for $i = 1, \dots, n-1$ it holds:

$$\begin{aligned}
&q_1(b_1, c_1) = p_1(b_1) = a_0 = a \text{ if } p_1(b_1) = a_0,\\
&q_1(b_1, c_1) = p_1(c_1) = a_0 = a \text{ otherwise},\\
&q_i(c_i, b_i) = p_i(c_i) = a_i \text{ if } p_i(b_i) = a_{i-1},\\
&q_i(c_i, b_i) = p_i(b_i) = a_i \text{ otherwise},\\
&q_{i+1}(b_{i+1}, c_{i+1}) = p_{i+1}(b_{i+1}) = a_i \text{ if } p_{i+1}(b_{i+1}) = a_i,\\
&q_{i+1}(b_{i+1}, c_{i+1}) = p_{i+1}(c_{i+1}) = a_i \text{ otherwise},\\
&q_n(c_n, b_n) = p_n(c_n) = a_n = b \text{ if } p_n(b_n) = a_{n-1} \text{ and}\\
&q_n(c_n, b_n) = p_n(b_n) = a_n = b \text{ otherwise}.
\end{aligned}$$

(iii) ⇒ (i):
$a = q_1(b_1, c_1)\, \Theta(M)\, q_1(c_1, b_1) = q_2(b_2, c_2)\, \Theta(M)\, \dots\, \Theta(M)\, q_n(c_n, b_n) = b$. □

Theorem 3. *For a variety $\mathcal{V}$ and a unary term t of $\mathcal{V}$ the following conditions are equivalent:*

(i) *$\mathcal{V}$ is restricted regular with respect to t.*

(ii) *There exists a positive integer n, ternary terms $t_1, \dots, t_n$ and 5-ary terms $s_1, \dots, s_n$ satisfying the following conditions:*
$t(x) = t(y)$ implies $t_1(x, y, z) = \dots = t_n(x, y, z) = z$,
$s_1(t_1(x, y, z), z, x, y, z) = t(x)$,

$s_i(z, t_i(x, y, z), x, y, z) = s_{i+1}(t_{i+1}(x, y, z), z, x, y, z)$ *for* $i = 1, \ldots, n-1$ *and* $s_n(z, t_n(x, y, z), x, y, z) = t(y)$.

(iii) *There exists a positive integer* n *and ternary terms* $t_1, \ldots, t_n$ *such that* $t_1(x, y, z) = \cdots = t_n(x, y, z) = z$ *is equivalent to* $t(x) = t(y)$.

Proof. (i) $\Rightarrow$ (ii):
Consider the free algebra $F_{\mathcal{V}}(x, y, z)$ of $\mathcal{V}$ generated by the free generators x, y, z. Then

$$[z]\Theta(t(x), t(y)) = [z]\Theta(\{z\} \times [z]\Theta(t(x), t(y)))$$

implies, according to (i),

$$(t(x), t(y)) \in \Theta(t(x), t(y))|t(F_{\mathcal{V}}(x, y, z)) = \Theta(\{z\} \times [z]\Theta(t(x), t(y)))|t(F_{\mathcal{V}}(x, y, z)).$$

In virtue of Theorem 2, there exists a positive integer n, $t_1, \ldots, t_n \in [z]\Theta(t(x), t(y))$ and binary polynomial functions $q_1, \ldots, q_n$ over $F_{\mathcal{V}}(x, y, z)$ with $q_1(z, t_1) = t(x)$, $q_i(t_i, z) = q_{i+1}(z, t_{i+1})$ for $i = 1, \ldots, n-1$ and $q_n(t_n, z) = t(y)$.

Now there exist 5-ary terms $s_1, \ldots, s_n$ with $s_i(u, v, x, y, z) = q_i(v, u)$ for $i = 1, \ldots, n$ and the elements $t_1, \ldots, t_n$ of $[z]\Theta(t(x), t(y))$ can be considered as ternary terms $t_1(x, y, z), \ldots \ldots, t_n(x, y, z)$. These terms satisfy the identities of (ii). Since $t_1, \ldots, t_n \in [z]\Theta(t(x), t(y))$ one obtains $t_1(x, y, z) = \cdots = t_n(x, y, z) = z$ provided $t(x) = t(y)$.
(ii) $\Rightarrow$ (iii):
$t_1(x, y, z) = \cdots = t_n(x, y, z) = z$ implies
$t(x) = s_1(z, z, x, y, z) = \cdots = s_n(z, z, x, y, z) = t(y)$.
(iii) $\Rightarrow$ (i):
If $\mathcal{A} = (A, F) \in \mathcal{V}$, $a, b, c \in A$, $\Theta \in \mathrm{Con}\mathcal{A}$, $[a]\Theta = \{a\}$ and $(t(b), t(c)) \in \Theta$ then

$$t_i(b, c, a) \in [t_i(b, c, a)]\Theta = t_i([b]\Theta, [c]\Theta, [a]\Theta) = [a]\Theta = \{a\}$$

for all $i = 1, \ldots, n$ according to (iii) whence $t(b) = t(c)$ again by (iii) which shows $\Theta|t(A) = \omega_{t(A)}$. In virtue of Theorem 1, $\mathcal{V}$ is restricted regular with respect to t. $\square$

Example 1. *The variety* $\mathcal{V}$ *of pseudocomplemented lattices* $(L, \vee, \wedge, ^*)$ *is restricted regular with respect to* * *but not regular* (*cf.* [**1**]). This can be seen as follows: The terms

$$\begin{aligned} t_1(x, y, z) &:= (x^* \wedge y)^* \wedge z, \\ t_2(x, y, z) &:= (x^* \wedge y) \vee z, \\ t_3(x, y, z) &:= (x \wedge y^*)^* \wedge z \text{ and} \\ t_4(x, y, z) &:= (x \wedge y^*) \vee z \end{aligned}$$

satisfy condition (iii) of Theorem 3: If $t_1(x, y, z) = \cdots = t_4(x, y, z) = z$ then

$$x^* \wedge y \le z \le (x^* \wedge y)^* \text{ and } x \wedge y^* \le z \le (x \wedge y^*)^*$$

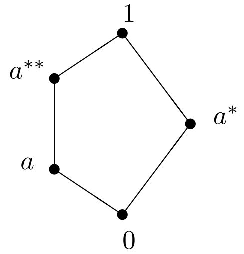
which implies

$$x^* \wedge y \le (x^* \wedge y) \wedge (x^* \wedge y)^* = 0 \text{ and } x \wedge y^* \le (x \wedge y^*) \wedge (x \wedge y^*)^* = 0.$$

From this it follows $x^* \leq y^*$ and $y^* \leq x^*$ and therefore $x^* = y^*$. If, conversely, $x^* = y^*$ then

$$\begin{aligned} t_1(x,y,z) &= (y^* \wedge y)^* \wedge z = z, \\ t_2(x,y,z) &= (y^* \wedge y) \vee z = z, \\ t_3(x,y,z) &= (x \wedge x^*)^* \wedge z = z \text{ and} \\ t_4(x,y,z) &= (x \wedge x^*) \vee z = z. \end{aligned}$$

That $\mathcal{V}$ is not regular can be seen from the following example: The equivalence relation on N_5



having the classes $\{0\}$, $\{a, a^{**}\}$, $\{a^*\}$ and $\{1\}$ is a congruence on the pseudocomplemented lattice $\mathcal{N}_5$ which has three classes in common with ω_{N_5}, but is different from ω_{N_5}.

Definition 3. An **algebra with** 0 is an algebra with a distinguished element 0 of its base set. A **variety with** 0 is a variety with a constant unary term 0.

Definition 4. Let $\mathcal{A} = (A, F)$ be an algebra with 0 and $M \subseteq A$. $\mathcal{A}$ is called **restricted weakly regular with respect to** M if for all $\Theta, \Phi \in \operatorname{Con}\mathcal{A}$, $[0]\Theta = [0]\Phi$ implies $\Theta|M = \Phi|M$.

Remark 3. *The case* $M = A$ *yields the concept of a weakly regular algebra with* 0 (*see* [**3**]). *Every algebra with* 0 *is restricted weakly regular with respect to every at most one-element subset of its base set.*

Lemma 2. *If* $\mathcal{A} = (A, F)$ *is an algebra with* 0, $M \subseteq A$, $\mathcal{A}$ *is restricted weakly regular with respect to* M *and* Θ *is a congruence on* $\mathcal{A}$ *with* $[0]\Theta = \{0\}$ *then* $\Theta|M = \omega_M$.

Proof. $[0]\Theta = \{0\} = [0]\omega_A$ implies $\Theta|M = \omega_A|M = \omega_M$. □

Definition 5. Let $\mathcal{V}$ be a variety with 0 and t a unary term of $\mathcal{V}$. $\mathcal{V}$ is called **restricted weakly regular with respect to** t if every member (A, F) of $\mathcal{V}$ is restricted weakly regular with respect to $t(A)$.

Remark 4. *The case* $t(x) = x$ *yields the concept of a weakly regular variety with* 0 (*see* [**3**]). *Every variety* $\mathcal{V}$ *with* 0 *is restricted weakly regular with respect to every constant unary term of* $\mathcal{V}$.

Theorem 4. *A variety* $\mathcal{V}$ *with* 0 *and with unary term* t *is restricted weakly regular with respect to* t *if and only if for each member* $\mathcal{A} = (A, F)$ *of* $\mathcal{V}$ *and each* $\Theta \in \operatorname{Con}\mathcal{A}$ *with* $[0]\Theta = \{0\}$ *it holds* $\Theta|t(A) = \omega_{t(A)}$.

Proof. If $\mathcal{V}$ satisfies the condition of the theorem, $\mathcal{A} = (A, F) \in \mathcal{V}$; $\Theta, \Phi \in \mathrm{Con}\mathcal{A}$ and $[0]\Theta = [0]\Phi$ then

$$|[[0](\Theta \cap \Phi)](\Theta/(\Theta \cap \Phi))| = |[[0](\Theta \cap \Phi)](\Phi/(\Theta \cap \Phi))| = 1$$

and hence

$$(\Theta/(\Theta \cap \Phi))|t(A/(\Theta \cap \Phi)) = (\Phi/(\Theta \cap \Phi))|t(A/(\Theta \cap \Phi))$$

which implies $\Theta|t(A) = \Phi|t(A)$. The rest of the proof follows from Lemma 2. □

Theorem 5. *For a variety $\mathcal{V}$ with* 0 *and a unary term t of $\mathcal{V}$ the following conditions are equivalent:*

(i) *$\mathcal{V}$ is restricted weakly regular with respect to t.*

(ii) *There exists a positive integer n, binary terms $d_1, \dots, d_n$ and 4-ary terms $s_1, \dots, s_n$ satisfying the following conditions:*
$t(x) = t(y)$ implies $d_1(x, y) = \dots = d_n(x, y) = 0$,
$s_1(d_1(x, y), 0, x, y) = t(x)$,
$s_i(0, d_i(x, y), x, y) = s_{i+1}(d_{i+1}(x, y), 0, x, y)$ for $i = 1, \dots, n-1$ and
$s_n(0, d_n(x, y), x, y) = t(y)$.

(iii) *There exists a positive integer n and binary terms $d_1, \dots, d_n$ such that $d_1(x, y) = \dots = d_n(x, y) = 0$ is equivalent to $t(x) = t(y)$.*

Proof. (i) ⇒ (ii):
Consider the free algebra $F_{\mathcal{V}}(x, y)$ of $\mathcal{V}$ generated by the free generators x, y. Then

$$[0]\Theta(t(x), t(y)) = [0]\Theta(\{0\} \times [0]\Theta(t(x), t(y)))$$

implies, according to (i),

$$(t(x), t(y)) \in \Theta(t(x), t(y))|t(F_{\mathcal{V}}(x, y)) = \Theta(\{0\} \times [0]\Theta(t(x), t(y)))|t(F_{\mathcal{V}}(x, y)).$$

In virtue of Theorem 2, there exists an integer $n \geq 1$, $d_1, \dots, d_n \in [0]\Theta(t(x), t(y))$ and binary polynomial functions $q_1, \dots, q_n$ over $F_{\mathcal{V}}(x, y)$ with

$q_1(0, d_1) = t(x)$,
$q_i(d_i, 0) = q_{i+1}(0, d_{i+1})$ for $i = 1, \dots, n-1$ and
$q_n(d_n, 0) = t(y)$.

Now there exist 4-ary terms $s_1, \dots, s_n$ with $s_i(u, v, x, y) = q_i(v, u)$ for $i = 1, \dots, n$ and the elements $d_1, \dots, d_n$ of $[0]\Theta(t(x), t(y))$ can be considered as binary terms $d_1(x, y), \dots, d_n(x, y)$. These terms satisfy the identities of (ii). Since $d_1, \dots, d_n \in [0]\Theta(t(x), t(y))$ one obtains $d_1(x, y) = \dots = d_n(x, y) = 0$ provided $t(x) = t(y)$.
(ii) ⇒ (iii):
$d_1(x, y) = \dots = d_n(x, y) = 0$ implies $t(x) = s_1(0, 0, x, y) = \dots = s_n(0, 0, x, y) = t(y)$.
(iii) ⇒ (i):
If $\mathcal{A} = (A, F) \in \mathcal{V}$, $a, b \in A$, $\Theta \in \mathrm{Con}\mathcal{A}$, $[0]\Theta = \{0\}$ and $(t(a), t(b)) \in \Theta$ then

$$d_i(a, b) \in [d_i(a, b)]\Theta = d_i([a]\Theta, [b]\Theta) = [0]\Theta = \{0\}$$

for all $i = 1, \dots, n$ according to (iii) whence $t(a) = t(b)$ again by (iii) which shows $\Theta|t(A) = \omega_{t(A)}$. In virtue of Theorem 4, $\mathcal{V}$ is restricted weakly regular with respect to t. □

Example 2. *The variety $\mathcal{V}$ of pseudocomplemented semilattices $(S, \wedge, ^*)$ is restricted weakly regular with respect to * but not weakly regular (cf.* [**1**]*).* This can be seen as follows: The terms $d_1(x,y) := x^* \wedge y$ and $d_2(x,y) := x \wedge y^*$ satisfy condition (iii) of Theorem 5: If $d_1(x,y) = d_2(x,y) = 0$ then $x^* \leq y^*$ and $y^* \leq x^*$ which yields $x^* = y^*$. If, conversely, $x^* = y^*$ then $d_1(x,y) = y^* \wedge y = 0$ and $d_2(x,y) = x \wedge x^* = 0$. That $\mathcal{V}$ is not weakly regular can be seen from the example mentioned in Example 1.

Remark 5. *The results obtained in this paper are natural generalizations of the corresponding results on (weak) regularity (see* [**2**]*,*[**3**]*,*[**5**] *and* [**7**]*).*

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