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ON CERTAIN FORMULAS FOR THE MULTIVARIABLE HYPERGEOMETRIC FUNCTIONS

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Abstract. We present relatively simple and direct proofs of the integral representations established recently in [**7**]. An algorithm is then furnished and applied to obtain new classes of integral formulas for the multivariable hypergeometric functions, thereby, providing generalizations to the results of [**7**]. Also, an operational formula involving fractional calculus operators for an analytic function is derived and its usefulness illustrated by considering some examples.

1. Preliminaries and Definitions

The multivariable generalized Lauricella function due to Srivastava and Daoust [**8**, p. 454] is a generalization of the Wright function ${}_p\Psi_q$ in several variables, and is defined by ([**1**, p. 107]),

$$(1)\qquad S^{p\,:\,p_1;\dots;p_n}_{q\,:\,q_1;\dots;q_n}\left[\begin{matrix}[(a_p)\colon(\alpha^1_p),\dots,(\alpha^n_p)]\colon[(c^1_{p_1})\colon(\gamma^1_{p_1})];\dots;[(c^n_{p_n})\colon(\gamma^n_{p_n})];\\ [(b_p)\colon(\beta^1_q),\dots,(\beta^n_q)]\colon[(d^1_{q_1})\colon(\delta^1_{q_1})];\dots;[(d^n_{q_n})\colon(\delta^n_{q_n})];\end{matrix}\right.$$
$$\left. z_1,\dots,z_n\right] = \sum_{m_1,\dots,m_n=0}^{\infty} A(m_1,\dots,m_n)\prod_{j=1}^{n}\frac{z_j^{m_j}}{m_j!}\,,$$

where

$$(2)\qquad A(m_1,\dots,m_n)=\frac{\prod\limits_{j=1}^{p}\Gamma\Big(a_j+\sum\limits_{k=1}^{n}m_k\alpha_j^k\Big)}{\prod\limits_{j=1}^{q}\Gamma\Big(b_j+\sum\limits_{k=1}^{n}m_k\beta_j^k\Big)}\prod_{k=1}^{n}\left\{\frac{\prod\limits_{j=1}^{p_k}\Gamma(c_j^k+m_k\gamma_j^k)}{\prod\limits_{j=1}^{q_k}\Gamma(d_j^k+m_k\delta_j^k)}\right\}.$$

The coefficients α_j^k $(j=1,\dots,p)$, β_j^k $(j=1,\dots,q)$, γ_j^k $(j=1,\dots,p_k)$ and δ_j^k $(j=1,\dots,q_k)$, $\forall\, k=1,\dots,n$, are real and positive, and (a_p) means the array of p-parameters $a_1,\dots,a_p$; with similar interpretations for (b_q), $(\gamma^1_{p_1})$, (α^1_p),

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etc., and $(a)_n = \Gamma(a+n)/\Gamma(a)$ denotes the usual Pochhammer symbol. For the precise conditions under which the multiple series (1) converges absolutely, see [**9**, pp. 157–158].

The generalized hypergeometric function is defined by

$$ {}_pF_q\left[\begin{matrix}(a)_p; \\ (b_q);\end{matrix}\ z\right] = \sum_{m=1}^{\infty} \frac{\prod_{j=1}^{p}(a_j)_m}{\prod_{j=1}^{q}(b_j)_m}\,\frac{z^m}{m!}, \tag{3}$$

for $p \le q+1$ (cf. [**10**, p. 42]), and its generalization known as the Wright's hypergeometric function ${}_p\Psi_g$ [**10**, p. 50] is defined by

$$ {}_p\Psi_q\left[\begin{matrix}(a_1,\alpha_1),\dots,(a_p,\alpha_p); \\ (b_1,\beta_1),\dots,(b_q,\beta_q);\end{matrix}\ z\right] = \sum_{m=0}^{\infty} \frac{\prod_{j=1}^{p}\Gamma(a_j+\alpha_j m)}{\prod_{j=1}^{q}\Gamma(b_j+\beta_j m)}\,\frac{z^m}{m!}, \tag{4}$$

where $1+\sum_{j=1}^{q}\beta_j - \sum_{j=1}^{p}\alpha_j \ge 0$, α_j $(j=1,\dots,p)$ and β_j $(j=1,\dots,q)$ are positive real numbers.

The Fox's H-function is defined by

$$ H^{m,n}_{p,q}[z] = H^{m,n}_{p,q}\left[z \,\middle|\, \begin{matrix}\{a_p,\alpha_p\} \\ \{b_q,\beta_q\}\end{matrix}\right] = (2\pi i)^{-1}\int_L \theta(s) z^{-s}\,ds, \tag{5}$$

where

$$ \theta(s) = \frac{\prod_{j=1}^{m}\Gamma(b_j+\beta_j s)\prod_{j=1}^{n}\Gamma(1-a_j-\alpha_j s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j-\beta_j s)\prod_{j=n+1}^{p}\Gamma(a_j+\alpha_j s)}, \tag{6}$$

where $\{a_p,\alpha_p\}$ abbreviates the p-parameters $(a_1,\alpha_1),\dots,(a_p,\alpha_p)$. We refer to [**3**, p. 626] (see also [**10**, p. 49]) for the details regarding the type of the contour L, and the conditions of existence of the H-function. If $\alpha_j = 1$ $(j=1,\dots,p)$ and $\beta_j = 1$ $(j=1,\dots,q)$ in (5), then we have the relation

$$ H^{m,n}_{p,q}\left[z \,\middle|\, \begin{matrix}\{a_p,1\} \\ \{b_q,1\}\end{matrix}\right] = G^{m,n}_{p,q}(z), $$

where the G-function is the familiar Meijer's G-function ([**3**, p. 617]).

2. Introduction

In their paper, Saigo and Tuan [**7**] established two integral representations for the generalized Kampé de Fériet function (a particular case of (1)) given by

$$F^{p:\,p_1;\dots;p_n}_{q:\,q_1;\dots;q_n}\left[\begin{matrix}(a_p)\colon (c^1_{p_1});\dots;(c^n_{p_n});\\ (b_q)\colon (d^1_{q_1});\dots;(d^n_{q_n});\end{matrix}\; x_1,\dots,x_n\right] \tag{7}$$

$$= \prod_{k=1}^{n}\left\{\frac{\prod_{j=1}^{q_k}\Gamma(d^k_j)}{\prod_{j=1}^{p_k}\Gamma(c^k_j)}\right\}\int_0^\infty\cdots\int_0^\infty {}_pF_q\left[\begin{matrix}(a_p);\\ (b_q);\end{matrix}\; x_1t_1+\cdots+x_nt_n\right]$$

$$\times\prod_{k=1}^{n}\left\{G^{\,p_k,0}_{\,q_k,p_k}\left(t_k\,\middle|\,\begin{matrix}(d^k_{q_k})\\ (c^k_{p_k})\end{matrix}\right)\right\}\frac{dt_1\dots dt_n}{t_1\dots t_n},$$

where $p \le q+1$, $p_k \ge q_k$ $(k=1,\dots,n)$; and the one-dimensional representation by

$$F^{p:\,p_1;\dots;p_n}_{q:\,q_1;\dots;q_n}\left[\begin{matrix}(a_p):(c^1_{p_1});\dots;(c^n_{p_n});\\ (b_q):(d^1_{q_1});\dots;(d^n_{q_n});\end{matrix}\; x_1,\dots,x_n\right] \tag{8}$$

$$= \frac{\prod_{j=1}^{q}\Gamma(b_j)}{\prod_{j=1}^{p}\Gamma(a_j)}\int_0^\infty G^{\,p,0}_{\,q,p}\left(t\,\middle|\,\begin{matrix}(b_q)\\ (a_p)\end{matrix}\right)\prod_{k=1}^{n}\left\{{}_{p_k}F_{q_k}\left[\begin{matrix}(c^k_{p_k});\\ (d^k_{q_k});\end{matrix}\; x_kt\right]\right\}\frac{dt}{t},$$

where $p \ge q$, $p_k \le q_k+1$ $(k=1,\dots,n)$.

The formulas (7) and (8) are derived in a rather longish manner by reverting to the analysis of Mellin transform (and its inverse) and invoking the Parseval theorem in the process.

This paper has two parts. First we derive direct (alternative) proofs of (7) and (8), and then furnish a simple straightforward algorithm which is applied in deriving more general classes of integral formulas than (7) and (8) for the multivariable hypergeometric functions. The second part of this paper gives an operational formula (eqn. (22) below) involving the fractional calculus operator of Saigo (see, e.g., [**5**] and [**6**]) for an analytic function, and some examples are deduced illustrating the applications.

3. Direct Proofs of (7) and (8)

Expanding the ${}_pF_q$ function on the right side of (7) in terms of the defining series (3), using the elementary identity [**10**, p. 52])

$$\sum_{m_1,\dots,m_n=0}^{\infty}\phi(m_1+\cdots+m_n)\frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!} = \sum_{m=0}^{\infty}\frac{\phi(m)}{m!}(x_1+\cdots+x_n)^m, \tag{9}$$

and interchanging the order of summation and integration (formally), we have

$$\text{R.H.S. of (7)} = \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{\prod_{j=1}^{p} (a_j)_{\sum_{k=1}^{n} m_k}}{\prod_{j=1}^{q} (b_j)_{\sum_{k=1}^{n} m_k}} \prod_{k=1}^{n} \left\{ \frac{x_k^{m_k}}{m_k!} \right\}$$

$$\times \int_0^\infty \cdots \int_0^\infty \prod_{k=1}^{n} \left\{ t_k^{m_k-1}\, G\,{}^{p_k,0}_{q_k,p_k} \left(t_k \,\middle|\, \begin{matrix} (d^k_{q_k}) \\ (c^k_{p_k}) \end{matrix} \right) \right\} dt_1 \ldots dt_n \,. \tag{10}$$

Appealing to the Mellin transforms of the Meijer's G-function [**3**, p. 728, eqn. (9)], we are lead to the formula (7) as a consequence of the definition (1).

Similarly, for proving (8) directly, we expand each function ${}_{p_k}F_{q_k}$ $(k = 1, \ldots, n)$ on the right side, invert the order of summation and integration, and apply the result [**3**, p. 728, eqn. (9)] to arrive at the result (8).

4. Generalizations of (7) and (8)

With a view to demonstrating the algorithm used in our derivation of the generalizations of the integral representations (7) and (8), we first define a multivariable function.

Suppose a function $f(z_1, \ldots, z_n)$ is analytic in the domain $D = D_1 \times D_2 \times \ldots \times D_n$ $(z_i \in D_i,\ i = 1, \ldots, n)$ possessing the power series expansion

$$f(z_1, \ldots, z_n) = \sum_{m_1,\ldots,m_n=0}^{\infty} C(m_1, \ldots, m_n) z_1^{m_1} \ldots z_n^{m_n}, \tag{11}$$

where $|z_i| < R_i$ $(R_i > 0, i \in \{1, \ldots, n\})$, and $C(m_1, \ldots, m_n)$ is a bounded sequence of real (or complex) numbers.

Let us replace z_i by $t_i x_i$ $(i = 1, \ldots, n)$ in (11), multiply the equation so obtained both sides by

$$\prod_{k=1}^{n} \left\{ t_k^{-1} H\,{}^{p_k,0}_{q_k,p_k} \left[t_k \,\middle|\, \begin{matrix} \{d^k_{q_k}, \delta^k_{q_k}\} \\ \{c^k_{p_k}, \gamma^k_{p_k}\} \end{matrix} \right] dt_k \right\}.$$

Then the repeated (n-fold) integration of the resulting equation between the limits 0 to ∞, and use of the Mellin transform of the H-function [**3**, p. 729, eqn. (11)] (with the assumption of the change in the order of summation and integrations) readily yields the following assertion:

$$\int_0^\infty \cdots \int_0^\infty \prod_{k=1}^{n} H\,{}^{p_k,0}_{q_k,p_k} \left[t_k \,\middle|\, \begin{matrix} \{d^k_{q_k}, \delta^k_{q_k}\} \\ \{c^k_{p_k}, \gamma^k_{p_k}\} \end{matrix} \right] f(t_1 x_1, \ldots, t_n x_n) \frac{dt_1 \ldots dt_n}{t_1 \ldots t_n} \tag{12}$$

$$= \sum_{m_1,\ldots,m_n=0}^{\infty} C(m_1, \ldots, m_n) \prod_{k=1}^{n} \left\{ \frac{\prod_{j=1}^{p_k} \Gamma(c_j^k + \gamma_j^k m_k)}{\prod_{j=1}^{q_k} \Gamma(d_j^k + \delta_j^k m_k)} x_k^{m_k} \right\},$$

where $p_k \geq q_k$, $\mathrm{Re}\,(c_j^k) > 0$, $\gamma_j^k > 0$ $(j = 1, \dots, p_k)$, and $\delta_j^k > 0$ $(j = 1, \dots, q_k)$, $\forall\, k = 1, \dots, n$; such that both the members of (12) exist.

Proceeding with the same steps as indicated above, we would also be led to the following result:

$$\tag{13} \int_0^\infty H_{q,p}^{p,0}\left[t \,\middle|\, \begin{matrix}\{b_q, \beta_q\}\\ \{a_p, \alpha_p\}\end{matrix}\right] f(x_1 t^{h_1}, \dots, x_n t^{h_n})\,\frac{dt}{t}$$
$$= \sum_{m_1,\dots,m_n=0}^{\infty} C(m_1, \dots, m_n) \frac{\prod_{j=1}^{p} \Gamma(a_j + \alpha_j \sum_{k=1}^n h_k m_k)}{\prod_{j=1}^{q} \Gamma(b_j + \beta_j \sum_{k=1}^n h_k m_k)} \prod_{k=1}^{n} x_k^{m_k},$$

where $p \geq q$, $\mathrm{Re}\,(h_j) > 0$ $(j = 1, \dots, n)$, $\mathrm{Re}\,(a_j) > 0$, $\alpha_j > 0$ $(j = 1, \dots, p)$, and $\beta_j > 0$ $(j = 1, \dots, q)$ such that both sides of (13) exist.

If we set

$$\tag{14} C(m_1, \dots, m_n) = \frac{\prod_{j=1}^{p} \Gamma(a_j + \sum_{k=1}^n m_k)}{\prod_{j=1}^{q} \Gamma(b_j + \sum_{k=1}^n m_k) \prod_{j=1}^{n} (m_j)!},$$

in (12), then in view of definition (1), and identity (9), we get

$$\tag{15} S_{q\,:\,q_1;\dots;q_n}^{p\,:\,p_1;\dots;p_n}\left[\begin{matrix}[(a_p){:}1, \dots, 1] : [(c_{p_1}^1), \gamma_{p_1}^1)]; \dots; [(c_{p_n}^n), (\gamma_{p_n}^n)];\\ [(b_q){:}1, \dots, 1] : [(d_{q_1}^1), \delta_{q_1}^1)]; \dots; [(d_{q_n}^n), (\delta_{q_n}^n)];\end{matrix}\; x_1, \dots, x_n\right]$$
$$= \int_0^\infty \cdots \int_0^\infty {}_pF_q\left[\begin{matrix}(a_p);\\ (b_q);\end{matrix}\; x_1 t_1 + \dots + x_n t_n\right]$$
$$\times \prod_{k=1}^{n} H_{q_k, p_k}^{p_k, 0}\left[t_k \,\middle|\, \begin{matrix}\{d_{q_k}^k, \delta_{q_k}^k\}\\ \{c_{p_k}^k, \gamma_{p_k}^k\}\end{matrix}\right] \frac{dt_1 \dots dt_n}{t_1 \dots t_n},$$

where $p \leq q + 1$, $p_k \geq q_k$, $\gamma_j^k > 0$ $(j = 1, \dots, p_k)$, $\delta_j^k > 0$ $(j = 1, \dots, q_k)$, $\forall\, k$ $1, \dots, n$.

Next, we put the sequence

$$\tag{16} C(m_1, \dots, m_n) = \prod_{k=1}^{n} \left\{ \frac{\prod_{j=1}^{p_k} \Gamma(c_j^k + \gamma_j^k m_k)}{m_k! \prod_{j=1}^{q_k} \Gamma(d_j^k + \delta_j^k m_k)} \right\},$$

$\alpha_j = 1$ $(j = 1, \dots, p)$, $\beta_j = 1$ $(j = 1, \dots, q)$ in (13), then we have by virtue of (1) the following result:

$$\tag{17} S_{q\,:\,q_1;\dots;q_n}^{p\,:\,p_1;\dots;p_n}\left[\begin{matrix}[(a_p){:}h_1, \dots, h_n] : [(c_{p_1}^1), \gamma_{p_1}^1)]; \dots; [(c_{p_n}^n), (\gamma_{p_n}^n)];\\ [(b_q){:}h_1, \dots, h_n] : [(d_{q_1}^1), \delta_{q_1}^1)]; \dots; [(d_{q_n}^n), (\delta_{q_n}^n)];\end{matrix}\; x_1, \dots, x_n\right]$$
$$= \int_0^\infty G_{q,p}^{p,0}\left(t \,\middle|\, \begin{matrix}(b_q)\\ (a_p)\end{matrix}\right) \prod_{k=1}^{n} \left\{ {}_{p_k}\Psi_{q_k}\left[\begin{matrix}(c_1^k, \gamma_1^k), \dots, (c_{p_k}^k, \gamma_{p_k}^k);\\ (d_1^k, \delta_1^k), \dots, (d_{q_k}^k, \delta_{q_k}^k);\end{matrix}\; x_k t^{h_k}\right]\right\} \frac{dt}{t},$$

where $h_i > 0$ $(i = 1, \ldots, n)$, $\gamma_j^k > 0$ $(j = 1, \ldots, p_k)$, $\delta_j^k > 0$ $(j = 1, \ldots, q_k)$, $1 + \sum_{j=1}^{q_k} \delta_j^k - \sum_{j=1}^{p_k} \gamma_j^k \geq 0, \forall\, k \in \{1, \ldots, n\}$.

It may be noted that the integral representations (7) and (8) are recoverable from our formulas (15) and (17), respectively, in the special case when $\gamma_j^k = 1$ $(j = 1, \ldots, p_k)$, $\delta_j^k = 1$ $(j = 1, \ldots, q_k)$, and $h_i = 1$ $(i = 1, \ldots, n)$.

5. Operational Formulas

In this section we establish an operational formula involving Saigo's fractional calculus operator $I_{0,x}^{\alpha,\beta,\eta}$ which is defined by (see [**5**, p. 15] and [**6**, p. 53])

$$I_{0,x}^{\alpha,\beta,\eta} f(x) = \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} F\Big(\alpha+\beta, -\eta; \alpha; 1 - \frac{t}{x}\Big) f(t)\, dt, \tag{18}$$

where $\mathrm{Re}\,(\alpha) > 0$, β and η are complex numbers, the F-function is the Gauss's function which is a special case of (3).

If $\mathrm{Re}\,(\alpha) \leq 0$, then

$$I_{0,x}^{\alpha,\beta,\eta} f(x) = \frac{d^n}{dx^n} I_{0,x}^{\alpha+n,\beta-n,\eta-n} f(x), \tag{19}$$

provided that n is a positive integer such that

$$-\mathrm{Re}\,(\alpha) < n \leq -\mathrm{Re}\,(\alpha) + 1\,.$$

Two special cases of (18) emerge, giving the Riemann-Liouville (R-L) and Erdélyi-Kober (E-K) fractional calculus operators. Indeed, for $\beta = -\alpha$, (18) gives the R-L operator

$$R_{0,x}^{\alpha} f(x) = I_{0,x}^{\alpha,-\alpha,\eta} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)\, dt, \tag{20}$$

and, for $\beta = 0$, (18) yields the E-K operator

$$E_{0,x}^{\alpha,\eta} f(x) = I_{0,x}^{\alpha,0,\eta} f(x) = \frac{x^{-\alpha-\eta}}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} t^{\eta} f(t)\, dt. \tag{21}$$

For an analytic function $f(z_1, \ldots, z_n)$ defined by (11), we have the following operational formula involving the fractional calculus operator (18) for a real variable x and complex variables $z_1, \ldots, z_n$:

Theorem. *Corresponding to the sequence $C(m_1, \ldots, m_n)$, let the function $f(z_1, \ldots, z_n)$ be defined by* (11), *then*

$$T\{f(xz_1, \ldots, xz_n)\} = x^{\beta_p - 1} \sum_{m_1,\ldots,m_n=0}^{\infty} C(m_1, \ldots, m_n) \times \prod_{j=1}^{p} \left\{ \frac{\Gamma(\alpha_j + M)\Gamma(\beta_j + \mu_j + M)}{\Gamma(\beta_j + M)\Gamma(\alpha_j + \lambda_j + \mu_j + M)} (xz_j)^{m_j} \right\}, \tag{22}$$

where $Re(\alpha_j) > 0$, $Re(\beta_j + \mu_j) > 0$ $(j = 1, \dots, p)$, $\max\{|xz_1|, \dots, |xz_n|\} < R$, T *is a chain of fractional calculus operators defined by*

$$T = I_{0,x}^{\lambda_p, \alpha_p - \beta_p, \mu_p} x^{\alpha_p - \beta_{p-1}} \dots I_{0,x}^{\lambda_2, \alpha_2 - \beta_2, \mu_2} x^{\alpha_2 - \beta_1} I_{0,x}^{\alpha_1, \alpha_1 - \beta_1, \mu_1} x^{\alpha_1 - 1}, \tag{23}$$

such that both sides of (22) *exist, and*

$$M = m_1 + \dots + m_n . \tag{24}$$

Proof. In view of defining equations (11) and (23), we have on replacing each z_i by xz_i $(i = 1, \dots, n)$:

$$\begin{aligned} T\{f(xz_1, \dots, xz_n)\} &= T\left\{ \sum_{m_1, \dots, m_n = 0}^{\infty} C(m_1, \dots, m_n) x^M \prod_{i=1}^{p} z_i^{m_i} \right\} \\ &= \sum_{m_1, \dots, m_n = 0}^{\infty} C(m_1, \dots, m_n) \prod_{i=1}^{p} z_i^{m_i} T\{x^M\}, \end{aligned} \tag{24a}$$

under of course the assumptions stated with (11), and with the above theorem, permitting the interchange in the order of the multiple summation and fractional differential operator $I_{0,x}^{\alpha,\beta,\eta}$; T and M being defined by (23) and (24), respectively.

Applying the known formula [**5**, p. 16, Lemma 1]:

$$I_{0,x}^{\alpha,\beta,\eta} x^{\lambda} = \frac{\Gamma(1+\lambda)\Gamma(1+\lambda-\beta+\eta)}{\Gamma(1+\lambda-\beta)\Gamma(1+\lambda+\alpha+\eta)} x^{\lambda-\beta}, \tag{25}$$

$\mathrm{Re}\,(\lambda) > \max[0, \mathrm{Re}\,(\beta - \eta)] - 1$, succesively p times on the right of (24a), we arrive at (22).

If $\lambda_j = \beta_j - \alpha_j$ $(j = 1, \dots, p)$, then in view of (20), the above theorem in terms of R-L operators gives

$$\begin{aligned} &R_{0,x}^{\beta_p - \alpha_p} x^{\alpha_p - \beta_{p-1}} \dots R_{0,x}^{\beta_2 - \alpha_2} x^{\alpha_2 - \beta_1} R_{0,x}^{\beta_1 - \alpha_1} x^{\alpha_1 - 1} \{f(xz_1, \dots, xz_n)\} \\ &\qquad = x^{\beta_p - 1} \sum_{m_1, \dots, m_n = 0}^{\infty} C(m_1, \dots, m_n) \prod_{i=1}^{p} \frac{\Gamma(\alpha_i + M)}{\Gamma(\beta_i + M)} (xz_i)^{m_i}, \end{aligned} \tag{26}$$

where $\mathrm{Re}\,(\alpha_i) > 0$ $(i = 1, \dots, p)$, and M is given by (24).

On the other hand, if $\beta_i = \alpha_i$ $(i = 1, \dots, p)$ in (22), then using (21), we get

$$\begin{aligned} &E_{0,x}^{\lambda_p, \mu_p} x^{\alpha_p - \alpha_{p-1}} \dots E_{0,x}^{\lambda_2, \mu_2} x^{\alpha_2 - \alpha_1} E_{0,x}^{\lambda_1, \mu_1} x^{\alpha_1 - 1} \{f(xz_1, \dots, xz_n)\} \\ &\quad = x^{\alpha_p - 1} \sum_{m_1, \dots, m_n = 0}^{\infty} C(m_1, \dots, m_n) \\ &\qquad \times \prod_{j=1}^{p} \left\{ \frac{\Gamma(\alpha_j + \mu_j + M)}{\Gamma(\alpha_j + \lambda_j + \mu_j + M)} (xz_j)^{m_j} \right\}, \end{aligned} \tag{27}$$

where $\mathrm{Re}\,(\alpha_j + \mu_j) > 0$ $(j = 1, \dots, p)$, and M is given by (24).

By letting $z_i \to 0$ $(i = 2, \dots, n)$ in (26), and putting

$$C(m_1, 0, \dots, 0) = (-m)_{m_1}/m_1! \qquad (m \text{ is a positive integer}),$$

so that

$$f(xz_1, 0, \dots, 0) = (1 - xz_1)^m ,$$

we receive the formula of Misra [**2**]. It may also be observed that when $p = 1$, then (26) would evidently correspond to the result due to Raina [**4**, p. 185, Corollary 1].

Lastly, we consider deducing certain examples illustrating the usefulness of the operational formula (22).

Example 1. Put

$$C(m_1, \dots, m_n) = \prod_{j=1}^{n} \left\{ \frac{(\gamma_j)}{(m_j)!} \right\}, \tag{28}$$

in (22), so that

$$f(xz_1, \dots, xz_n) = \prod_{j=1}^{n} (1 - xz_j)^{-\gamma_j}, \tag{29}$$

then in terms of the generalized Kampé de Fériet function, (22) gives

$$T\left\{ \prod_{j=1}^{n} (1 - xz_j)^{-\gamma_j} \right\} = \Omega x^{\beta_p - 1} F^{\,2p:1;\dots,1}_{\,2p:0;\dots;0} \left[\begin{matrix} (\alpha_p), (\beta_p + \mu_p) : \gamma_1; \dots; \gamma_n; \\ (\beta_p), (\alpha_p + \lambda_p + \mu_p) : \text{—}; \dots; \text{—}; \end{matrix} \; xz_1, \dots, xz_n \right], \tag{30}$$

where

$$\Omega = \prod_{j=1}^{p} \left\{ \frac{\Gamma(\alpha_j)\Gamma(\beta_j + \mu_j)}{\gamma(\beta_j)\Gamma(\alpha_j + \lambda_j + \mu_j)} \right\}. \tag{31}$$

The formula of Saigo and Raina [**6**, p. 56, eqn. (2.5)] is at once obtainable from (30) when $p = 1$.

Example 2. Let us set

$$C(m_1, \dots, m_n) = (\alpha_n)_M \prod_{j=1}^{n} \left\{ \frac{(\gamma_j)_{m_j}}{(\rho_j)_{m_j} (m_j)!} \right\}, \tag{32}$$

in (22), where m is defined by (24), we get the following operational formula for the Lauricella function $F_A^{(n)}$:

(33)
$$T\{F_A^{(n)}[\alpha,\gamma_1,\ldots,\gamma_n;\mu_1,\ldots,\mu_n;xz_1,\ldots,xz_n]\} = \Omega x^{\beta_p-1} F^{2p+1:\,1;\ldots;1}_{2p\quad\;:\,1;\ldots;1} \left[\begin{matrix}(\alpha_p),(\beta_p+\mu_p),\alpha\quad\;:\gamma_1;\ldots;\gamma_n;\\(\beta_p),(\alpha_p+\lambda_p+\mu_p):\rho_n;\ldots;\rho_n;\end{matrix}\; xz_1,\ldots,xz_n\right],$$

where $\operatorname{Re}(\alpha_j)>0$, $\operatorname{Re}(\beta_j+\mu_j)>0$ $(j=1,\ldots,p)$, $|xz_1+\cdots+xz_n|<1$, and Ω is given by (31).

Example 3. If we set the sequence

$$C(m_1,\ldots,m_n)=\frac{(\alpha)_M}{(\mu)_M}\prod_{j=1}^{n}\left\{\frac{(\sigma_j)_{m_j}}{(m_j)!}\right\}, \tag{34}$$

where, as before, M is given by (24), then (22) yields the following operational formula involving Lauricella function $F_D^{(n)}$ and the Kampé de Fériet function of n variables:

$$T\{F_D^{(n)}[\alpha,\sigma_1,\ldots,\sigma_n;\mu;xz_1,\ldots,xz_n]\} = \Omega x^{\beta_p-1} F^{2p+1:1;\ldots;1}_{2p+1:0;\ldots;0} \left[\begin{matrix}(\alpha_p),(\beta_p+\mu_p),\alpha\quad\;:\sigma_1;\ldots;\sigma_n;\\(\beta_p),(\alpha_p+\lambda_p+\mu_p),\mu:—;\ldots;—;\end{matrix}\; xz_1,\ldots,xz_n\right], \tag{35}$$

where $\operatorname{Re}(\alpha_j)>0$, $\operatorname{Re}(\beta_j+\mu_j)>0$ $(j=1,\ldots,p)$, $\max\{|xz_i|\}<1$, for $i=1,\ldots,n$, and Ω is given by (31).

On replacing z_k by z_k/α, σ_k by $-r_k$, for all $k=1,\ldots,n$ in (35), and letting $|\alpha|\to\infty$, we are led to the operational formula for the generalized Laguerre polynomials of several variables ([**10**, p. 464]),

$$T\{L^{(\mu)}_{r_1,\ldots,r_n}(xz_1,\ldots,xz_n)\} = \Omega x^{\beta_p-1}\frac{(1+\mu)_{r_1+\cdots+r_n}}{r_1!\ldots r_n!} F^{2p\quad\;:\,1;\ldots;1}_{2p+1:\,0;\ldots;0} \left[\begin{matrix}(\alpha_p),(\beta_p+\mu_p)\quad\;:-r_1;\ldots;-r_n;\\(\beta_p),(\alpha_p+\lambda_p+\mu_p),\mu+1:—;\;\ldots;—;\end{matrix}\; xz_1,\ldots,xz_n\right], \tag{36}$$

where Ω is given by (31).

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