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# POSITIVE SOLUTIONS OF QUASILINEAR ELLIPTIC SYSTEMS WITH STRONG DEPENDENCE ON THE GRADIENT

D. ŽUBRINIĆ

Abstract. We study existence and nonexistence of positive, spherically symmetric solutions of diagonal quasilinear elliptic systems involving equations with $p$-Laplacians, and with strong dependence on the gradient on the right-hand side. The existence proof is constructive, with solutions possessing explicit integral representation. Also, we obtain critical exponents of the gradient. We introduce the notion of cyclic elliptic systems in order to study nonsolvability of general elliptic systems. The elliptic system is studied by relating it to the corresponding system of singular ordinary integro-differential equations of the first order.

## 1. Introduction

This article is motivated by the fact that very little is known about solvability and nonsolvability of elliptic systems with strong dependence on the gradient. We consider diagonal quasilinear elliptic systems involving $p$-Laplacians on the left-hand side. The main difficulty is the presence of gradients of unknown functions on right-hand sides with powers of arbitrary positive order. We study existence and nonexistence of positive, spherically symmetric solutions in a ball. Diagonal quasilinear elliptic systems involving two equations have been considered in a number of papers, let us cite De Figueiredo **[3]**, Clément, Manásevich, Mitidieri **[2]**, and the references therein. All these papers consider problems without gradients on the right-hand side.

In our previous paper **[11]** we have studied quasilinear elliptic systems involving only two equations and with the natural growth in the gradient. The method exploited there does not permit us to extend existence and nonexistence results to systems with more general right-hand sides, involving all unknown functions and their gradients. In this paper we use a different approach which enables us to consider also this case. Although the results obtained here are less explicit than in **[11]**, the question of solvability of a quasilinear elliptic system is reduced to question of solvability of a simple system of algebraic inequalities, see Theorems 1 and 2. In **[11]** the basic tool was to study fixed points of a system of two singular ODEs of the first order. This method was exploited in **[6]** in the scalar case.

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Solutions of the system of singular ODEs in **[11]** are obtained by means of fixed points of a composition of two integral operators of Volterra type corresponding to the system of singular ODEs. In analogous way we have studied the question of nonsolvability. Here we use a different approach, and study existence of solutions by means of fixed points of an ordered pair of integral operators on the corresponding product function space. Of course, when dealing with an elliptic system of $n$ equations as is the case in this paper, we introduce an operator represented as $n$-tuple of integral operators. Solvability of the system is studied by means of fitting the domain of this operator, in order to be able to apply Schauder's theorem, or using method of monotone iterations. The question of nonsolvability for general quasilinear elliptic systems is studied by means of cyclic elliptic systems that we introduce in Section 3.

Let us introduce some notation. Throughout this paper $B = B_R(0)$ will be a ball of radius $R$ in $\mathbf{R}^N$, $N \geq 1$. The Lebesgue measure of $B$ is denoted by $|B|$, $\partial B$ is the boundary of $B$, the Lebesgue measure of the unit ball is denoted by $C_N$. If $1 < p < \infty$, we define $p$-Laplacian $\Delta_p$ by $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$. By a strong solution of an elliptic system we mean a vector function whose components are in $C^2(B \setminus \{0\}) \cap C(\overline{B})$ and satisfy the system pointwise in $B \setminus \{0\}$. By $p' = \frac{p}{p-1}$ we denote the conjugate exponent of $p$.

Rather than formulating the most general result, we illustrate a special case of Theorems 3 and  for the following cyclic system of three quasilinear elliptic equations:

$$
\begin{cases}
-\Delta_p u = \tilde{g}_1|x|^{m_1} + \tilde{f}_1|\nabla v|^{e_1} & \text{in } B \setminus \{0\}, \\
-\Delta_q v = \tilde{g}_2|x|^{m_2} + \tilde{f}_2|\nabla w|^{e_2} & \text{in } B \setminus \{0\}, \\
-\Delta_r w = \tilde{g}_3|x|^{m_3} + \tilde{f}_3|\nabla u|^{e_3} & \text{in } B \setminus \{0\}, \\
u > 0,\ v > 0,\ w > 0 \text{ on } B, \text{ spherically symmetric, decreasing}, \\
u = v = w = 0 \text{ on } \partial B.
\end{cases} \tag{1}
$$

Here $e_i$ are positive constants, $p, q, r \in (1, \infty)$, $m_i \in \mathbf{R}$, $\tilde{f}_i > 0$, $\tilde{g}_i > 0$. We seek for strong solutions $(u, v, w)$ of (1), that is, a vector function with components in $C^2(B \setminus \{0\}) \cap C(\overline{B})$ satisfying (1) pointwise in $B \setminus \{0\}$. The following theorem shows that the critical case is when the product of exponents $e_i$ is equal to $(p-1)(q-1)(r-1)$. It will be convenient to denote $p_1 = p$, $p_2 = q$, $p_3 = r$,

$$
\gamma_i = 1 + \frac{m_i}{N}, \quad \delta_i = \frac{e_i}{p_{i+1} - 1}, \quad \varepsilon_i = \delta_i \left(1 - \frac{1}{N}\right), \tag{2}
$$

$$
g_i = \frac{\tilde{g}_i}{C_N^{\frac{m_i + p_i}{N}} N^{p_i - 1}(m_i + N)}, \quad f_i = \frac{\tilde{f}_i}{N^{p_i - e_i} C_N^{\frac{p_i - e_i}{N}}}, \quad T = |B|, \tag{3}
$$

and

$$b_i = f_i \frac{T^{\delta_i \gamma_{i+1} - \varepsilon_i + 1 - \gamma_i}}{\delta_i \gamma_{i+1} - \varepsilon_i + 1}, \tag{4}$$

where $i = 1, 2, 3$. Here we compute $i + 1$ modulo 3.

**Theorem 1. (Existence and Nonexistence of Solutions)** *Assume that* $m_i > -N$ *and*

$$e_i(m_{i+1} + 1) \geq m_i(p_{i+1} - 1), \tag{5}$$

*for* $i = 1, 2, 3$.

(a) *If* $e_1 e_2 e_3 < (p-1)(q-1)(r-1)$ *then for any positive* $\tilde{f}_i$ *and* $\tilde{g}_i$ *there exists a strong solution of quasilinear elliptic system* (1).

(b1) *Assume that* $e_1 e_2 e_3 > (p-1)(q-1)(r-1)$. *If there exist positive numbers* $M_i$, $i = 1, 2, 3$, *satisfying the following cyclic system of algebraic inequalities:*

$$g_i + b_i \cdot M_{i+1}^{\delta_i} \leq M_i, \tag{6}$$

$i = 1, 2, 3$, *then there exists a strong solution of quasilinear elliptic system* (1).

(b2) *Assume that* $e_1 > p - 1$, $e_2 > q - 1$, $e_3 > r - 1$. *We also assume that technical condition* (33) *is fulfilled with* $k = 3$. *There exist explicit positive constants* $\tilde{H}_i$ *independent of* $\tilde{g}_i$ *and* $\tilde{f}_i$, *such that if*

$$\tilde{g}_i^{\delta_1 \delta_2 \delta_3 - 1} \tilde{f}_i > \tilde{H}_i \quad \textit{for some } i \in \{1, 2, 3\},$$

*then system* (1) *has no strong solutions.*

As we see, contrary to case $e_1 e_2 e_3 < (p-1)(q-1)(r-1)$, if $e_1 e_2 e_3 > (p-1)(q-1)(r-1)$ then we have existence-nonexistence breaking with respect to coefficients $\tilde{f}_i$, $\tilde{g}_i$. For this reason we say that the case when

$$e_1 e_2 e_3 = (p-1)(q-1)(r-1)$$

is critical. This situation is analogous to that of scalar quasilinear elliptic equations, see [**10**]. Regarding nonexistence result stated in Theorem 1(b2), we do not know anything about nonsolvability of (1) when $e_1 e_2 e_3 > (p-1)(q-1)(r-1)$ and, say, $e_1 < p - 1$. Also, the question of solvability and nonsolvability for system (1) modelled on arbitrary bounded domain is an open problem.

**Remark.** It is possible to impose sufficient conditions that will gaurantee existence of classical solution of (1) on the whole domain, that is, $u, v, w \in C^2(B)$. This can be done using integral representation (19) of solutions, similarly as in [**7**, Proposition 6], [**10**, Theorem 7] and [**11**, Theorem 6]. For example, assuming that conditions of Theorem 1(b1) are satisfied with $p_i = e_i = 2$ (i.e., we have ordinary Laplacian and quadratic growth in the gradient in (1)), and $m_i \geq 0$ for $i = 1, 2, 3$, then there exists a classical solution of (1). Note that we do not claim that all solutions are classical in this case, that is, we have only a-posteriori regularity. Using methods from cited papers it is also possible to study existence of weak and bounded solutions of (1) on $B$.

## 2. Existence of Solutions

We study the following diagonal quasilinear elliptic system of $n$ equations with strong dependence on the gradient, whose very special case is system (1):

$$\begin{cases} -\Delta_p u = F(|x|, u, |\nabla u|_*) \quad \text{in } B \setminus \{0\}, \\ u > 0 \text{ on } B,\ u = 0 \text{ on } \partial B, \\ u_i \text{ spherically symmetric and decreasing,} \end{cases} \tag{7}$$

where $u = (u_1, \dots, u_n)$. We say that $u > 0$ on $B$ if $u_i > 0$ for all $i = 1, \dots, n$. Here we denote $p = (p_1, \dots p_n)$, $1 < p_i < \infty$,

$$\Delta_p u = (\Delta_{p_1} u_1, \dots, \Delta_{p_n} u_n), \quad |\nabla u|_* = (|\nabla u_1|, \dots, |\nabla u_n|),$$

and assume that

$$F = (F_1, \dots, F_n) : (0, R] \times \mathbf{R}_+^n \times \mathbf{R}_+^n \to \mathbf{R}^n$$

is continuous, where $\mathbf{R}_+ = [0, \infty)$. We consider strong solutions of (7), that is, $u = (u_1, \dots, u_n)$ such that $u_i \in C^2(B \setminus \{0\}) \cap C(\overline{B})$. Our basic assumption on the right-hand side of (7) is

$$0 \le F_i(r, u, \xi) \le \tilde{g}_i r^{m_i} + \sum_{j=1}^n \tilde{f}_{ij} \xi_j^{e_{ij}}, \tag{8}$$

for all $r \in (0, R)$ and $\xi \in \mathbf{R}_+^n$, where $\tilde{g}_i$, $\tilde{f}_{ij}$ are given nonnegative numbers. We also assume that

$$\forall a > 0,\ \exists r \in (0, a),\ \forall \eta \ge 0,\ \forall \xi \ge 0,\ \forall i,\ F_i(r, \eta, \xi) > 0. \tag{9}$$

The role of (9) will be to ensure that the solution $u$ of (7) be positive, that is, $u_i > 0$ on $B$ for all $i$. If we seek only for nonnegative solutions, then condition (9) can be dropped. Now we define

$$\gamma_i = 1 + \frac{m_i}{N}, \quad \delta_{ij} = \frac{e_{ij}}{p_j - 1}, \quad \varepsilon_{ij} = \delta_{ij}(1 - \frac{1}{N}), \tag{10}$$

$$g_i = \frac{\tilde{g}_i}{C_N^{\frac{m_i + p_i}{N}} N^{p_i - 1}(m_i + N)}, \quad f_{ij} = \frac{\tilde{f}_{ij}}{N^{p_i - e_{ij}} C_N^{\frac{p_i - e_{ij}}{N}}}, \quad T = |B|, \tag{11}$$

where $i, j = 1, \dots, n$, and

$$b_{ij} = f_{ij} \frac{T^{\gamma_j \delta_{ij} - \varepsilon_{ij} + 1 - \gamma_i}}{\gamma_j \delta_{ij} - \varepsilon_{ij} + 1}. \tag{12}$$

We say that a function $F_i(r, \eta, \xi)$ is nondecreasing in $\eta$ and $\xi$ if it is nondecreasing with respect to each component of $\eta$ and $\xi \in \mathbf{R}_+^n$.

**Theorem 2. (Existence of Solutions)** *Assume that* (8) *and* (9) *are fulfilled, and let* $m_i > -N$, $e_{ij} > 0$,

$$\gamma_j \delta_{ij} - \varepsilon_{ij} + 1 \geq \gamma_i \tag{13}$$

*for all* $i, j = 1, \ldots, n$. *Assume that* $\tilde{g}_{ij} \geq 0$ *and* $\tilde{f}_{ij} \geq 0$ *are such that the following system of algebraic inequalities is solvable:*

$$\exists M_1 > 0, \ldots, \exists M_n > 0,\ \forall i \in \{1, \ldots, n\},\ g_i + \sum_{j=1}^{n} b_{ij} M_j^{\delta_{ij}} \leq M_i. \tag{14}$$

*Then* (7) *possesses at least one strong solution. If* $F_i(r, \eta, \xi)$ *are nondecreasing in* $\eta$ *and* $\xi$ *for all* $i = 1, \ldots, n$, *then there exists a strong solution which can be obtained constructively using monotone iterations.*

The idea of the proof is to assign to quasilinear elliptic problem (7) the corresponding system of singular, integro-differential equations in the following way. Let $D = D_1 \times \cdots \times D_n$, where

$$D_i = \{\varphi \in C([0, T]) : 0 \leq \varphi(t) \leq M_i t^{\gamma_i}\}, \tag{15}$$

with constants $M_i > 0$ to be determined later. For fixed $\omega \in D$ we define a function

$$f_i^{\omega}(t) := \frac{1}{N^{p_i} C_N^{p_i/N}} F_i(\,(tC_N^{-1})^{\frac{1}{N}}, V^{\omega}(t), W(t, \omega(t))\,). \tag{16}$$

where $V^{\omega}(t) = (V_j^{\omega_j}(t))_{j=1,\ldots,n}$, $W(t, \omega(t)) = (W_j(t, \omega(t))_{j=1,\ldots,n}$, with

$$V_j^{\omega_j}(t) = \int_t^T \frac{\omega_j(s)^{p_j'-1}}{s^{p_j'(1-\frac{1}{N})}}\, ds, \quad W_j(t, \omega(t)) = N C_N^{1/N} \frac{\omega_j(t)^{p_j'-1}}{t^{\frac{p_j'}{p_j}(1-\frac{1}{N})}}.$$

Note that the operator $\omega_j \mapsto V_j^{\omega_j}$ is not of Nemytski type. The growth condition (8) implies that

$$0 \leq f_i^{\omega}(t) \leq g_i \gamma_i t^{\gamma_i - 1} + \sum_{j=1}^{n} f_{ij} \frac{\omega_j(t)^{\delta_{ij}}}{t^{\varepsilon_{ij}}}. \tag{17}$$

Let us consider the following system of singular ordinary integro-differential equations:

$$\frac{d\omega_i}{dt} = f_i^{\omega}(t), \quad t \in (0, T], \tag{18}$$

for $i = 1, \ldots, n$. Note that $\omega \mapsto f_i^{\omega}$ is not an operator of the Nemytski type. Using the analogous proof as in [**7**, Lemma 1], we obtain the following result which enables to generate solutions of system (7) by means of solutions of (18).

**Lemma 1.** *Assume that* $\omega$ *is a solution of singular system of integro-differential equations* (18)*. Then*

$$u_i(x) = V_i^{\omega_i}(C_N|x|^N) = \int_{C_N|x|^N}^{|B|} \frac{\omega_i(s)^{p_i'-1}}{s^{p_i'(1-\frac{1}{N})}}\, ds, \quad i = 1,\dots,n \tag{19}$$

*is a strong solution of quasilinear elliptic system* (7)*,* $u_i(0) < \infty$*.*

*Proof of Theorem* 2. To prove existence of solutions of (7) it suffices to prove solvability of (18), see Lemma 3. Let us define the operator

$$\begin{cases} K : D \subset C([0,T], \mathbf{R}^n) \to C([0,T], \mathbf{R}^n), \\ K = (K_1, \dots, K_n), \quad K_j\omega(t) = \displaystyle\int_0^t f_j^{\omega}(s)\, ds. \end{cases} \tag{20}$$

It suffices to show that $K$ possesses a fixed point in $D$. We assume that $C([0,T], \mathbf{R}^n)$ is endowed with uniform topology. Compactness of $K$ will follow from Schauder's fixed point theorem.

To prove that the set $K(D)$ is relatively compact, we use vector valued version of Ascoli's theorem. The domain $D$ defined via (15) is bounded. To show that the family of vector functions $K(D)$ is uniformly equicontinuous on $[0,T]$, we take any $\omega \in D$ and $a$, $b$ such that $0 \le a < b \le T$. Using (17) we obtain that

$$|K_i\omega(b) - K_i\omega(a)| \le \int_a^b [g_i\gamma_i s^{\gamma_i - 1} + \sum_{j=1}^n f_{ij} \frac{\omega_j(s)^{\delta_{ij}}}{s^{\varepsilon_{ij}}}]\, ds. \tag{21}$$

The fact that $\omega_j \in D_j$ together with (13) yields after a short computation that the right-hand side of (21) converges to 0 uniformly for all $\omega \in D$ as $b - a \to 0$. In the similar way we prove uniform boundedness:

$$0 \le K_i\omega(t) \le (g_i + \sum_{j=1}^n b_{ij} M_j^{\delta_{ij}}) t^{\gamma_i}. \tag{22}$$

where we have used (13) again in order to have $t^{\gamma_j\delta_{ij} - \varepsilon_{ij} + 1} \le T^{\gamma_j\delta_{ij} - \varepsilon_{ij} + 1 - \gamma_i} t^{\gamma_i}$ for $t \in [0,T]$. Therefore $K$ is compact by Ascoli's theorem. Conditions (14) and (22) imply that $K(D) \subset D$, so that the claim of Theorem 2 follows from Schauder's theorem.

To prove the constructive part of Theorem 2 we introduce an operator $K_0 = (K_{01}, \dots, K_{0n})$ defined analogously as $K$, see (20), by

$$K_{0i}\omega(t) = g_i t^{\gamma_i} + \sum_{j=1}^n f_{ij} \int_0^t \frac{\omega_j(s)^{\delta_{ij}}}{s^{\varepsilon_{ij}}}\, ds, \quad i = 1, \dots n.$$

Using the above proof with $K_0$ instead of $K$, we see that there exists a fixed point $\overline{\omega} \in D$ of $K_0$. Now we view the space $C([0,T],\mathbf{R}^n)$ as an ordered Banach space with the usual componentwise partial ordering. Since $0 \le K \le K_0$, see (17), and since due to our monotonicity assumption on $F_i$s the operator $K$ is nondecreasing in the sense of [**1**], we see that $0 \le K(0)$ and $K\overline{\omega} \le \overline{\omega}$, that is, 0 and $\overline{\omega}$ are ordered subsolution and supersolution of $K$ respectively. The claim follows from Amann [**1**, Theorem 6.1]. In other words, the sequence $(\omega^{(k)})$ of monotone iterations defined inductively by $\omega^{(k)} = K\omega^{(k-1)}$, $\omega^{(0)} = 0$, converges monotonically in $C([0,T],\mathbf{R}^n)$ to a fixed point $\omega \in D$ of operator $K$. This $\omega$ generates a strong solution $u(x)$ of (7) via (19). The sequence of successive approximations $u^{(k)}(x)$ generated by $\omega^{(k)}$ via (19) converges monotonically to $u(x)$.

Since $\omega = K\omega$, then (9) implies that $\omega > 0$ on $(0,T]$. Therefore $u > 0$ on $B$, see (19). □

Now we discuss a class of quasilinear elliptic systems which contains cyclic systems considered in Theorem 1 as a special case. We consider the following special case of (8):

$$0 \le F_i(r,u,\xi) \le \tilde{g}_i r^{m_i} + \tilde{f}_i \xi_{i+1}^{e_i}$$

for $i = 1,\dots,n$, where by definition $n+1 = 1$.

**Theorem 3. (Existence of Solutions)** *Assume that condition* (9) *is fulfilled and let* (8) *hold with* $\tilde{f}_{ij} = 0$ *for all* $i$ *and* $j \ne i+1$, $\tilde{f}_i := \tilde{f}_{i,i+1} > 0$, $e_i := e_{i,i+1} > 0$ *for all* $i$, *where* $i+1$ *is computed modulo* $n$. *Furthermore, assume that*

$$e_i(m_{i+1}+1) \ge m_i(p_{i+1}-1).$$

*and let the constants* $\gamma_i$, $\delta_i$, $\varepsilon_i$, $g_i$, $f_i$, $b_i$ *be defined by* (2), (3) *and* (4).

*(a) If*

$$e_1 \dots e_n < (p_1 - 1)\dots(p_n - 1), \tag{23}$$

*then quasilinear elliptic system* (7) *is solvable for any positive* $\tilde{g}_i$, $\tilde{f}_i$.

*(b) Let*

$$e_1 \dots e_n > (p_1 - 1)\dots(p_n - 1), \tag{24}$$

*and let there exist positive numbers* $M_i$ *satisfying condition* (6) *for all* $i = 1,\dots,n$. *Then there exists a strong solution of quasilinear elliptic system* (7).

*In both cases, if* $F_i(r,\eta,\xi)$ *is nondecreasing with respect to* $\eta$ *and* $\xi$ *for all* $r \in (0,R]$, $i = 1,\dots,n$, *then there exists a strong solution of* (7) *which can be obtained constructively using monotone iterations.*

*Proof.* (a) Condition (23) is equivalent to $\prod_{i=1}^n \delta_i < 1$, and it is easy to see that in this case condition (6) is fulfilled for suitable positive $M_i$. Indeed, we can find

$M_i > 0$ so that in (6) we have equalities. To see this, we substitute $M_{n-1}$ from the last equation into the preceding one, then $M_{n-2}$ from $(n-1)$st equation into the preceding one, and so on. The final equation acquires the form $f(M) = M$, where $M = M_1$ and

$$f(M) = g_1 + b_1(g_2 + b_2(\dots g_{n-1} + b_{n-1}(g_n + b_n M^{\delta_n})^{\delta_{n-1}} \dots)^{\delta_2})^{\delta_1}.$$

It is easy to see that $\frac{f(M)}{M} \to \infty$ as $M \to 0$, and since $\prod_i \delta_i < 1$ then $\frac{f(M)}{M} \to 0$ as $M \to \infty$. Continuity of $f(M)$ implies that there exists $M > 0$ such that $f(M) = M$. Condition (6) is then satisfied with $M_1 = M$, $M_n = g_n + b_n M_1^{\delta_n}$, $M_{n-1} = g_{n-1} + b_{n-1} M_n^{\delta_{n-1}}$ etc. This means that (14) is fulfilled, and the claim follows from Theorem 2.

Case (b) is an immediate consequence of Theorem 2. □

**Remark.** Algebraic system of inequalities in (14) defines a set of possible values of ($\tilde{g}_i$ and $\tilde{f}_i$) for which our elliptic system (7) is solvable. It is worth noting that if $\delta_{ij} < 1$ for all $i, j$ in Theorem 2, then condition (14) is clearly satisfied. Therefore, in case when $0 < e_{ij} < p_j - 1$ for all $i, j$, elliptic system (7) is solvable. We do not know any reasonably general sufficient condition on the coefficients of algebraic system of inequalities (14) for $n \geq 2$, that guarantees its solvability.

In the scalar case, i.e. when $n = 1$, we have the following characterization:

$$(\exists M > 0,\ g + b \cdot M^{\delta} \leq M) \iff g^{\delta-1} \cdot b \leq \frac{(\delta-1)^{\delta-1}}{\delta^{\delta}}, \tag{25}$$

where $\delta > 1$ and $g$ and $b$ are given positive real numbers, see **[11**, Lemma 5]. Furthermore, if $g^{\delta-1} \cdot b \leq (\delta-1)^{\delta-1}/\delta^{\delta}$ then $g + b \cdot M^{\delta} \leq M$ is satisfied with

$$M_0 = \left(\frac{g}{b(\delta-1)}\right)^{1/\delta}. \tag{26}$$

## 3. Nonsolvability of Cyclic Systems of Singular ODEs

The aim of this section is to study nonsolvability of the following cyclic system consisting of $k$ singular ODEs of the first order:

$$\frac{d\omega_i}{dt} = g_i \gamma_i t^{\gamma_i - 1} + f_i \frac{\omega_{i+1}(t)^{\delta_i}}{t^{\varepsilon_i}}, \quad i = 1, \dots, k, \quad \omega \in D_+^k, \tag{27}$$

where

$$\begin{aligned} D_+ &= \{\varphi \in C([0,T]) : \varphi(t) \geq 0, \text{and nondecreasing}\}, \\ D_+^k &= D_+ \times \cdots \times D_+, \end{aligned} \tag{28}$$

and $\gamma_i > 0$, $\delta_i > 0$, $\varepsilon_i \in \mathbf{R}$, $g_i$, $f_i > 0$ are given constants. We compute $i+1$ modulo $k$, that is, $k+1=1$. Here we use an approach recently introduced by Pašić in **[8]** for scalar quasilinear elliptic equations, see also an extension of his result in **[7]**.

To formulate the main result of this section we introduce some notation:

$$E_{mi} = 1 - \varepsilon_i + \delta_i \cdot E_{m-1,i+1}, \quad m = 2,\dots,k, \quad E_{1j} = 1 - \varepsilon_j, \tag{29}$$

$$\delta_{(i,j)} = \delta_i \delta_{i+1} \dots \delta_{i+j-1}, \quad \delta = \delta_{(1,k)} = \delta_1 \dots \delta_k, \quad \delta_i^+ = \sum_{j=1}^{k-1} \delta_{(i,j)}, \tag{30}$$

$$H_i(\gamma) = \frac{\delta^{\frac{\delta_i^+ + \delta}{\delta-1}} \gamma^{\delta_i^+ + 1}}{T^{(\delta-1)\gamma + E_{ki}}} \prod_{j=1}^{k-1} \left( \frac{f_{i+j}}{\delta_{(i+j,i+k-j)}} \right)^{-\delta_{(i,j)}}. \tag{31}$$

Summation of indices in the definition of $\delta_{(i,j)}$ and $E_{ki}$ is performed modulo $k$, and $\delta_{(i,j)}$ is the product of $j$ terms.

**Theorem 4.** *Let $k \geq 2$ be a given natural number. Assume that*

$$\delta_i \geq 1, \quad \delta := \delta_1 \dots \delta_n > 1, \quad \gamma_i > 0, \tag{32}$$

$$\min\{\gamma_{i+1}, \gamma_{i+2}\} \cdot \delta_i - \varepsilon_i + 1 \geq \gamma_{i+1} > 0, \tag{33}$$

*for all $i = 1,\dots,k$, where indices are summed modulo $k$. Let $E_{ki}$ be constants defined by* (29) *and $H_i$ by* (4)*. Assume that any of the following four conditions holds:*

*(a)* $\exists i \in \{1,\dots,k\},\ E_{ki} \leq 0,\ \forall j \in \{1,\dots,k-1\},\ E_{k-j,i+j} \leq 0,$

$$g_i^{\delta-1} f_i \geq H_i(\gamma_i),$$

*(b)* $\exists i \in \{1,\dots,k\},\ E_{ki} > 0,\ \forall j \in \{1,\dots,k-1\},\ E_{k-j,i+j} \leq 0,$

$$g_i^{\delta-1} f_i \geq H_i(\gamma_i + E_{ki}/(\delta-1)),$$

*Then the singular system of $k$ integro-differential equations* (27) *has no solutions in $D_+^k$.*

In order to prove Theorem 4, we define simultaneously $k$ sequences of functions $(z_{im})_{m\geq 0}$, $i = 1,\dots,k$, by:

$$z_{i,m+1}(t) = f_i \int_0^t \frac{z_{i+1,m}(s)^{\delta_i}}{s^{\varepsilon_i}}\, ds, \quad z_{i0}(t) = g_i t^{\gamma_i}. \tag{34}$$

We have the following a priori estimate.

**Lemma 2.** *Let $\delta_i \geq 1$, $i = 1, \ldots, k$, and let $\omega = (\omega_1, \ldots, \omega_k)$ be a solution of (27) in $D = D_+^k$. Then for each index $i$ we have*

$$\sum_{m=0}^{\infty} z_{im}(t) \leq \omega_i(t). \tag{35}$$

*Proof.* It suffices to prove that

$$\sum_{m=0}^{n} z_{im}(t) \leq \omega_i(t), \tag{36}$$

for all $n$. This can easily be proved by induction with respect to $n$, simultaneously for all $i$. For $n = 0$ the claim is clear, see (27). Assume that (36) holds for some fixed $n$ and all $i = 1, \ldots, k$. Since $\omega_i$ is nondecreasing and nonnegative, we have $\omega_i(t) \geq \omega_i(0) + \int_0^t \omega_i'(s)\, ds \geq \int_0^t \omega_i'(s)\, ds$. Using (27) and $\delta_i \geq 1$, we have

$$\begin{aligned}
\omega_i(t) &\geq g_i t^{\gamma_i} + f_i \int_0^t \frac{\omega_{i+1}(s)^{\delta_i}}{s^{\varepsilon_i}}\, ds \\
&\geq z_{i0}(t) + f_i \int_0^t \frac{(\sum_{m=0}^n z_{i+1,m}(s))^{\delta_i}}{s^{\varepsilon_i}}\, ds \\
&\geq z_{i0}(t) + \sum_{m=0}^{n} f_i \int_0^t \frac{z_{i+1,m}(s)^{\delta_i}}{s^{\varepsilon_i}}\, ds \\
&= \sum_{m=0}^{n+1} z_{im}(t). \qquad \square
\end{aligned}$$

**Lemma 3.** *Assume* (33), $\gamma_i > 0$, *and let $(z_{im})_{m\geq 0}$ be $k$ sequences defined by* (34), $i = 1, \ldots, k$. *Then*

$$z_{im}(t) = a_{im} t^{b_{im}}, \quad i = 1, \ldots, k, \quad m = 0, 1, 2, \ldots \tag{37}$$

*where*

$$a_{i,km} = a_{i0}^{\delta^m} \prod_{j=1}^{m} A_{i,m-j}^{\delta^{j-1}}, \quad (a_{i0}, b_{i0}) = (g_i, \gamma_i), \tag{38}$$

*with*

$$A_{i,m} = \frac{f_i}{\delta \cdot b_{i,km} + E_{ki}} \prod_{j=1}^{k-1} \left( \frac{f_{i+j}}{b_{i+j,km+k-j}} \right)^{\delta_{(i,j)}}, \tag{39}$$

$$b_{i,km} = \delta^m b_{i0} + \frac{\delta^m - 1}{\delta - 1} \cdot E_{ki}, \quad i = 1, \ldots, k, \ m = 0, 1, 2, \ldots \tag{40}$$

$$b_{im} \geq \gamma_{i+1}.$$

*Proof.* It is clear that sequences $((a_{im}(t))_{m\geq 0}$, $i=1,\ldots,k$, defined by (34) have the form (37). From (34) we see that

$$a_{i,m+1}=\frac{f_i}{b_{i,m+1}}\,a_{i+1,m}^{\delta_i},\quad b_{i,m+1}=b_{i+1,m}\delta_i-\varepsilon_i+1. \tag{41}$$

The claim is proved using induction with respect to $m$ simultaneously for all $i=1,\ldots,k$. We omit the details. □

*Proof of Theorem* 4. It suffices to prove that under conditions of the theorem we have for each $i=1,\ldots,k$:

$$\sum_{m=0}^{\infty} z_{i,km}(T)=\infty. \tag{42}$$

Indeed, assume, contrary to the claim of the theorem, that there exists a solution $\omega$ of (27). Then (35) implies that $\omega_i(T)=\infty$, which is a contradiction.

(a) To prove (42), assume that condition (a) in the theorem holds. It is easy to see that, cf. (41),

$$b_{i+j,km+k-j}=\left(\delta^m b_{i0}+\frac{\delta^m-1}{\delta-1}\cdot E_{ki}\right)\prod_{s=j}^{k-1}\delta_{i+s}+E_{k-j,i+j}. \tag{43}$$

From this, and using (39), (40), $E_{ki}\leq 0$ and $E_{k-j,i+j}\leq 0$ for all $j=1,\ldots,k-1$, we obtain

$$\begin{aligned} A_{i,m} &\geq \frac{f_i}{\delta^{m+1}b_{i0}}\prod_{j=1}^{k-1}\left(\frac{f_{i+j}}{\delta^m b_{i0}\cdot\delta_{(i+j,k-j)}}\right)^{\delta_{(i,j)}}\\ &\geq \frac{f_i}{\delta^{m(\delta_i^+ +1)+1}}P_i(b_{i0}), \end{aligned} \tag{44}$$

where we have denoted

$$P_i(b_{i0})=\frac{1}{b_{i0}^{\delta_i^+ +1}}\prod_{j=1}^{k-1}\left(\frac{f_{i+j}}{\delta_{(i+j,k-j)}}\right)^{\delta_{(i,j)}}.$$

This implies that

$$\prod_{j=1}^{m}A_{i,m-j}^{\delta^{j-1}}\geq\frac{[f_iP_i(b_{i0})]^{\frac{\delta^m-1}{\delta-1}}}{\delta^{(\delta_i^+ +1)S_m+\frac{\delta^m-1}{\delta-1}}},$$

where

$$\begin{aligned} S_m &= \sum_{j=1}^{m-1}(m-j)\delta^{j-1}=m\cdot\frac{\delta^{m-1}-1}{\delta-1}-\frac{d}{d\delta}\left(\frac{\delta^m-\delta}{\delta-1}\right)\\ &=\frac{\delta^m-1}{(\delta-1)^2}-\frac{m}{\delta-1}. \end{aligned}$$

Hence, see (38),

$$a_{i,km} \geq a_{i0}^{\delta^m} \frac{[f_i P_i(b_{i0})]^{\frac{\delta^m-1}{\delta-1}}}{\delta^{(\delta_i^+ +1)S_m + \frac{\delta^m-1}{\delta-1}}}.$$

Now we substitute this and $b_{i,km} = \delta^m b_{i0} + \frac{\delta^m-1}{\delta-1} \cdot E_{ki}$ into $z_{i,km}(T) = a_{i,km} T^{b_{i,km}}$. Using $S_m \leq \delta^m/(\delta-1)^2$ and separating the terms containing $\frac{\delta^m}{\delta-1}$ from the rest we obtain that

$$z_{i,km}(T) \geq C_i(b_{i0}) \cdot S_i(b_{i0})^{\frac{\delta^m}{\delta-1}}, \tag{45}$$

where

$$C_i(b_{i0}) = P_i(b_{i0})^{\frac{-1}{\delta-1}} \cdot T^{\frac{-E_{ki}}{\delta-1}} \cdot (\delta/f_i)^{\frac{1}{\delta-1}}, \tag{46}$$

$$S_i(b_{i0}) = g_i^{\delta-1} f_i \cdot \frac{P_i(b_{i0}) \cdot T^{(\delta-1)b_{i0}+E_{ki}}}{\delta^{\frac{\delta_i^+ + \delta}{\delta-1}}}. \tag{47}$$

Condition $g_i^{\delta-1} f_i \geq H_i(\gamma_i)$ is equivalent to $S_i(b_{i0}) \geq 1$, and (42) follows from (45).

(b) If $E_{ki} \geq 0$ for some $i$, then

$$\delta^m b_{i0} + \frac{\delta^m - 1}{\delta - 1} \cdot E_{ki} \leq \delta^m \left( b_{i0} + \frac{E_{ki}}{\delta - 1} \right),$$

which we use in (40) and (43) in order to estimate $A_{i,m}$ from below. We can proceed in the same way as in (a), with $b_{i0} + E_{ki}/(\delta - 1)$ instead of $b_{i0}$. $\square$

## 4. Nonexistence of Solutions of Elliptic Systems

Here we study the problem of nonsolvability of quasilinear elliptic system (7). To formulate the main result of this section, we introduce some notation and terminology.

We say that a quasilinear elliptic system (7) possesses a $k$-cycle, $2 \leq k \leq n$, if there exist indices $i_1 < \cdots < i_k$ such that

$$\begin{aligned}
F_{i_1}(|x|, u, |\nabla u|_*) &\geq \tilde{g}_{i_1} |x|^{m_{i_1}} + \tilde{f}_{i_1} |\nabla u_{i_2}|^{e_{i_1}}, \\
F_{i_2}(|x|, u, |\nabla u|_*) &\geq \tilde{g}_{i_2} |x|^{m_{i_2}} + \tilde{f}_{i_2} |\nabla u_{i_3}|^{e_{i_2}}, \\
&\dots \\
F_{i_k}(|x|, u, |\nabla u|_*) &\geq \tilde{g}_{i_k} |x|^{m_{i_k}} + \tilde{f}_{i_k} |\nabla u_{i_1}|^{e_{i_k}},
\end{aligned}$$

for all $x \in B$ and $u \in C^2(B \setminus \{0\})$. When speaking about $k$-cycles, we can assume without loss of generality that $i_1 = 1, \ldots, i_k = k$. We can also define 1-cycle at index $i$ if we have $F_i(|x|, u, |\nabla u|_*) \geq \tilde{g}_i |x|^{m_i} + \tilde{f}_i |\nabla u_i|^{e_i}$. With this convention, we can state the following fairly general nonexistence result.

**Theorem 5. (Nonexistence of Solutions)** *Let the coefficients $\gamma_i$, $\delta_i$, $\varepsilon_i$, $g_i$, $f_i$ be defined by* (2) *and* (3), *$i = 1,\dots,k$. Assume that* (7) *possesses a $k$-cycle, $1 \le k \le n$, with the following properties. If $k \ge 2$, we assume that the corresponding coefficients of the $k$-cycle satisfy conditions of Theorem* 4. *If $k = 1$, then we only change the definition of $\delta_1$ in* (2) *to $\delta_1 = \frac{e_1}{p_1-1}$, and assume that $\gamma_1\delta_1 - \varepsilon_1 + 1 > \gamma_1$, $\delta_1 > 1$, $\gamma_1 > 0$, and $f_1$, $g_1$, see* (3), *are positive real numbers such that*

$$g_1^{\delta_1-1} f_1 \ge \begin{cases} \dfrac{[\gamma_1(\delta_1-1)-\varepsilon_1+1]\delta_1^{\delta_1'}}{(\delta_1-1)T^{\gamma_1(\delta_1-1)-\varepsilon_1+1}} & \text{for } \varepsilon_1 < 1, \\ \dfrac{\gamma_1\,\delta_1^{\delta_1'}}{T^{\gamma_1(\delta_1-1)-\varepsilon_1+1}} & \text{for } \varepsilon_1 \ge 1. \end{cases} \tag{48}$$

*Then quasilinear elliptic system* (7) *has no strong solutions.*

It is easy to see that system (7) can have at most $2^n - 1$ cycles, and this number can be achieved. Our Theorem 5 gives rise to nonexistence test which we can formulate in the form of algorithm with the following two steps: 1. find all cycles, 2. check if conditions of the theorem are satisfied for any of them. If so, quasilinear elliptic system (7) has no strong solutions.

To prove Theorem 5 we use the following lemma. Its proof is analogous to that of [**7**, Lemma 2], and therefore we omit it.

**Lemma 4.** *Assume that $u(x)$ is a strong solution of quasilinear elliptic system* (7). *Let us define constants $\gamma_i$, $\delta_i$, $\varepsilon_i$ by* (2), *$g_i$, $f_i$ by* (3), *and*

$$\omega_i(t) = t^{p_i(1-\frac{1}{N})}\left|\frac{dV_i}{dt}\right|^{p_i-1}, \quad t \in (0,T],$$

*where*

$$V_i(t) = u_i((tC_N^{-1})^{\frac{1}{N}}),$$

*and let $f_i(t)$ be defined by*

$$f_i(t) = \frac{1}{N^p C_N^{p/N}} F_i(\,(tC_N^{-1})^{\frac{1}{N}}, u((tC_N^{-1})^{\frac{1}{N}}), |\nabla u((tC_N^{-1})^{\frac{1}{N}})|_*\,).$$

*Then the functions $\omega_i(t)$ satisfy the following system of equations:*

$$\frac{d\omega_i}{dt} = f_i(t), \quad t \in (0,T), \quad \omega \in D_+^n, \quad i = 1,\dots,n, \tag{49}$$

*with $D_+$ defined by* (28).

*Proof of Theorem* 5. Assume, contrary to the claim of Theorem 5, that there exists a strong solution $u(x)$ of elliptic system (7). Let us consider the case of

$k \geq 2$ first. Using Lemma 6 we obtain a solution $\omega$ of (49) in $D_+^n$. Since by the assumption the system (7) is cyclic with respect to first $k$ equations, we conclude that

$$f_i(t) \geq g_i \gamma_i t^{\gamma_i - 1} + \frac{\omega_{i+1}(t)^{\delta_i}}{t^{\varepsilon_i}}, \quad i = 1, \dots, k. \tag{50}$$

Since $\omega_i(t)$ is nondecreasing, (49) implies that

$$\omega_i(t) \geq \omega_i(0) + \int_0^t f_i(s)\, ds \geq_i t^{\gamma_i} + f_i \int_0^t \frac{\omega_{i+1}(s)^{\delta_i}}{s^{\varepsilon_i}}\, ds =: K_i \omega_i(t). \tag{51}$$

Let us define the operator

$$K_0 : D_+^k \subset C([0,T], \mathbf{R}^k) \to C([0,T], \mathbf{R}^k),$$

by

$$K_0 \varphi = (K_1 \varphi_1, \dots, K_k \varphi_k), \quad \varphi = (\varphi_1, \dots, \varphi_k).$$

The space $C([0,T], \mathbf{R}^k)$ is an ordered Banach space with respect to the usual componentwise ordering. By (51) we have $K_0 P_k \omega \leq P_k \omega$, where $P_k$ is the projection operator defined on $\mathbf{R}^n$ by $P_k \omega = (\omega_1, \dots, \omega_k)$. Also, it is obvious that $0 \leq K_0(0)$, that is, 0 and $P_k \omega$ are ordered subsolution and supersolution of $K_0$ respectively. Since $K_0$ is compact and nondecreasing, we can use Amann [**1**, Theorem 6.1] to conclude that $K_0$ possesses a fixed point $\overline{\omega}$ in $D_+^k$. However, this contradicts Theorem 4.

If $k = 1$ then we can proceed in the same way as for $k \geq 2$, using [**7**, Theorem 7] instead of Theorem 4. □

Using minor modifications in the proof of Theorem 4 when $\delta = 1$, it is possible to treat also the critical case. This enables to study nonsolvability of elliptic system which has a $k$-cycle such that $e_i = p_i - 1$ for all $i = 1, \dots, k$. Here is the corresponding result which we state without proof.

**Theorem 6.** *Assume that quasilinear elliptic system* (7) *possesses a $k$-cycle, $k \geq 2$, such that $e_j = p_j - 1$ for all $j = 1, \dots, k$. Retaining the notation from Theorem* 5*, let condition* (33) *be satisfied and let there exist $i \in \{1, \dots, k\}$ such that $E_{ki} \leq 0$. If*

$$\prod_{j=1}^{k} f_j \geq T^{-E_{ki}} \gamma_i \prod_{j=1}^{k-1} (\gamma_i + E_{k-j,i+j}), \tag{52}$$

*then for all $\tilde{g}_i > 0$ quasilinear elliptic system* (7) *has no strong strong solutions.*

Condition $E_{ki} \leq 0$ in the above result is not artificial. Indeed, let us compare the critical case of system (7) considered in Theorem 6 with the following scalar elliptic equation:

$$
\begin{cases} -\Delta_p u = \tilde{g}_1|x|^m + \tilde{f}_1|\nabla u|^{p-1} \quad \text{in } B \setminus \{0\}, \\ \qquad u = 0 \quad \text{on } \partial B, \\ u(x) \text{ spherically symmetric and decreasing,} \end{cases} \tag{53}
$$

which has the critical exponent $e_1 = p - 1$ on the gradient. In this case the condition $E_{11} \leq 0$ is equivalent to $p \leq N$, while $E_{11} > 0$ is equivalent to $p > N$. It is possible to prove the following precise result.

**Theorem 7.** (*see* [**10**, Theorem 2]) *Assume that* $m > -N$, $\tilde{g}_1 > 0$, $\tilde{f}_1 > 0$, *and* $1 < p < \infty$.

*(a) If* $p > N$ *then* (53) *has a continuum of explicit strong solutions.*

*(b) If* $p < N$ *then* (53) *has no strong solutions.*

*(c) If* $p = N$ *then for* $\tilde{f}_1 < (m + N)C_N^{1/N}$ *and arbitrary* $\tilde{g}_1 > 0$ *equation* (53) *possess a continuum of explicit strong solutions, while for* $\tilde{f}_1 \geq (m+N)C_N^{1/N}$ *there are no strong solutions.*

As we see, Theorem 6 is in accordance with Theorem 7.

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D. Žubrinić, Department of Applied Mathematics, Faculty of Electrical Engineering, Unska 3, 10000 Zagreb, Croatia; *e-mail*: darko.zubrinic@fer.hr