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DIAMETER IN PATH GRAPHS

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Abstract. If G is a graph, then its path graph, $P_k(G)$, has vertex set identical with the set of paths of length k in G, with two vertices adjacent in $P_k(G)$ if and only if the corresponding paths are "consecutive" in G. We construct bounds on the diameter of every component of $P_k(G)$ in form $diam(G) + f(k)$, where $f(k)$ is a function depending only on k. We have a general lower bound with $f(k) = -k$; upper bound for trees with $f(k) = k(k-2)$; and an upper bound for graphs with large diameter with $f(k) = k^2 - 2$, if $2 \le k \le 4$. All bounds are best possible.

1. Introduction

In this paper we consider only connected graphs G without loops and multiple edges. Let G be a graph, $k \ge 1$, and let $\mathcal{P}_k$ be the set of all subgraphs of G which form a path of length k (i.e., with $k+1$ vertices). The **path graph** $P_k(G)$ of G has vertex set $\mathcal{P}_k$. Let $A, B \in \mathcal{P}_k$. The vertices of $P_k(G)$ that correspond to A and B are joined by an edge in $P_k(G)$ if and only if the edges of $A \cap B$ form a path on k vertices and $A \cup B$ is either a path of length $k+1$ or a cycle of length $k+1$.

Path graphs were investigated by Broersma and Hoede in [**1**], as a natural generalization of line graphs (observe that $P_1(G)$ is a line graph of G). In [**1**] and [**5**] P_2-graphs are characterized, and in [**6**] traversability of P_2-graphs is studied. Centers of path graphs are studied in [**3**] and the behavior of the diameter of iterated P_2-graphs is studied in [**2**]. As proved in [**4**], for connected graph G it holds

$$\operatorname{diam}(G) - 1 \le \operatorname{diam}(P_1(G)) \le \operatorname{diam}(G) + 1,$$

where $\operatorname{diam}(H)$ denotes the diameter of H. In this paper we extend this result to path graphs. We show that

$$\operatorname{diam}(G) - k \le \operatorname{diam}(P_k(G))$$

for arbitrary graph G and $k \ge 1$. If $k \ge 2$, it is easy to find a connected graph G such that $P_k(G)$ is not connected. For this reason, we stress to find an upper bound for the diameter of every component of $P_k(G)$, instead of finding the diameter of

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$P_k(G)$. Let G be a graph and let H be an arbitrary component of $P_k(G)$. If G is a tree, we have

$$\operatorname{diam}(H) \le \operatorname{diam}(G) + k(k-2);$$

and if $2 \le k \le 4$ and $\operatorname{diam}(G) \ge \frac{1}{2}k^2 + 5k - 2$, we have

$$\operatorname{diam}(H) \le \operatorname{diam}(G) + k^2 - 2.$$

As shown by examples, all results are best possible in a sense. Moreover, all values from the interval determined by the lower and the upper bound, are attainable.

2. Lower Bound

We use standard graph-theoretical terminology, so that $V(G)$ denotes the vertex set, and $E(G)$ the edge set, of a graph G. By $d_G(u,v)$ we denote the distance from u to v in G, and $d_G(U,V)$ denotes the distance between sets of vertices U and V. To distinguish a path of length k in G, that results to a vertex in $P_k(G)$, from a shortest path in G connecting two vertices, we call the later a shortest walk. We remark that throughout the paper we use k only for the length of paths for path graph $P_k(G)$.

The vertices of path graph are adjacent if and only if one can be obtained from the other by "shifting" the corresponding path in G. For easier handling with paths of length k in G (i.e. the vertices in $P_k(G)$) we make the following agreement. We denote the vertices of $P_k(G)$ (as well as the vertices of G) by small letters a, b, ..., while the corresponding paths of length k in G we denote by capital letters A, B, ... It means that if A is a path of length k in G and a is a vertex in $P_k(G)$, then a is necessary the vertex corresponding to the path A.

Let A be a path of length k in G. By $A(i)$, $0 \le i \le k$, we denote the i-th vertex of A. If A and B are the same paths of length k in G, then either $A(i) = B(k-i)$, $0 \le i \le k$, or $A(i) = B(i)$, $0 \le i \le k$. To distinguish these situations we write $A = B$ if A and B are the same paths, while $A \equiv B$ if $A(i) = B(i)$ for all i, $0 \le i \le k$.

However, if a and b are adjacent vertices in $P_k(G)$, we always assume that the paths are denoted so that either $A(i) = B(i+1)$, $0 \le i < k$, or $A(i) = B(i-1)$, $0 < i \le k$. Thus, if $\mathcal{T} = (a_0, a_1, \dots, a_l)$ is a walk of length l in $P_k(G)$, then $A_0(i), A_1(i), \dots, A_l(i)$ are walks in G, $0 \le i \le k$.

Lemma 1. *Let G be a graph and let a and b be vertices in $P_k(G)$. Then*

$$d_{P_k(G)}(a,b) \ge \min\Big\{ \max\{d_G(A(0),B(0)), d_G(A(k),B(k))\}, \max\{d_G(A(0),B(k)), d_G(A(k),B(0))\}\Big\}.$$

Proof. Let $(a, a_1, \dots, a_l)$ be a shortest walk in $P_k(G)$ such that $A_l = B$. Then both $A(0), A_1(0), \dots, A_l(0)$ and $A(k), A_1(k), \dots, A_l(k)$ are walks in G. If $A_l(0) = B(0)$ and $A_l(k) = B(k)$, then

$$d_{P_k(G)}(a,b) = l \geq d_G(A(0), B(0)) \quad \text{and} \quad d_{P_k(G)}(a,b) \geq d_G(A(k), B(k)),$$

and hence, $d_{P_k(G)}(a,b) \geq \max\{d_G(A(0),B(0)), d_G(A(k),B(k))\}$. On the other hand, if $A_l(0) = B(k)$ and $A_l(k) = B(0)$, then

$$d_{P_k(G)}(a,b) \geq d_G(A(0), B(k)) \quad \text{and} \quad d_{P_k(G)}(a,b) \geq d_G(A(k), B(0)),$$

and hence, $d_{P_k(G)}(a,b) \geq \max\big\{d_G(A(0),B(k)), d_G(A(k),B(0))\big\}$. □

Theorem 2. *Let G be a graph such that $P_k(G)$ is not empty. Then*

$$\operatorname{diam}(P_k(G)) \geq \operatorname{diam}(G) - k.$$

Proof. Let u_0 and u_l be vertices in G such that $d_G(u_0, u_l) = l = \operatorname{diam}(G)$. Moreover, let $\mathcal{T} = (u_0, u_1, \dots, u_l)$ be a walk of length l in G. Without loss of generality we may assume that $l \geq k$. Denote $A \equiv (u_0, u_1, \dots, u_k)$ and $B \equiv (u_{l-k}, u_{l-k+1}, \dots, u_l)$. Since $\mathcal{T}$ is a diametric path in G, we have $d_G(A(0), B(0)) = l{-}k$, $d_G(A(k), B(k)) = l{-}k$, $d_G(A(0), B(k)) = l$, and $d_G(A(k), B(0)) = |l{-}2k|$. By Lemma 1, we have $\operatorname{diam}(P_k(G)) \geq d_{P_k(G)}(a,b) \geq \min\{\max\{(l{-}k),(l{-}k)\}, \max\{l, |l{-}2k|\}\} = l - k = \operatorname{diam}(G) - k$. □

Since $\operatorname{diam}(P_k(G)) = \operatorname{diam}(G) - k$ if G is a path of length $l \geq k$, the bound in Theorem 2 is best possible.

3. Upper Bounds

In this part we give an upper bound for the diameter of some path graphs. For this we need one more notion.

Let G be a graph and let $\mathcal{T} = (a_0, a_1, \dots, a_l)$ be a walk in $P_k(G)$. Assume that $A_0(i) = A_1(i{-}1)$ for all i, $0 < i \leq k$. Then A_j is a **turning path** if

(i) $j = 0$;
(ii) $A_{j-1}(i{+}1) = A_j(i) = A_{j+1}(i{+}1)$ for all i, $0 \leq i < k$;
(iii) $A_{j-1}(i{-}1) = A_j(i) = A_{j+1}(i{-}1)$ for all i, $0 < i \leq k$.

The vertex $A_0(k)$ in the case (i), $A_j(0)$ in the case (ii), and $A_j(k)$ in the case (iii), is a **turning point** of $\mathcal{T}$.

Let a and a' be vertices in $P_k(G)$. Suppose that A and A' are edge-disjoint, and denote $\mathcal{T} = (u_0, u_1, \dots, u_l)$ a shortest trail in G beginning in a vertex from A and terminating in a vertex from A'. In some situations it is possible to construct a walk from a to a' in $P_k(G)$ in the following way: first "shift" A "forwards and backwards" several times to get the path A into a path B such that one endvertex of B is u_0, then utilize the walk $\mathcal{T}$, and repeat the same process with A' in a reverse order. In the next fundamental lemma we count the distance from a to b in $P_k(G)$.

Lemma 3. *Let G be a graph, such that G does not contain a cycle of length at most k if $k \geq 5$. Let a be a vertex in $P_k(G)$ and $v \in V(A)$. Finally, let b be a vertex at the shortest distance from a, such that one endvertex of B is v. If $d_{P_k(G)}(a,b) < \infty$, then*

$$d_{P_k(G)}(a,b) \leq \frac{k(k-1)}{2}.$$

Proof. Let $\mathcal{T} = (a_0, a_1, a_2, \dots, a_l)$ be a shortest walk in $P_k(G)$, such that $A_0 \equiv A$ and $A_l = B$. Let $A_{i_0}, A_{i_1}, A_{i_2}, \dots, A_{i_{p-1}}$ be the turning paths, $0 = i_0 < i_1 < i_2 < \dots < i_{p-1} < l$, and let $v_0, v_1, v_2, \dots, v_{p-1}$ be corresponding turning points. Assume that $v_0 = A(k)$. Then $v_1 = A_{i_1}(0)$, $v_2 = A_{i_2}(k)$, $v_3 = A_{i_3}(0), \dots$. Set $v_p = v$ and $A_{i_p} \equiv A_l = B$. By induction we prove the following statement:

(∗) All $v_0, v_1, \dots, v_p$ are vertices of A; v_j is between v_{j-2} and v_{j-1} on A, $2 \leq j \leq p$; and A_{i_j} contains the subpath of A between v_{j-1} and v_j, $1 \leq j \leq p$. Moreover, if $1 \leq j \leq p$, then $d_{P_k(G)}(a_{i_{j-1}}, a_{i_j}) = k - d_A(v_{j-1}, v_j)$.

1° Clearly, v_0 and v_1 are vertices of A. Thus, $d_{P_k(G)}(a, a_{i_1}) = k - d_A(v_0, v_1)$.

2° Suppose that (∗) is valid for all j', $j' < j \leq p$, and denote by a_q the first vertex on $\mathcal{T}$, such that A_q has one endvertex, say z, between v_{j-2} and v_{j-1}. By induction $q > i_{j-1}$, and $q \leq l$ as $j-1 < p$. Since a_q is the first vertex on $\mathcal{T}$ with an endvertex of A_q between v_{j-2} and v_{j-1}, one endvertex of A_{q-1} is either v_{j-1} or v_{j-2}. We will solve both these cases separately, see Figure 1 and Figure 2.

(a) A_{q-1} has one endvertex in v_{j-1}. We will prove that this case cannot occur if $\mathcal{T}$ is a shortest walk. Since $A_{i_{j-1}}$ contains the subpath of A between v_{j-2} and v_{j-1}, by induction, A_q contains the subpath of A between v_{j-2} and z, see Figure 1.

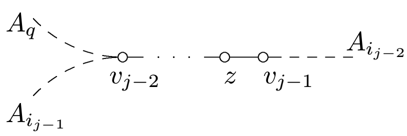

Figure 1.

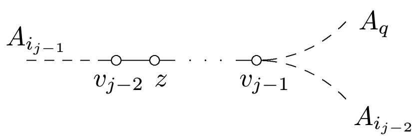

Figure 2.

As $A_{i_{j-1}}$ is a turning path with turning point v_{j-1}, the endvertex of $A_{i_{j-1}+1}$ adjacent to v_{j-1} is different from z. Thus,

$$d_{P_k(G)}(a_{i_{j-1}}, a_q) \geq 3.$$

Since $\mathcal{T}$ is a shortest walk in $P_k(G)$, we have

$$\begin{aligned} d_{P_k(G)}(a_{i_{j-2}}, a_q) &= d_{P_k(G)}(a_{i_{j-2}}, a_{i_{j-1}}) + d_{P_k(G)}(a_{i_{j-1}}, a_q) \\ &\geq k - d_A(v_{j-2}, v_{j-1}) + 3, \end{aligned} \tag{1}$$

by induction. Consider the graph $H = A_{i_{j-2}} \cup A_q$. If G does not contain a cycle of length at most k, then in H we can "shift" the path $A_{i_{j-2}}$ towards A_q step by step, and such a "shifted" path is a path again. Thus,

$$d_{P_k(G)}(a_{i_{j-2}}, a_q) \leq k - d_A(v_{j-2}, z) = k - d_A(v_{j-2}, v_{j-1}) + 1,$$

which contradicts (1). Hence, it remains to solve the cases $k = 3$ and $k = 4$. Here an obstacle can occur only if there is a cycle $\mathcal{C}$ of length at most k in H.

Suppose that $k = 3$. Since A_q and $A_{i_{j-2}}$ are paths, $\mathcal{C} = (v_{j-2}, z, c)$ for some vertex c in H, see Figure 3. As v_{j-1} is a vertex adjacent to z in $A_{i_{j-2}}$, we have $c = v_{j-1}$. Thus, $A_q = (v_{j-1}, z, v_{j-2}, v_{j-1})$, a contradiction.

Suppose that $k = 4$. If A_{q-1} does not contain a cycle, then $\mathcal{C}$ has the length four and v_{j-2} and z are adjacent vertices on $\mathcal{C}$, see Figure 4. But then $z = v$ and $A_q = B$. Let $A_{i_{j-1}} = (v_{j-1}, v, v_{j-2}, c_4, c_5)$ for some $c_4, c_5 \in V(G)$. By induction, we have $d_{P_4(G)}(a_{i_{j-2}}, a_{i_{j-1}}) = 2$, and hence $c_0 \neq c_4$. We have $d_{P_4(G)}(a_{i_{j-1}}, a_{q-1}) \geq 4$, since on a shortest walk from $a_{i_{j-1}}$ to a_{q-1} there must be a vertex x in $P_4(G)$ such that v_{j-2} is an endvertex of X. Moreover, by the definition of a_q, X has the form $X = (v_{j-2}, v, v_{j-1}, c_6, c_7)$ for some $c_6, c_7 \in V(G)$, $c_6 \neq c_0$. Since $\mathcal{T}$ is a shortest walk from a to b in $P_4(G)$, we have

$$\begin{aligned} d_{P_4(G)}(a_{i_{j-2}}, b) &= d_{P_4(G)}(a_{i_{j-2}}, a_q) \\ &= d_{P_4(G)}(a_{i_{j-2}}, a_{ij-1}) + d_{P_4(G)}(a_{i_{j-1}}, a_{q-1}) + d_{P_4(G)}(a_{q-1}, a_q) \\ &\geq 2 + 4 + 1 = 7. \end{aligned}$$

Denote $X_1 = (c_4, v_{j-2}, v, v_{j-1}, c_0)$, $X_2 = (v_{j-2}, v, v_{j-1}, c_0, c_2)$ and $B' = (v, v_{j-1}, c_0, c_2, c_3)$. If $c_3 \neq v_{j-1}$, then $(a_{i_{j-2}}, x_1, x_2, b')$ is a shorter walk in $P_4(G)$ such that one endvertex of B' is v, which contradicts the choice of b.

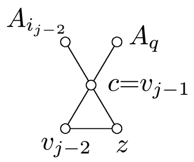

Figure 3.

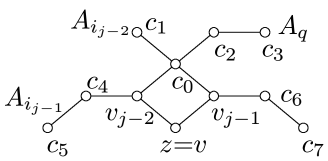

Figure 4.

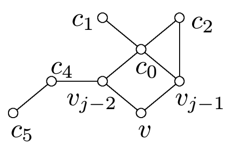

Figure 5.

Thus, suppose that $c_3 = v_{j-1}$, see Figure 5. Denote $X_1 = (c_4, v_{j-2}, v, v_{j-1}, c_0)$, $X_2 = A_{i_{j-1}} = (c_5, c_4, v_{j-2}, v, v_{j-1})$, $X_3 = (c_4, v_{j-2}, v, v_{j-1}, c_2)$, $X_4 = (v_{j-2}, v, v_{j-1}, c_2, c_0)$ and $B' = (v, v_{j-1}, c_2, c_0, v_{j-2})$. If $c_2 \neq c_4$, then $(a_{i_{j-2}}, x_1, x_2, x_3, x_4, b')$ is a shorter walk in $P_4(G)$ such that one endvertex of B' is v, which contradicts the choice of b.

Finally, suppose that $c_3 = v_{j-1}$ and $c_2 = c_4$, see Figure 6. Then $c_2 \neq c_6$. Denote $X_1 = (c_2, v_{j-2}, v, v_{j-1}, c_0)$, $X_2 = A_{i_{j-1}} = (c_5, c_2, v_{j-2}, v, v_{j-1})$, $X_3 =$

$(c_2, v_{j-2}, v, v_{j-1}, c_6)$, $X_4 = (c_0, c_2, v_{j-2}, v, v_{j-1})$ and $B' = (v_{j-1}, c_0, c_2, v_{j-2}, v)$. Then $(a_{i_{j-2}}, x_1, x_2, x_3, x_4, b')$ is a shorter walk in $P_4(G)$ such that one endvertex of B' is v, which contradicts the choice of b.

(b) A_{q-1} has one endvertex in v_{j-2}, see Figure 2. We will prove that $A_{i_{j-1}+1}$, $A_{i_{j-1}+2}, \dots, A_{q-1}$ are not turning paths in this case. Consider the graph $H = A_{i_{j-1}} \cup A_q$. If G does not contain a cycle of length at most k, then we can "shift" the path $A_{i_{j-1}}$ towards A_q step by step, and such a "shifted" path is a path again. Thus,

$$d_{P_k(G)}(a_{i_{j-1}}, a_q) \leq k - d_A(v_{j-1}, z).$$

Since $\mathcal{T}$ is a shortest walk and all $A_{i_{j-1}}, A_{i_{j-1}+1}, \dots, A_q$ contain the subpath of A between v_{j-1} and z, we have

$$d_{P_k(G)}(a_{i_{j-1}}, a_q) = k - d_A(v_{j-1}, z),$$

and $A_{i_{j-1}+1}, A_{i_{j-1}+2}, \dots, A_{q-1}$ are not turning paths.

However, if H contains a cycle $\mathcal{C}$ of length at most k, an obstacle can occur. As A_{q-1} does not contain a cycle, it remains to solve the case $k = 4$ when $\mathcal{C}$ has the length four and v_{j-1} and z are adjacent vertices on $\mathcal{C}$, see Figure 7. But then $z = v$ and $A_q = B$. Since $A_{i_{j-1}}$ and A_{q-1} share a path of length two and $A_{i_{j-1}} \cup A_{q-1}$ contains a cycle of length four, we have $d_{P_4(G)}(a_{i_{j-1}}, a_{q-1}) \geq 6$, and hence,

$$d_{P_4(G)}(a_{i_{j-2}}, a_q) \geq 9.$$

Moreover, on a shortest walk from $a_{i_{j-1}}$ to a_{q-1} there is a vertex x in $P_4(G)$, such that v_{j-1} is an endvertex of X. Thus, $X = (v_{j-1}, v, v_{j-2}, c_6, c_7)$ for some $c_6, c_7 \in V(G)$.

Denote $X_1 = (c_4, v_{j-1}, v, v_{j-2}, c_0)$, $X_2 = (v_{j-1}, v, v_{j-2}, c_0, c_2)$ and $B' = (v, v_{j-2}, c_0, c_2, c_3)$. If $c_3 \neq v_{j-2}$, then $(a_{i_{j-2}}, x_1, x_2, b')$ is a shorter walk in $P_4(G)$ such that one endvertex of B' is v, which contradicts the choice of b.

Thus, suppose that $c_3 = v_{j-2}$, see Figure 8. Then the problem is reduced to that in the case (a), since all walks constructed there passed through $a_{i_{j-1}}$.

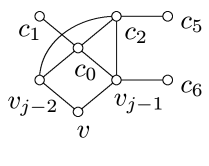

Figure 6.

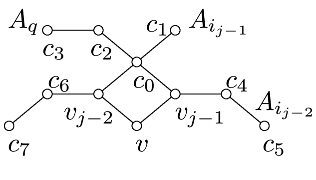

Figure 7.

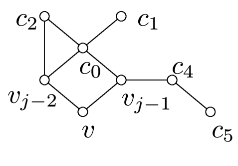

Figure 8.

It means, that on a shortest walk $\mathcal{T}$ no obstacles with short cycles can occur, so that $A_{i_{j-1}+1}, A_{i_{j-1}+2}, \dots, A_{q-1}$ are not turning paths. Thus, v_j is between v_{j-2}

and v_{j-1} on A, and A_{i_j} contains the subpath of A between v_j and v_{j-1}. Hence

$$d_{P_k(G)}(a_{i_{j-1}}, a_{i_j}) = k - d_A(v_{j-1}, v_j),$$

so that $(*)$ is proved.

By $(*)$ for the length of $\mathcal{T}$ we have

$$l = \sum_{j=1}^{p} d_{P_k(G)}(a_{i_{j-1}}, a_{i_j}) = \sum_{j=1}^{p} k - d_A(v_{j-1}, v_j).$$

Since v_j is a vertex between v_{j-2} and v_{j-1} on A, by $(*)$, we have $k > d_A(v_0, v_1) > d_A(v_1, v_2) > \cdots > d_A(v_{p-1}, v_p) > 0$. Thus, $k - d_A(v_0, v_1), k - d_A(v_1, v_2), \ldots, k - d_A(v_{p-1}, v_p)$ are all different values from $[1, k-1]$. Hence,

$$\sum_{j=1}^{p} k - d_A(v_{j-1}, v_j) \leq \sum_{i=1}^{k-1} i = \frac{k(k-1)}{2}. \qquad \square$$

We remark, that the restriction: "G does not contain a cycle of length at most k if $k \geq 5$", is necessary in Lemma 3, as shown by graph G in Figure 9. If $A = (u_1, u_2, u_3, u_4, u_5, u_6)$ and $v = u_4$, then the required vertex at the shortest distance from a is b, $B = (u_4, u_5, u_7, u_2, u_8, u_9)$. However, $d_{P_5(G)}(a, b) = 13 > 10 = \frac{5 \cdot 4}{2}$.

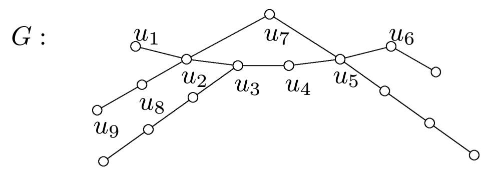

Figure 9.

Theorem 4. *Let G be a tree and let H be a component of $P_k(G)$. Then*

$$\operatorname{diam}(H) \leq \operatorname{diam}(G) + k(k-2).$$

Proof. Let H be a component of $P_k(G)$ and let a and a' be vertices in H, such that $d_{P_k(G)}(a, a') = \operatorname{diam}(H)$. Let $\mathcal{T} = (a_0, a_1, \ldots, a_l)$ be a shortest walk in $P_k(G)$ such that $A_0 \equiv A$ and $A_l = A'$. Distinguish two cases:

(a) A and A' are edge-disjoint. Let $\mathcal{W} = (v_0, v_1, \ldots, v_r)$ be a shortest walk in G beginning in a vertex from A and terminating in a vertex from A' (i.e., $d_G(V(A), V(A')) = r$). Since G is a tree, there must be a vertex, say a_b, in $\mathcal{T}$, such that v_0 is an endvertex of A_b. Let b be a vertex at the shortest distance from a, such that v_0 is an endvertex of B. Then $d_{P_k(G)}(a, b) < \infty$. Analogously, let b'

be a vertex at the shortest distance from a', such that v_r is an endvertex of B'. By $(*)$ in Lemma 3, v_0 and v_r are the unique vertices of B and B', respectively, in $\mathcal{W}$. By Lemma 3, we have

$$\begin{aligned} d_{P_k(G)}(a,a') &\le d_{P_k(G)}(a,b) + d_{P_k(G)}(b,b') + d_{P_k(G)}(b',a') \\ &\le \frac{k(k-1)}{2} + (k+r) + \frac{k(k-1)}{2} = (2k+r) + k(k-2). \end{aligned}$$

Since G is a tree, there is an endvertex of B, say u, and an endvertex of B', say u', such that $d_G(u,u') = 2k+r$. Thus, $\operatorname{diam}(G) \ge 2k+r$, and hence,

$$\operatorname{diam}(H) = d_{P_k(G)}(a,a') \le \operatorname{diam}(G) + k(k-2).$$

(b) A and A' share a path $\mathcal{W}$ of length $r \ge 1$ in G, $\mathcal{W} = (v_0, v_1, \dots, v_r)$. Distinguish three subcases:

(b1) Suppose that for every vertex c in H, the c contains a subpath $\mathcal{V} = (v_i, v_{i+1}, \dots, v_j)$ of $\mathcal{W}$, $i<j$. (The subpath $\mathcal{V}$ is maximal with this property.) Then there is a vertex b^* in H, such that v_i is an endvertex of B^*. Let b be a vertex at the shortest distance from a, such that v_i is an endvertex of B. Then $d_{P_k(G)}(a,b) < \infty$ and $B = (v_i, v_{i+1}, \dots, v_j, \dots)$. Analogously, let b' be a vertex at the shortest distance from a', such that v_j is an endvertex of B'. Then $B' = (v_j, v_{j-1}, \dots, v_i, \dots)$. By Lemma 3, we have

$$\begin{aligned} d_{P_k(G)}(a,a') &= d_{P_k(G)}(a,b) + d_{P_k(G)}(b,b') + d_{P_k(G)}(b',a') \\ &\le \frac{k(k-1)}{2} + (k-(j-i)) + \frac{k(k-1)}{2} = (2k-(j-i)) + k(k-2). \end{aligned}$$

Since G is a tree, there is an endvertex of B, say u, and an endvertex of B', say u', such that $d_G(u,u') = 2k-(j-i)$. Thus, $\operatorname{diam}(G) \ge 2k-(j-i)$, and hence,

$$\operatorname{diam}(H) = d_{P_k(G)}(a,a') \le \operatorname{diam}(G) + k(k-2).$$

Now suppose that every vertex v of $\mathcal{W}$ is an endvertex of some C, such that $d_{P_k(G)}(a,c) < \infty$. For every vertex v_j in $\mathcal{W}$ denote by c_{i_j} a vertex in H, such that v_j is an endvertex of C_{i_j} and $d_{P_k(G)}(a,c_{i_j})$ is minimum. Let v_s be a vertex in $\mathcal{W}$ such that

$$d_{P_k(G)}(a,c_{i_s}) = \max\bigl\{d_{P_k(G)}(a,c_{i_j});\ v_j \in V(\mathcal{W})\bigr\}.$$

Denote $B = A_{i_s}$. The edge of B incident to v_s lies in A, by $(*)$ in Lemma 3.

(b2) Suppose that the edge of B incident to v_s lies in $\mathcal{W}$, and assume that $B = (v_s, v_{s+1}, \dots)$. Let b' be a vertex at the shortest distance from a', such that one endvertex of B' is v_{s+1}. If $B' = (v_{s+1}, v_s, \dots)$, then analogously as in (b1) we have $d_{P_k(G)}(a,a') \le (2k-1) + k(k-2)$ and $\operatorname{diam}(G) \ge 2k-1$, so that $\operatorname{diam}(H) \le \operatorname{diam}(G) + k(k-2)$. Thus, suppose that $B' = (v_{s+1}, v^*, \dots)$, $v^* \ne v_s$. On a shortest $a-b$ walk in $P_k(G)$ there is a vertex d, such that $D=(v_{s+1},v_s,\dots)$, by $(*)$ in Lemma 3. Then analogously as in (b1) we have $d_{P_k(G)}(a,a')\le d_{P_k(G)}(a,d)+d_{P_k(G)}(d,b')+d_{P_k(G)}(b',a')\le 2k+k(k-2)$ and $\operatorname{diam}(G)\ge 2k$, so that $\operatorname{diam}(H)\le \operatorname{diam}(G)+k(k-2)$.

(b3) Suppose that the edge of B incident to v_s lies in $A-\mathcal{W}$. Assume that $v_s=v_0$ and $B=(v_0,v^*,\dots)$, $v^*\ne v_1$. Let b' be a vertex at the shortest distance from a', such that one endvertex of B' is v_0, $B'=(v_0,v^{*\prime},\dots)$. Since $v^*\in V(A)-V(\mathcal{W})$, $v^{*\prime}\ne v^*$. Then analogously as above we have $d_{P_k(G)}(a,a')\le 2k+k(k-2)$ and $\operatorname{diam}(G)\ge 2k$, so that $\operatorname{diam}(H)\le \operatorname{diam}(G)+k(k-2)$. □

In Corollary 5 we prove that the bound in Theorem 4 is best possible. Moreover, we show that the diameter of a component of $P_k(G)$ can achieve all values from the range bounded by Theorem 2 and Theorem 4, if G is a tree.

Corollary 5. *Let $r\ge 2k$ and $-k\le s\le k(k-2)$. Then there is a tree $G_{r,s}$ with diameter r such that for one component H of $P_k(G_{r,s})$ we have*

$$\operatorname{diam}(H)=\operatorname{diam}(G_{r,s})+s.$$

Proof. First we construct a graph G_r with diameter r. Let $A_0,A_1,\dots,A_{k-1}$ be a collection of vertex-disjoint paths, such that the length of A_i is i, $0\le i\le k-1$. Let a graph G be obtained from a path $(v_0,v_2,v_4,\dots,v_k,\dots,v_5,v_3,v_1)$ by identifying one endvertex of A_i with v_i, $0\le i\le k-1$, see Figure 10 for the case $k=6$. Moreover, let G' be a copy of G, consisting from a path $(v_0',v_2',v_4',\dots,v_k',\dots,v_5',v_3',v_1')$ and $A_0',A_1',\dots,A_{k-1}'$. Denote by G_r a graph consisting from G, G', and a path $\mathcal{W}$ of length $r-2k$ joining v_k with v_k'. Then G_r is a tree and $\operatorname{diam}(G_r)=r$, since one endvertex of A_{k-1} has distance $(k-1)+1+(r-2k)+1+(k-1)=r$ from one endvertex of A_{k-1}'. Let $A=(v_0,v_2,\dots,v_k,\dots,v_3,v_1)$ and $A'=(v_0',v_2',\dots,v_k',\dots,v_3',v_1')$. Denote by H the component of $P_k(G)$ containing a. Since every vertex of H (except a and a') has degree two, H is a path. Thus, $d_{P_k(G)}(a,a')=\frac{k(k-1)}{2}+((r-2k)+k)+\frac{k(k-1)}{2}=\operatorname{diam}(G_r)+k(k-2)$.

Now we order the $k(k-1)$ vertices of $A_0,A_1,\dots,A_{k-2},A_0',A_1',\dots,A_{k-2}'$. Let

$$A_0=(u_1{=}v_0),\quad A_1=(u_2,u_3{=}v_1),\quad A_2=(u_4,u_5,u_6{=}v_2),\quad \dots,$$
$$A_{k-2}=(u_{\frac{k(k-1)}{2}-k+2},u_{\frac{k(k-1)}{2}-k+3},\dots,u_{\frac{k(k-1)}{2}}{=}v_{k-2}),$$
$$A_0'=(u_{\frac{k(k-1)}{2}+1}{=}v_0'),\quad A_1=(u_{\frac{k(k-1)}{2}+2},u_{\frac{k(k-1)}{2}+3}{=}v_1'),\quad \dots,$$
$$A_{k-2}'=(u_{k(k-1)-k+2},u_{k(k-1)-k+3},\dots,u_{k(k-1)}{=}v_{k-2}'),$$

see Figure 10 for ordering the vertices of $A_0,A_1,\dots,A_{k-2}$ in the case $k=6$.

Let $G_{r,s}$ be a graph obtained from G_r by deleting the vertices $u_1,u_2,\dots,u_{k(k-2)-s}$. Since $G_{r,s}$ is a subgraph of G_r, and $G_{r,s}$ contains A_{k-1}, A_{k-1}', and the path $\mathcal{W}$ joining v_k with v_k', we have $\operatorname{diam}(G_{r,s})=\operatorname{diam}(G_r)=r$. Denote

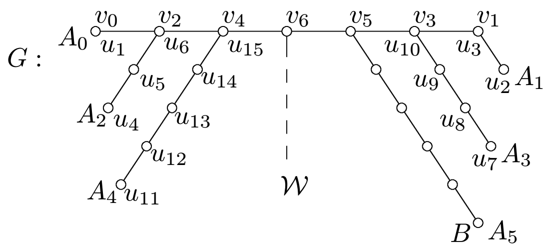

Figure 10.

by B the path of length k consisting from A_{k-1} and the edge $v_{k-1}v_k$, and denote H the component of $P_k(G_{r,s})$ containing b. Since H is a path of length $r + k(k-2) - (k(k-2) - s)$, we have $\operatorname{diam}(H) = \operatorname{diam}(G_{r,s}) + s$. □

If $2 \le k \le 4$, then using Lemma 3 we are able to determine a good upper bound for the diameter in $P_k(G)$ for arbitrary graph G, provided that G has sufficiently large diameter.

Theorem 6. *Let G be a graph such that* $\operatorname{diam}(G) \ge \frac{1}{2}k^2 + 5k - 2$, *and* $2 \le k \le 4$. *Then for any component H of $P_k(G)$ we have*

$$\operatorname{diam}(H) \le \operatorname{diam}(G) + k^2 - 2$$

Proof. Let H be a component of $P_k(G)$, and let $\operatorname{diam}(H) = d_{P_k(G)}(a, a') = l$. Suppose that $\mathcal{T} = (a_0, a_1, \dots, a_l)$ is a shortest walk in $P_k(G)$, such that $A_0 \equiv A$ and $A_l = A'$. Let $\mathcal{W} = (v_0, v_1, \dots, v_r)$ be a shortest walk in G beginning in a vertex from A and terminating in a vertex from A' (i.e., $d_G(V(A), V(A')) = r$). Distinguish two cases.

(a) $r \ge k - 1$. Since A and A' are edge-disjoint, there are vertices a_b and $a_{b'}$ in $\mathcal{T}$, such that v_0 is an endvertex of A_b and v_r is an endvertex of $A_{b'}$. Let b and b' be vertices at the shortest distance from a and a', respectively, such that v_0 is an endvertex of B and v_r is an endvertex of B'. Then $d_{P_k(G)}(a, b) \le \frac{k(k-1)}{2}$ and $d_{P_k(G)}(b', a') \le \frac{k(k-1)}{2}$, by Lemma 3. Moreover, the edge e of B incident to v_0 is in A, and the edge e' of B' incident to v_r is in A', by $(*)$ in Lemma 3. If there is not a cycle of length at most k in $\mathcal{W} \cup B$ and $\mathcal{W} \cup B'$, then $d_{P_k(G)}(b, b') \le r + k$, and hence, $d_{P_k(G)}(a, a') \le d_{P_k(G)}(a, b) + d_{P_k(G)}(b, b') + d_{P_k(G)}(b', a') \le k^2 + r$.

Thus, suppose that there is a "short" cycle in $\mathcal{W} \cup B$. Since e is an edge of B, the "short" cycle necessarily contains v_1. Hence, on a shortest $a - b$ walk there is a vertex c, such that one endvertex of C is v_1. Assume that c is the first vertex on a shortest $a - b$ walk with this property. If $c = b$, then the endvertices of C are v_0 and v_1, so that there cannot be a "short" cycle in $\mathcal{W} \cup B$. Thus $c \ne b$, and hence, C contains at least two edges of A. It means that if $k = 3$, then v_1 is the unique vertex of C outside A; and if $k = 4$, then at most one vertex of C different from

v_1 can have a nonzero distance at most one from A. If $\mathcal{W} \cup B'$ does not contain a "short" cycle, then $d_{P_k(G)}(c, b') \leq (r-1) + k$. Thus, suppose that $\mathcal{W} \cup B'$ contains a cycle of length at most k and construct C' analogously as C. Since $r \geq k-1$, $C \cup (v_1, v_2, \ldots, v_{r-1}) \cup C'$ form a path in G, so that $d_{P_k(G)}(c, c') \leq (r-2) + k$. As $d_{P_k(G)}(a, c) < d_{P_k(G)}(a, b)$ and $d_{P_k(G)}(c', a') < d_{P_k(G)}(b', a')$, in all cases we have

$$d_{P_k(G)}(a, a') \leq \frac{k(k-1)}{2} + (k+r) + \frac{k(k-1)}{2} = k^2 + r.$$

Now we bound the diameter of G. Consider three cases.

(i) Suppose that there is a walk $\mathcal{V}$ of length r from an endvertex of A to an endvertex of A' in G. Then $d_{P_k(G)}(a, a') \leq k+r$, and $\operatorname{diam}(G) \geq r$. Thus, $\operatorname{diam}(H) \leq \operatorname{diam}(G) + k \leq \operatorname{diam}(G) + k^2 - 2$, since $k \geq 2$.

(ii) Suppose that there is a walk $\mathcal{V}$ of length r from an endvertex of A to a vertex of A', but there is no walk of type (i) in G (we remark, that the case when there is a walk $\mathcal{V}$ of length r from a vertex of A to an endvertex of A' can be solved analogously). Then $d_{P_k(G)}(a, a') \leq \frac{k(k-1)}{2} + k + r$, and $\operatorname{diam}(G) \geq r + 1$. Thus, $\operatorname{diam}(H) \leq \operatorname{diam}(G) + \frac{k(k-1)}{2} + k - 1 \leq \operatorname{diam}(G) + k^2 - 2$, since $k \geq 2$.

(iii) Suppose that there is a walk $\mathcal{V}$ of length r from a vertex of A to a vertex of A', but there are no walks of type (i) or (ii) in G. If there is a walk $\mathcal{V}'$ of length $r{+}1$ from an endvertex of A to an endvertex of A', then $\mathcal{V}'$ contains only one vertex from A and only one vertex from A'. Hence, $d_{P_k(G)}(a, a') \leq k + (r+1)$ and $\operatorname{diam}(G) \geq r+1$, so that $\operatorname{diam}(H) \leq \operatorname{diam}(G) + k \leq \operatorname{diam}(G) + k^2 - 2$, since $k \geq 2$. Thus, suppose that $\operatorname{diam}(G) \geq r+2$. As $d_{P_k(G)}(a, a') \leq k(k-1) + k + r$, we have $\operatorname{diam}(H) \leq \operatorname{diam}(G) + k^2 - 2$.

(b) $r \leq k-2$. Let w_0 and w_1 be vertices in G such that $d_G(w_0, w_1) = \operatorname{diam}(G)$. Assume that $d_G(w_0, V(A))$ is the shortest distance from $d_G(w_0, V(A))$, $d_G(w_0, V(A'))$, $d_G(w_1, V(A))$, and $d_G(w_1, V(A'))$ (the other cases can be proved analogously). Since

$$\begin{aligned} &2 \cdot d_G(w_1, V(A')) + (3k-2) \\ &\quad \geq d_G(w_0, V(A)) + k + d_G(V(A), V(A')) + k + d_G(V(A'), w_1) \\ &\quad \geq d_G(w_0, w_1) = \operatorname{diam}(G), \end{aligned}$$

we have

$$\begin{aligned} d_G(w_1, V(A')) \geq \frac{\operatorname{diam}(G) - (3k-2)}{2} & \frac{(\frac{1}{2}k^2 + 5k - 2) - (3k-2)}{2} \\ &= \frac{1}{4}k^2 + k \geq 2k - 1, \end{aligned}$$

and $d_G(w_1, V(A)) \geq 2k-1$ as well. Assume that $d_G(w_1, V(A')) \leq d_G(w_1, V(A))$, and denote $\mathcal{V} = (u_0, u_1, \ldots, u_s = w_1)$ a shortest walk beginning in a vertex from A (i.e., $d_G(V(A'), w_1) = s$). Denote $A^* = (u_{k-1}, u_k, \ldots, u_{2k-1})$. Then $d_G(V(A'), V(A^*)) = k-1$ and $d_G(V(A), V(A^*)) \geq k{-}1$.

Suppose that there is a vertex b' in $P_k(G)$, such that one endvertex of B' is u_0 and $d_{P_k(G)}(a', b') < \infty$. As $d_G(V(A'), V(A^*)) = k-1$, analogously as in the case (a) can be shown

$$d_{P_k(G)}(a', a^*) \leq \frac{k(k-1)}{2} + ((k{-}1) + k) = \frac{1}{2}k^2 + \frac{3}{2}k - 1.$$

Since $d_{P_k(G)}(a', a^*) < \infty$, we have $d_{P_k(G)}(a, a^*) < \infty$. As $d_G(V(A), V(A^*)) \leq d_G(V(A), V(A')) + k + d_G(V(A'), V(A^*)) \leq (k{-}2) + k + (k{-}1) = 3k-3$, we have

$$d_{P_k(G)}(a, a^*) \leq \frac{k(k-1)}{2} + ((3k{-}3) + k) + \frac{k(k-1)}{2} = k^2 + 3k - 3,$$

and hence

$$d_{P_k(G)}(a, a') \leq d_{P_k(G)}(a, a^*) + d_{P_k(G)}(a^*, a') \leq \frac{3}{2}k^2 + \frac{9}{2}k - 4.$$

Thus,

$$d_{P_k(G)}(a, a') \leq \operatorname{diam}(G) + k^2 - \frac{1}{2}k - 2 < \operatorname{diam}(G) + k^2 - 2,$$

since $\operatorname{diam}(G) \geq \frac{1}{2}k^2 + 5k - 2$.

Now suppose that there is not b' in $P_k(G)$ such that one endvertex of B' is u_0 and $d_{P_k(G)}(a', b') < \infty$, and denote this fact by ($\triangle$). Since $d_{P_k(G)}(a, a') < \infty$, A and A' share two incident edges in G. If $k = 2$, then $d_{P_k(G)}(a, a') = 0$; and if $k = 3$, then $d_{P_k(G)}(a, a') \leq 2$, by ($\triangle$). Thus, suppose that $k = 4$. Then $\operatorname{diam}(G) \geq \frac{1}{2}k^2 + 5k - 2 = 26$ and $\operatorname{diam}(G) + k^2 - 2 \geq 40$. Assume that $\operatorname{diam}(H) \geq 41$. Since $\mathcal{T}$ is a shortest walk from a to a' in $P_4(G)$, there are vertices a_i and $a_{i'}$ in $\mathcal{T}$, such that $A_i = (y_0, x_1, x_2, x_3, y_4)$ and $A_{i'} = (y_0', x_1, x_2, x_3, y_4')$, $x_2 = u_0$, $0 \leq i \leq 2$ and $l{-}2 \leq i' \leq l$, see Figure 11. Clearly, $y_0 \neq y_0'$ and $y_4 \neq y_4'$. By Lemma 3, there is a vertex c in $P_4(G)$ such that one endvertex of C is x_3 and $d_{P_4(G)}(a_i, c) \leq 3$. Analogously, there is a vertex c' in $P_4(G)$ such that one endvertex of C' is x_1 and $d_{P_4(G)}(c', a_{i'}) \leq 3$. If $C \cup C'$ does not contain a cycle of length four, then $l = d_{P_4(G)}(a, a') \leq d_{P_4(G)}(a, a_i) + d_{P_4(G)}(a_i, c) + d_{P_4(G)}(c, c') + d_{P_4(G)}(c', a_{i'}) + d_{P_4(G)}(a_{i'}, a') \leq 2 + 3 + 2 + 3 + 2 = 12$.

Thus, suppose that there is a cycle of length four in $C \cup C'$, and denote the vertices as indicated in Figure 12. By ($\triangle$), x_5 is not adjacent to any vertex from $V(G) - \{x_1, x_2, x_4\}$, and x_6 is not adjacent to any vertex from $V - \{x_2, x_3, x_4\}$. By ($*$) in Lemma 3, there is a vertex d in $P_4(G)$ such that one endvertex of D is x_1 and $d_{P_4(G)}(c, d) \leq 2$. Analogously, there is a vertex d' in $P_4(G)$ such that one

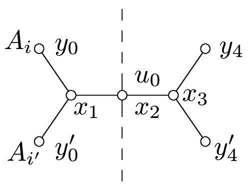

Figure 11.

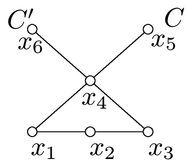

Figure 12.

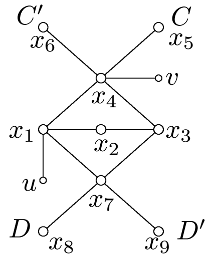

Figure 13.

endvertex of D' is x_3 and $d_{P_4(G)}(d', c') \leq 2$. If $D \cup D'$ does not contain a cycle of length four, then $d_{P_4(G)}(a, a') \leq 2 + 3 + 2 + 2 + 2 + 3 + 2 = 16$.

Thus, suppose that there is a cycle of length four in $D \cup D'$, see Figure 13. Note that not all vertices $x_1, x_2, \ldots, x_9$ are necessarily distinct. Denote $X = \{x_1, x_2, \ldots, x_9\}$ and consider the vertices from $V^* = V(G) - X$. By $(\triangle)$, no vertex from V^* is adjacent to x_5, x_6, x_8 or x_9. Moreover, if $v \in V^*$ and v is adjacent to x_4 (or to x_7), then v may be adjacent only to x_2 and x_4 (only to x_2 and x_7), by $(\triangle)$. Finally, if $u \in V^*$ and u is adjacent to x_1 (or to x_3), then u may be adjacent only to x_1, x_2 and x_3, as otherwise $d_{P_4(G)}(d, c') \leq 4$ (or $d_{P_4(G)}(c, d') \leq 4$), so that $d_{P_4(G)}(c, c') \leq 6$ and $d_{P_4(G)}(a, a') \leq 16$. But then $d_{P_4(G)}(c, c') = \infty$, and hence $d_{P_4(G)}(a, a') = \infty$, a contradiction. □

In Corollary 7 we prove that the bound in Theorem 6 is best possible. Moreover, we show that the diameter of a component of $P_k(G)$ can achieve all values from the range bounded by Theorem 2 and Theorem 6, if $2 \leq k \leq 4$.

Corollary 7. *Let $r \geq 2k$ and $-k \leq s \leq k^2 - 2$. Then there is a graph $G^*_{r,s}$ with diameter r such that for one component H of $P_k(G^*_{r,s})$ we have*

$$\operatorname{diam}(H) = \operatorname{diam}(G^*_{r,s}) + s.$$

Proof. First we construct a graph G^*_r with diameter r. Let G be a graph constructed from a collection $A_0, A_1, \ldots, A_{k-1}$ of paths and the path $A = (v_0, v_2, \ldots, v_k, \ldots, v_3, v_1)$ in Corollary 5. Let a graph G^* be obtained from G by joining every vertex of G with v_k, see Figure 14 for the case $k = 4$, and let $G^{*\prime}$ be a copy of G^*. Denote by G^*_r a graph consisting from G^*, $G^{*\prime}$, and a path $\mathcal{W}$ of length $r-2$ joining v_k with v'_k. Then the diameter of G^*_r equals r.

Let b be a vertex of $P_k(G^*)$ at the shortest distance from a, such that v_k is an endvertex of B. By $(*)$ in Lemma 3, the edge of B incident to v_k lies in A, and hence, B consists from the edge $v_k v_{k-1}$ and A_{k-1}. In Corollary 5 we showed that $d_{P_k(G)}(a, b) = \frac{k(k-1)}{2}$. Since v_k is an interior vertex of every C such that c is a vertex of $P_k(G)$, $c \neq b$, the shortest $a-b$ walk cannot be shorten in $P_k(G^*)$. Thus, $d_{P_k(G^*)}(a,b) = \frac{k(k-1)}{2}$, and hence $d_{P_k(G^*_r)}(a,a') = \frac{k(k-1)}{2} + ((r-2)+k) + \frac{k(k-1)}{2} = r+k^2-2$, $A' = (v'_0, v'_2, \dots, v'_k, \dots, v'_3, v'_1)$. Hence, if we denote by H the component of G^*_r containing a, then diam$(H) =$ diam$(G^*_r) + k^2 - 2$.

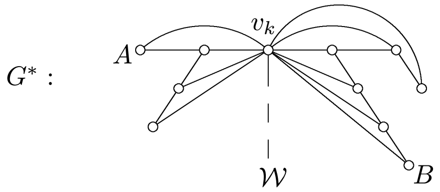

Figure 14.

Now remove from G^*_{r-i-j} i edges joining the last i vertices of B with v_k, and j edges joining the last j vertices of B' with v'_k, and denote the resulting graph by $G^*_{r,k^2-i-j-2}$, $0 \le i \le k-1$ and $0 \le j \le k-1$. Denote $s = k^2 - i - j - 2$. Then $k(k-2) \le s \le k^2-2$ and diam$(G^*_{r,s}) = r$. Moreover, $d_H(a,a') = (r-i-j)+k^2-2$, and hence, diam$(H) =$ diam$(G^*_{r,s})+s$ (recall that H is the component of $P_k(G^*_{r,s})$ containing a). Combining this with Corollary 5 we obtain the result. $\square$

We remark, that using more gentle techniques, the bound on the diameter of G can be decreased in Theorem 6. In fact, for $k = 2$ the statement of Theorem 6 is valid for all graphs, see [**2**, Theorem 6]. However, for $k \ge 3$ some bound is necessary, as shown by graph G pictured in Figure 15. If $A = (v_2, v_3, v_1, v_9)$ and $A' = (v_5, v_4, v_1, v_6)$, then $d_{P_3(G)}(a,a') = 10 > 2 + 7 =$ diam$(G) + k^2 - 2$.

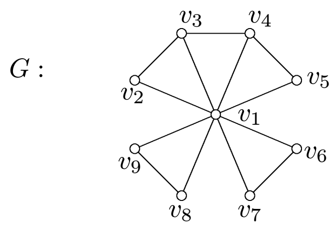

Figure 15.

The problem of bounding the diameter of a component of $P_k(G)$ if $k \ge 5$ remains open.

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