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PREVALENCE OF SOME KNOWN TYPICAL PROPERTIES

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Abstract. In this paper, some known typical properties of function spaces are shown to be prevalent in the sense of the measure-theoretic notion of Haar null.

1. Introduction

In 1994, B. R. Hunt, T. Sauer and J. A. Yorke [**8**] rediscovered the idea of defining Haar null sets in a Banach space (see such sets in [**6**] and [**16**]). They termed a Borel Haar null set and all its subsets **shy**. However, Hunt, Sauer and Yorke emphasized the applications of shy sets to the study of properties of function spaces. An interesting result [**9**] shows that the known typical property of nowhere differentiability in the space of continuous functions is also a prevalent property in the sense of shyness. The purpose of this paper is to show that some known typical properties of function spaces are also prevalent. Of course, generally a typical property may or may not be prevalent in a function space, and **vice verse** (see [**14**] for example).

Throughout this paper we use the universal measurability of the small sets as required by Christensen [**6**]. We record several definitions from [**6**], [**8**] and [**16**] as follows.

A universally measurable set S in a Banach space B is said to be **shy** if there exists a tight, Borel probability measure μ on X such that $\mu(S + x) = 0$ for any $x \in B$.

A set in a Banach space is **prevalent** if it is the complement of a shy set.

A prevalent property in a Banach space is a property which holds for all points except for a shy set.

A probability measure μ on a Banach space X is **transverse** to a set S if $\mu(\{z \in X : \ x + z \in S\}) = 0$ for all $x \in X$.

2. A Prevalent Property in $C[a, b]$

A continuous function is called **nowhere monotonic type** on an interval $[a, b]$ if for any $\gamma \in \mathbb{R}$ the function $f(x) - \gamma x$ is not monotonic on any subinterval of $[a, b]$.

It is well known (e.g. see [**4**, pp. 461–464]) that being nowhere monotonic type is a typical property in the space $C[a, b]$. We will show that this typical property

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is also a prevalent property in $C[a,b]$. This result follows easily from the results of Hunt [**9**]. Our proof, however, is more elementary.

Theorem 2.1. *The prevalent function $f \in C[a,b]$ is of nowhere monotonic type.*

Proof. Given any interval I, let

$$G(I) = \{f \in C[a,b] : \ f \text{ is of monotonic type on } I\},$$

and let

$$G_n(I) = \left\{ f \in C[0,1] : \begin{array}{l} \text{there exists a } \gamma \in [-n,n] \text{ such that} \\ f - \gamma x \text{ is monotonic on } I \end{array} \right\}.$$

Then

$$G(I) = \bigcup_{n=1}^{\infty} G_n(I).$$

It is easy to show that $G_n(I)$ is a closed set and so $G(I)$ is a Borel set. Now we show that $G_n(I)$ is a shy set. Choose a function $g \in C[a,b]$ that is nowhere differentiable on I. For any $f \in C[a,b]$, let

$$G_{n,g} = \{\lambda \in \mathbb{R} : \ f + \lambda g \in G_n\}.$$

Then $G_{n,g}$ is a singleton or empty set. If not, there exist $\lambda_1, \lambda_2 \in G_{n,g},\ \lambda_1 \neq \lambda_2$. Then there exist γ_1 and γ_2 such that $f+\lambda_1 g-\gamma_1 x$ and $f+\lambda_2 g-\gamma_2 x$ are monotonic on I. Therefore

$$(f + \lambda_1 g - \gamma_1 x) - (f + \lambda_2 g - \gamma_2 x) = (\lambda_1 - \lambda_2)g + (\gamma_2 - \gamma_1)x$$

is differentiable almost everywhere on I. This contradicts our assumption that the function g is nowhere differentiable. Thus $G_{n,g}$ is a singleton or empty set and therefore the set $G_n(I)$ is a shy set. The countable union of $G(I_k)$ over all rational intervals I_k is also shy. The result follows. □

3. A Prevalent Property in $C^\infty[0,1]$

$C^\infty[0,1]$ is the space of all infinitely differentiable functions on $[0,1]$, equipped with the metric d defined by

$$d(f,g) = \sum_{n=0}^{\infty} \frac{1}{2^n} \frac{p_n(f-g)}{1+p_n(f-g)},$$

where $p_n(f) = \sup\{|f^{(n)}(x)| : \ x \in [0,1]\}$. $C^\infty[0,1]$ is complete under the metric d. A function $f \in C^\infty[0,1]$ is said to be **analytic** at a point x_0 if its Taylor series

$$T(f,x_0) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

converges to f in an open neighbourhood of x_0. There are several papers (e.g. [**1**], [**5**], and [**12**]) devoted to the constructions of nowhere analytic functions in $C^\infty[0,1]$, or to showing that nowhere analytic functions in $C^\infty[0,1]$ are typical.

In this section a very simple method will be given to show that nowhere analytic functions in $C^\infty[0,1]$ are both typical and prevalent.

Theorem 3.1. *Both the typical function and the prevalent function in* $C^\infty[0,1]$ *are nowhere analytic on* $[0,1]$.

Proof. Let $I \subseteq [0,1]$ be a closed interval and x_I be the center point of I. Let

$$C(I) = \{f \in C^\infty[0,1] : \ T(f, x_I) \text{ converges to } f \text{ on } I\}.$$

Then $C(I)$ is closed. In fact, let $\{f_n\}$ be a Cauchy sequence in $C(I)$. Then there exists a function $f \in C^\infty[0,1]$ such that $f_n \to f$ in the metric d. For arbitrary integers n, N and $x \in I$,

$$\begin{aligned}
&\left| f(x) - \sum_{k=0}^{N} \frac{f^{(k)}(x_I)}{k!}(x - x_I)^k \right| \\
&\quad \le \left(1 + \sum_{k=0}^{N} \frac{2^k |x - x_I|^k}{k!}\right) d(f_n, f) + \left| f_n(x) - \sum_{k=0}^{N} \frac{f_n^{(k)}(x_I)}{k!}(x - x_I)^k \right|.
\end{aligned}$$

Thus, for any n and $x \in I$,

$$\overline{\lim}_{N\to\infty} \left| f(x) - \sum_{k=0}^{N} \frac{f^{(k)}(x_I)}{k!}(x - x_I)^k \right| \le \left(1 + \sum_{k=0}^{\infty} \frac{2^k |x - x_I|^k}{k!}\right) d(f_n, f).$$

Let $n \to \infty$. Then, for all $x \in I$,

$$\overline{\lim}_{N\to\infty} \left| f(x) - \sum_{k=0}^{N} \frac{f^{(k)}}{k!}(x - x_I)^k \right| = 0.$$

Thus $f \in C(I)$ and so $C(I)$ is closed. Clearly, the set $C(I)$ is a linear, proper subspace of $C^\infty[0,1]$. Thus $C(I)$ is nowhere dense and shy. Notice that the union of all $C(I)$ over all rational intervals I consists of all functions in $C^\infty[0,1]$ that are analytic at some point of $[0,1]$. Thus the result follows. ☐

4. Prevalent Properties in $b\mathcal{A}$, $b\mathcal{DB}^1$, $b\mathcal{B}^1$

Following Bruckner [**2**] and Bruckner and Petruska [**3**] we use $b\mathcal{A}$, $b\mathcal{DB}^1$, $b\mathcal{B}^1$ to denote the spaces of bounded approximately continuous functions, bounded Darboux Baire 1 functions, and bounded Baire 1 functions defined on $[0,1]$ respectively, all of which are equipped with supremum norms. All these spaces are Banach spaces and form a strictly increasing system of closed subspaces (see [**2**], [**3**]). In [**3**] it was shown that for a given arbitrary Borel measure μ on $[0,1]$ the typical function in $\mathcal{F} = b\mathcal{A}$, $b\mathcal{DB}^1$, or $b\mathcal{B}^1$ is discontinuous μ almost everywhere. We will show that such typical properties in these three spaces are also prevalent properties for any σ-finite Borel measure.

Theorem 4.1. *Let* μ *be a* σ*-finite Borel measure on* $[0,1]$. *The prevalent function in* $\mathcal{F} = b\mathcal{A}$, $b\mathcal{DB}^1$, *or* $b\mathcal{B}^1$ *is discontinuous* μ *almost everywhere on* $[0,1]$.

Proof. Let

$$S = \{f \in \mathcal{F} : \ f \text{ is continuous on a set } F_\lambda, \ \mu(F_\lambda) > 0\}.$$

We show first that the set S is a Borel set. Note

$$S = \bigcup_{n=1}^{\infty} A_{\frac{1}{n}}$$

where

$$A_{\frac{1}{n}} = \left\{ f \in \mathcal{F} : \ \mu(C_f) \geq \frac{1}{n} \right\}$$

and C_f is the set of continuity points of f. Let $\{f_m\}$ be a Cauchy sequence in $A_{\frac{1}{n}}$, then $f_m \to f \in \mathcal{F}$ uniformly. Let

$$C = \bigcap_{N=1}^{\infty} \bigcup_{m=N}^{\infty} C_{f_m}.$$

Then $C \subseteq C_f$. Note

$$\mu(C) = \lim_{N\to\infty} \mu(\bigcup_{m=N}^{\infty} C_{f_m}) \geq \liminf_{m\to\infty} \mu(C_{f_m}) \geq \frac{1}{n}.$$

So $f \in A_{\frac{1}{n}}$. Thus $A_{\frac{1}{n}}$ is closed and the set S is a Borel set.

We now show that the set S is a shy set. It is well known that there is a function $g \in \mathcal{F}$ which is discontinuous μ almost everywhere on $[0,1]$. See [**3**, pp. 331, Theorem 2.4]. We will use this function g as a probe. For any given function $f \in \mathcal{F}$, let

$$S_g = \{\lambda \in \mathbb{R} : \ f + \lambda g \in S\}.$$

We claim that S_g is Lebesgue measure zero. For distinct $\lambda_1, \lambda_2 \in S_g$, if $\mu(F_{\lambda_1} \cap F_{\lambda_2}) > 0$, then

$$(f + \lambda_1 g) - (f + \lambda_2 g) = (\lambda_1 - \lambda_2) g$$

would be continuous on $F_{\lambda_1} \cap F_{\lambda_2}$. This contradicts the choice of the function g. Thus for distinct λ_1 and λ_2 the corresponding sets F_{λ_1} and F_{λ_2} satisfy $\mu(F_{\lambda_1} \cap F_{\lambda_2}) = 0$. Since μ is σ-finite on $[0,1]$ then $[0,1] = \bigcup_{i=1}^{\infty} X_i$ where $\mu(X_i) < \infty$ and $X_i \cap X_j = \phi, \ i \neq j$. Let

$$S_{mn} = \left\{ \lambda \in S_g : \ \mu(F_\lambda \cap X_m) \geq \frac{1}{n} \right\}.$$

Then S_{mn} is finite. If not, there exist countably many $\lambda_i \in S_{mn}$ such that

$$\infty = \sum_{i=1}^{\infty} \mu(F_{\lambda_i} \cap X_m) = \mu\left(X_m \cap \left(\bigcup_{i=1}^{\infty} F_{\lambda_i} \right) \right) \leq \mu(X_m) < \infty.$$

This is a contradiction. Hence S_{mn} is finite and $S_g = \bigcup_{m,n=1}^{\infty} S_{mn}$ is at most countable. Thus S_g is Lebesgue measure zero and the result follows. $\square$

In the proof of Theorem 4.1 we did not use any special propertes of functions in $b\mathcal{A}$, $b\mathcal{DB}^1$, $b\mathcal{B}^1$, except that in all these classes there are functions that are discontinuous μ almost everywhere on $[0,1]$. Thus, we can extend Theorem 4.1 in a general form as follows (see [**11**] for a typical version).

Theorem 4.2. *Let μ be a σ-finite Borel measure on $[0,1]$. Let $\mathcal{F}$ be a linear space of bounded functions $f\colon [0,1] \to \mathbb{R}$ with supremum metric. Suppose that there is a function $f \in \mathcal{F}$ that is discontinuous μ almost everywhere on $[0,1]$. Then the prevalent function in $\mathcal{F}$ is discontinuous μ almost everywhere on $[0,1]$.*

5. Prevalent Properties in $BSC[a,b]$

The space $BSC[a,b]$ of bounded symmetrically continuous functions equipped with the supremum norm is a complete space (see [**15**]). From [**13**] we know that the set of functions $f \in BSC[a,b]$, which have c-dense sets of points of discontinuity, is residual. In this section we show that such a set is also prevalent. Here we say that a set is **c-dense** in a metric space (X,ρ) if it has continuum many points in every non-empty open set.

In [**10**] Pavel Kostyrko showed the following theorem.

Theorem 5.1. *Let (X,ρ) be a metric space. Let F be a linear space of bounded functions $f\colon X \to \mathbb{R}$ furnished with the supremum norm $\|f\| = \sup_{x\in X}\{|f(x)|\}$. Suppose that in F there exists a function h such that its set $D(h)$ of points of discontinuity is uncountable. Then*

$$G = \{f \in F:\ D(f) \text{ is uncountable}\}$$

is an open residual set in (F,d), $d(f,g) = \|f-g\|$.

By modifying the methods in [**13**] we can get a stronger result in separable metric spaces.

Theorem 5.2. *Let (X,ρ) be a separable metric space. Let F be a complete metric linear space of bounded functions $f\colon X \to \mathbb{R}$ furnished with supremum norm $\|f\| = \sup_{x\in X}\{|f(x)|\}$. Suppose that there is a function h such that its set $D(h)$ of points of discontinuity is c-dense in (X,ρ). Then*

$$G = \{f \in F:\ D(f) \text{ is c-dense}\}$$

is a dense residual G_δ set in (F,d), where $d(f,g) = \|f-g\|$.

Proof. Given a non-empty set O, we can show that

$$A(O) = \{f \in F:\ D(f) \cap O \text{ is of power } c\}$$

is a dense open set by using the methods in [**13**]. In fact, let $\{f_n\} \subseteq F \setminus A(O)$ be a convergent sequence. Then there is a function $f \in F$ such that $f_n \longrightarrow f$ uniformly. Then $D(f_n) \cap O$ is at most countable and so the union $\bigcup_{n=1}^{\infty} D(f_n) \cap O$ is at most countable. Since f is continuous at each point $x \in O \setminus \bigcup_{n=1}^{\infty} D(f_n) \cap O$, so $f \in F \setminus A(O)$. Hence $F \setminus A(O)$ is closed and $A(O)$ is open.

Now we show that $A(O)$ is dense in F. For every ball $B(f,\epsilon) \subseteq F$, if $f \in A(O)$ there is nothing to prove. We assume $f \in F \setminus A(O)$, then f has at most countably many points of discontinuity in O. From the assumption there is a function $h \in F$ such that h has a c-dense set of points of discontinuity in O. Let M be a constant such that $|h(x)| \le M$ for all $x \in X$ and set $g = f + \frac{\epsilon}{2M}h$. Then $g \in F$ is discontinuous in continuum many points of O. It is easy to see $g \in A(O) \bigcap B(f,\epsilon)$ and hence $A(O)$ is dense.

Let $\{x_i\}$ be a dense countable subset of X, then

$$G = \bigcap_{i=1}^{\infty} \bigcap_{m=1}^{\infty} A(B(x_i, 1/m))$$

is a dense G_δ set where $B(x_i, 1/m)$ is the open ball centered at x_i and with radius $1/m$. Thus G is a dense residual G_δ set in F. □

Corollary 5.3. *The typical functions in $R[a,b]$, the space of Riemann integrable functions furnished with the supremum norm, have c-dense sets of points of discontinuity.*

Proof. The set of points of discontinuity of any bounded symmetrically continuous function is Lebesgue measure zero [**15**, pp. 27, Theorem 2.3]. Thus, such a function is Riemann integrable by the well known fact that a bounded Lebesgue measurable function is Riemann integrable iff its set of points of discontinuity is Lebesgue measure zero [**4**]. Tran in [**17**] has constructed a symmetrically continuous function whose set of points of discontinuity is c-dense. Hence the result follows. □

The following theorem gives a similar form of Theorem 5.2 in the sense of prevalence.

Theorem 5.4. *Let (X,ρ) be a separable metric space, μ be a σ-finite Borel measure that is non-zero on every open set in (X,ρ). Let F be a complete metric linear space of bounded functions $f\colon X \to \mathbb{R}$ furnished with the supremum norm $\|f\| = \sup_{x\in X}\{|f(x)|\}$. Suppose that in F there exists a function h such that its set of points of discontinuity is c-dense in (X,ρ). Then the prevalent function $f \in F$ has a c-dense set of points of discontinuity.*

Proof. Let

$$G = \{f \in F :\ D(f) \text{ is } c\text{-dense in } X\}.$$

By Theorem 5.2 the set G and its complement are Borel sets. We need only show that for every $f \in F$ the following set

$$S = \left\{ \lambda \in \mathbb{R} : \quad \begin{array}{l} f + \lambda h \text{ is discontinuous at most countably many} \\ \text{points in some non-empty open set } O_\lambda \subseteq (X,\rho) \end{array} \right\}.$$

is a Lebesgue measure zero set. Using the method in the proof of Theorem 4.1 we can show that the set S is at most countable. Therefore the set F is shy and the result follows. □

Corollary 5.5. *The prevalent function $f \in BSC_n[a,b]$, the space of bounded n-th symmetrically continuous functions furnished with the supremum norm, has a c-dense set of points of discontinuity.*

Proof. The space $BSC_n[a,b]$ is a complete metric space (see [**10**]). By Tran's results [**17**], there exist functions h_1 and h_2 in $BSC_1[a,b]$ and $BSC_2[a,b]$ respectively such that h_1 and h_2 have c-dense sets of points of discontinuity on $[a,b]$. Also note that $BSC_1[a,b] = BSC[a,b] \subseteq BSC_{2k-1}[a,b]$ and $BSC_2[a,b] \subseteq BSC_{2k}[a,b]$ (see [**10**] for details). Thus, the result follows from Theorem 5.4. $\square$

Applying Theorem 5.4 and Tran's results [**17**] we immediately obtain a prevalent property in the space $R[a,b]$, which is also a typical property in $R[a,b]$ as shown in Corollary 5.3.

Corollary 5.6. *The prevalent function $f \in R[a,b]$ has a c-dense set of points of discontinuity.*

In the following theorem we show that the non-countable continuity is both a typical property and a prevalent property in the space $BSC[a,b]$. Here a function $f\colon X \subseteq \mathbb{R} \to \mathbb{R}$ is called **countably continuous** if there is a countable cover $\{X_n : n \in \mathbb{N}\}$ of X (by arbitrary sets) such that each restriction $f|X_n$ is continuous.

By constructing an example of a non-countably continuous function Ciesielski [**7**] answered a question of L. Larson who asked whether every symmetrically continuous function is countably continuous. We will use this construction to show that the non-countable continuity is both a typical property and a prevalent property in the space $BSC[a,b]$.

Theorem 5.7. *Both the typical function and the prevalent function $f \in BSC[a,b]$ are not countably continuous.*

Proof. Let

$$S = \{f \in BSC[a,b] : \ f \text{ is countably continuous}\}.$$

Then the set S is a linear, closed and proper space. In fact, for all $f_1, f_2 \in S$, there are countable covers $\{X_n^1 : \ n \in \mathbb{N}\}$ and $\{X_n^2 : \ n \in \mathbb{N}\}$ of $[a,b]$ such that the restrictions $f_i|X_n^i$ are continuous. So $\{X_i^1 \cap X_j^2 : \ i,j \in \mathbb{N}\}$ is a countable cover of $[a,b]$ and the restrictions $f_i|X_i^1 \cap X_j^2$ are continuous for any $\alpha, \beta \in \mathbb{R}$. Therefore S in linear. Let $\{f_n\} \subseteq S$ be a convergent sequence. Then there is a function $f \in BSC[a,b]$ such that $f_n \to f$ uniformly. For each f_n there exists a countable cover $\{X_i^n\}$ of $[a,b]$ such that $f_n|X_i^n$ is continuous. so $\bigcap_{n=1}^{\infty} X_i^n$ is a countable cover of $[a,b]$ and the restriction $f_m|\bigcap_{n=1}^{\infty} X_i^n$ is continuous for any m,i. Thus the restriction $f|\bigcap_{n=1}^{\infty} X_i^n$ is continuous since f is the uniform limit of f_n. Therefore $f \in S$ and S is closed. S is proper from Ciesielski's construction of a non-countably continuous function [**7**]. Therefore, the set S is closed and nowhere dense. So the non-countable continuity is a typical property.

On the other hand, the intersection of the set S with any line with the direction $g \in BSC[a,b] \setminus S$ is Lebesgue measure zero. Therefore the set S is shy and the non-countable continuity is also a prevalent property. $\square$

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