UNIFORMLY WIGGLY DOMAINS AND RADIAL VARIATION

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1. INTRODUCTION

In a recent article [5] P. W. Jones and the author verified the so called Anderson conjecture for conformal maps on the unit disk; i.e., we proved that for each conformal map φ on the unit disk there exists an angle β such that

(1.1)
$$\int_0^1 |\varphi''(re^{i\beta})| \, dr < \infty.$$

The conjecture was formulated by Anderson in 1971 [1] and popularized by Ch. Pommerenke's book [8] and N. Makarov's survey [7]. For an early discussion of Anderson's conjecture see also the collection of problems edited by Ch. Pommerenke in 1972 [9].

In this note we present a class of conformal maps for which (1.1) can be given an elementary proof. We say that a domain Ω is uniformly wiggly if the corresponding Riemann map $\varphi \colon \mathbf{D} \to \Omega$ satisfies the following condition.

(1.2) For each
$$w \in \mathbf{D}$$
 there exists $z \in \mathbf{D}$ with $|z - w| \le (1 - |w|)/2$ and $(1 - |z|)|\varphi''(z)|/|\varphi'(z)| > \delta$, where $\delta > 0$ is independent of w .

In [4] one finds a geometric description of these domains that motivates the terminology. We consider these domains here for the following reasons.

- Important and well known examples like the von Koch curve, hyperbolic Julia sets and limit sets of degenerate Kleinian groups bound uniformly wiggly domains. (See [7], [10], [3].)
- (2) A lower estimate in the law of the interated logarithm for $b = \log |\varphi'|$ holds precisely when Ω is uniformly wiggly. (See [7].)
- (3) The special case of uniformly wiggly domains illustrates the proof of the general case given in [5]; in particular the role played by the stopping time Lipschitz domains and the estimates coming from J. Bourgain's theorem [2].

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2. UNIFORMLY WIGGLY DOMAINS AND RADIAL VARIATION

We start the proof by first collecting simple consequences of condition (1.2). Let $n = n(R, \delta)$ be a large integer, its value will be specified later. For $z = re^{i\theta}$ let

$$G(z) = \left\{ \rho e^{i\psi} : |\psi - \theta| < 1 - r, \ (1 - r)2^{-n} \le (1 - \rho) < 2(1 - r) \right\}.$$

Let R(z) denote the boundary of G(z) and let

$$O(z) = R(z) \cap \{w : 1 - |w| > (1 - |z|)/4\}$$

and

$$U(z) = R(z) \setminus O(z).$$

Let h_z be the harmonic measure of G(z) evaluated in z. Using square functions, Koebe's distortion estimates and (1.2) we verify that that $b = \log |\varphi'|$ satisfies the following estimates. (See [4].)

(2.1)
$$\left(\int_{R(z)} |b(w) - b(z)|^2 \, dh_z(w)\right)^{1/2} \ge C_0 n_z$$

(2.2)
$$\left(\int_{R(z)} |b(w) - b(z)|^4 \, dh_z(w)\right)^{1/2} \le C_0 n,$$

(2.3)
$$\left(\int_{O(z)} |b(w) - b(z)|^2 \, dh_z(w)\right)^{1/2} \le C_0,$$

(2.4)
$$\int_{R(z)} (b(w) - b(z)) dh_z(w) = 0.$$

Below we use the following simple observation which in the special case of uniformly wiggly domains replaces the following components used in [5] for the proof of the general case: Bourgain's estimates from [2], Beurling's estimates on harmonic measure and the analysis of stopping time Lipschitz-domains.

Lemma 1. Let (R, μ) be a probability space and let $b: R \to \mathbb{R}$ be square integrable, R^- denotes $\{w: b(w) \leq 0\}$. Let $O \subseteq R$ and $U = R \setminus O$. Suppose that the following conditions hold:

(2.5)
$$\left(\int_{R} b^2 d\mu\right)^{1/2} \le C_2 \int_{R} |b| d\mu,$$

(2.6)
$$\int_{O} b^2 d\mu \le \tau \int_{R} b^2 d\mu,$$

(2.7)
$$\int_R b \, d\mu = 0.$$

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Then, the following lower estimate holds

(2.8)
$$\int_{U \cap R^{-}} b^2 d\mu \ge \left(\frac{C_2^{-2}}{4} - \tau\right) \int_R b^2 d\mu.$$

Proof. By (2.5) and (2.7) we have

(2.9)
$$\left(\int_{R^{-}} b^2 d\mu\right)^{1/2} \ge \int_{R^{-}} |b| d\mu = \frac{1}{2} \int_{R} |b| d\mu \ge \frac{C_2^{-1}}{2} \left(\int_{R} b^2 d\mu\right)^{1/2}.$$

From the identity $R^- = \{R^- \cap U\} \cup \{R^- \cap O\}$ it follows that $R^- \cap U \subseteq R^- \setminus O$. Hence, by (2.7) and (2.9)

$$\int_{R^- \cap U} b^2 \, d\mu \ge \int_{R^-} b^2 \, d\mu - \int_O b^2 \, d\mu \ge \frac{c_2}{4} \int_R b^2 \, d\mu - \tau \int_R b^2 \, d\mu.$$

Remarks. 1. Condition (2.5) follows from

(2.10)
$$\left(\int_{R} b^{4} d\mu\right)^{1/4} \leq C_{4} \left(\int_{R} b^{2} d\mu\right)^{1/2}.$$

We can take $C_2 = C_4^6$. Indeed, Hölder's inequality applied to $u^2 = |u|^{2/3} |u|^{4/3}$ gives

$$\int_{R} b^2 d\mu \leq \left(\int_{R} |b| d\mu \right)^{2/3} \left(\int_{R} b^4 d\mu \right)^{1/3}.$$

By (2.10),

$$\int_{R} b^{2} d\mu \leq C_{4}^{4} \left(\int_{R} |b| d\mu \right)^{2/3} \left(\int_{R} b^{2} d\mu \right)^{1/3}.$$

Or

$$\left(\int_R b^2 d\mu\right)^{1/2} \le C_4^4 \left(\int_R |b| d\mu\right)^{2/3}.$$

This gives $C_2 = C_4^6$.

2. Let $\varphi: \mathbf{D} \to \Omega$ satisfy (1.2), and let $b = \log |\varphi'|$. For $z \in \mathbf{D}$ let w_z be the harmonic measure on the boundary of G(z), evaluated at z. Then the function $w \to b(w) - b(z)$ on the probability space $(R(z), h_z)$ satisfies the hypothesis of Lemma 1. This follows from (2.1)–(2.4), and the previous remark.

Theorem 1. When Ω is uniformly wiggly then the Riemann map $\varphi : \mathbf{D} \to \Omega$ satisfies (1.1).

Remark. As we pointed out above, this theorem holds true for any conformal map. (See [5].) The point made in this paper is that for uniformly wiggly domains we obtain an elementary proof which is given below. Its ingredients are the measure theoretic Lemma 2 and Remark 2.

Proof of Theorem 1. Fix $\eta > 0$, then for n large enough it follows from the meanvalue property of harmonic functions that there exists $z_1 \in \mathbb{D}$ with $2^{-n} \ge |z_1| > 1 - 1/8$ and

$$b(z_1) - b(0) < -\eta.$$

Suppose we have already constructed z_1, z_2, \ldots, z_l so that for $k \leq l$,

(2.11)
$$b(z_k) - b(z_{k-1}) < -\eta,$$
$$z_k \in U(z_{k-1}) \text{ and } 1 - |z_k| < \frac{1}{2}(1 - |z_{k-1}|).$$

Then consider $G(z_l)$. It follows from Remark 2 and Lemma 1 that there exists $z_{l+1} \in U(z_l)$ such that $b(z_{l+1}) - b(z_l) < -\eta$. Summing the corresponding telescoping series gives

$$(2.12) b(z_k) < -k\eta$$

for $k \in \mathbf{N}$. Next, by (2.11) the sequence z_k converges to a point in \mathbb{T} , call it $e^{i\beta}$. We fix k, and let $L_k = \{z \in (0, e^{i\beta}) : |z_k| \ge |z| \ge |z_{k+1}|\}$. Again by (2.11), for $z \in L_k$ we have $|z - z_k| \le (1 - |z|)2^n$. Now we recall that $|\nabla b(z)|(1 - |z|) \le 6$, for $z \in \mathbf{D}$, (see [8]). By (2.12) this gives $b(z) < -k\eta + n$, for $z \in L_k$, and $\int_{L_k} |\nabla b(\zeta)| |d\zeta| \le 24n$. Summing over k we have, for $z \in (0, e^{i\beta})$,

$$b(z) < -\frac{1}{4n} \int_{(0,z)} |\nabla b(\zeta)| \left| d\zeta \right| + 24n$$

A simple change of variables transforms this estimate into (1.1). Indeed, we observe that $2|\varphi''| = |\nabla b|e^b$ and write,

$$2\int_0^1 |\varphi''(re^{i\beta})| dr = \int_0^1 |\nabla b(re^{i\beta})| e^{b(re^{i\beta})} dr$$
$$\leq c_1 \int_0^1 |\nabla b(re^{i\beta})| e^{-c_2 \int_0^r |\nabla b(\rho e^{i\beta})| d\rho} dr$$
$$\leq c_3.$$

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