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# GENERATING HAMILTONIAN CYCLES IN COMPLETE GRAPHS

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Abstract. We prove that hamiltonian cycles of complete graphs can be generated in a Gray code manner by means of small local interchanges.

## 1. Introduction

Let $C$ and $C'$ be two hamiltonian cycles in a (simple) graph $G$. We say that $C$ and $C'$ are **switching-equivalent** (symbolically, $C \sim C'$) if the symmetric difference of their edge sets induces a quadrangle in $G$, i.e., if $E(C) \triangle E(C') = \{a, b, c, d\}$ where $a, b, c, d$ are the consecutive edges of a cycle of length 4 in $G$. It is easy to see that $C \sim C'$ if and only if the cyclic sequences of vertices representing $C$ and $C'$ have the form $C = (u_1u_2v_1 \dots v_ku_3u_4w_1 \dots w_m)$, $C' = (u_1u_3v_k \dots v_1u_2u_4w_1 \dots w_m)$; in this case $E(C) \triangle E(C')$ is the edge set of the quadrangle $u_1u_2u_4u_3$ in $G$. Roughly speaking, $C'$ is then obtained from $C$ by "switching" the pairs of edges $u_1u_2$, $u_3u_4$ and $u_1u_3$, $u_2u_4$. If $k = 0$, i.e., if $C = (u_1u_2u_3u_4 \dots)$ and $C' = (u_1u_3u_2u_4 \dots)$, then we say that $C$ and $C'$ are **strongly switching-equivalent**.

We note that analogous concepts have been studied in operations research in connection with the travelling salesman problem. For example, the transformation used to define switching-equivalent hamiltonian cycles is the basic operation on travelling salesman tours called "2-opt" (see e.g. [**L**]). Also, the transformation for strong switching-equivalence (called "2-swap" in [**J**]) has been considered in local optimization of travelling salesman algorithms.

Let $G$ be a hamiltonian graph. We associate with $G$ two new graphs $H(G)$ and $H_s(G)$ as follows: The vertices of both $H(G)$ and $H_s(G)$ are the hamiltonian cycles of $G$; two vertices of $H(G)$ or $H_s(G)$ are adjacent if the corresponding hamiltonian cycles are switching equivalent or strongly switching-equivalent, respectively.

The idea of defining $H(G)$ and $H_s(G)$ is to express how "close" two structures (in our case, hamiltonian cycles) are, and how the switching operation can be used in generating all hamiltonian cycles of a graph. Similar situations are often encountered in the theory of generating combinatorial objects: the task is to

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generate all structures of a certain type in a Gray Code manner, that is, where successively generated objects are "close" in some sense (they are obtained from each other by a "small local perturbation"). The combinatorial structures which have so far been studied from this point of view include, for example, permutations [**KL, RuS**], spanning trees of a given graph [**C, S**], eulerian trails and eulerian orientations [**MP, ZX1**], 0-1 matrices [**BL**], perfect matchings [**ZX2**], polyhedra [**NP**], and linear extensions of posets [**Ru**]. Most of those results say that the associated "local perturbation" graph is edge hamiltonian. We are interested in the same problem for the "hamiltonian cycle graph" $H_s(G)$.

Closely related to our problem, in fact we were inspired by it, is a problem of Gary Meisters and Janusz Olech concerning knight tours on chessboard, which were considered already by Euler. In our terminology, a knight tour (knight cycle) is a hamiltonian path (hamiltonian cycle) of a graph $G$ on 64 vertices corresponding to the squares of a chessboard where two vertices are adjacent if a knight can get from one to the other in one move. The Meisters and Olech problem asks: Is the graph $H(G)$ connected? How many components are there?

## 2. The Result

Observe that if $G$ is a hamiltonian graph of girth at least 5, then $H(G)$ consists of isolated vertices only. On the other hand, if $G$ is a complete graph, then one would expect that even $H_s(G)$ is a graph of fairly rich structure.

Our aim is to prove that for every $n \geq 4$ the graph $H_s(K_n)$ is edge-hamiltonian. Clearly, $H_s(K_4) \cong K_3$. It is an easy exercise to show that $H_s(K_5)$ is isomorphic to $K_{6,6}$ minus a perfect matching.

For the sake of convenience put $H_n = H_s(K_n)$. Let $x$ be an edge of $K_n$; consider the subgraph of $H_n$ induced by those vertices which correspond to hamiltonian cycles containing the edge $x$. Denote this subgraph by $H_n(x)$. Obviously, $H_n$ and $H_n(x)$ have $(n-1)!/2$ and $(n-2)!$ vertices, respectively.

Let $P_{n-2}$ be the path $u_1 u_2 \dots u_{n-2}$ ($n \geq 3$) on $n-2$ vertices. A bijection $f\colon V(P_{n-2}) \to \{1, 2, \dots, n-2\}$ is called a **labelling**. The graph $L(P_{n-2})$ of all labellings of $P_{n-2}$ is defined as follows. The vertices of $L(P_{n-2})$ are all the $(n-2)!$ labellings of $P_{n-2}$. Two labellings $f$ and $g$ are adjacent in $L(P_{n-2})$ if they differ "along" just one edge of $P_{n-2}$, i.e., if there is an edge $u_i u_{i+1}$ of $P_{n-2}$ such that $f(u_i) = g(u_{i+1})$, $f(u_{i+1}) = g(u_i)$, and $f(u) = g(u)$ for every $u \notin \{u_i, u_{i+1}\}$. Our first observation relates the labelling graph $L(P_{n-2})$ to $H_n(x)$.

**Lemma 1.** *$H_n(x) \cong L(P_{n-2})$ for $n \geq 3$.*

*Proof.* Let $V(K_n) = \{v_1, v_2, \dots, v_n\}$ and let $x = v_{n-1} v_n$. The mapping $v_i \mapsto u_i$, $1 \leq i \leq n-2$ induces a bijection $\Phi$ of the vertex sets of $H_n(x)$ and $L(P_{n-2})$ which assigns to a hamiltonian cycle $C = (v_n v_{n-1} v_{i_1} v_{i_2} \dots v_{i_{n-2}})$ the labelling $f = \Phi(C)$ for which $f(u_j) = i_j$, $1 \leq j \leq n-2$. Moreover, it is easy to check that

$C$ and $C'$ are adjacent in $H_n(x)$ if and only if $\Phi(C)$ and $\Phi(C')$ are adjacent in $L(P_{n-2})$. □

By a result of [**RuS**] (see also [**RSZ**]) we know that there is a hamiltonian cycle in $L(P_{n-2})$ through any two specified edges. We thus have:

**Corollary.** *If $n \geq 4$ then for any two given edges of $H_n(x)$ there is a hamiltonian cycle in $H_n(x)$ containing the two edges.*

This corollary will be of central importance in the proof of the next theorem.

**Theorem 2.** *For $n \geq 4$ the graph $H_n = H_s(K_n)$ is edge-hamiltonian.*

*Proof.* As we have already seen, the statement is true for $n = 4$ or 5. We proceed as follows. Fix a vertex $u \in V(K_{n+1})$, $n \geq 5$, and consider a hamiltonian cycle, say, $(vuw\dots)$ in $K_{n+1}$. Suppressing the vertex $u$ in the cycle yields the hamiltonian cycle $(vw\dots)$ in the graph $K_{n+1} - u \cong K_n$ which passes through the edge $x = vw$. Clearly, the subgraph $H_n^x$ of $H_{n+1}$ induced by those hamiltonian cycles of $K_{n+1}$ that pass through the edges $vu$ and $uw$ is isomorphic to $H_n(x)$ (note again that $x = vw$). Moreover, if $y = v'w'$ is another edge not incident to $u$, then the subgraphs $H_n^x$ and $H_n^y$ are vertex-disjoint. We thus may put $V(H_{n+1}) = \cup_x V(H_n^x)$ where $x$ runs through all edges of $K_n = K_{n+1} - u$; the union here is considered as a disjoint union.

Now, for convenience we assume that $V(K_{n+1}) = \{0, 1, \dots, n\}$ and that $u = 0$. Consider the following cyclic sequence $S$ of all edges of $K_n = K_{n+1} - 0$:

$$(*) \qquad S = (1n, 1n-1, \dots, 12, 2n, 2n-1, \dots, 23, 3n, \dots, \dots, n-1n).$$

Note that any two consecutive edges in $S$ are adjacent in $K_{n+1}$; this also holds for the last and first edge. In what follows, this sequence will play only an auxiliary role.

Let $x$ and $y$ be two consecutive edges in the cyclic ordering of $S$, say, $x = ij$ and $y = jk$. Consider the following four hamiltonian cycles of $K_{n+1}$: $C_1 = (j0ik\dots lm\dots)$, $C_2 = (j0ik\dots ml\dots)$, $D_1 = (j0ki\dots lm\dots)$, $D_2 = (j0ki\dots ml\dots)$. Obviously, $(C_1, C_2, D_2, D_1)$ forms a 4-cycle in the graph $H_{n+1}$. Notice that by suppressing the vertex 0 we obtain the cycles $C_1(x) = (jik\dots lm\dots)$ and $C_2(x) = (jik\dots ml\dots)$ which are adjacent in $H_n^x$; the same holds true in $H_n^y$ with respect to the cycles $D_1(y) = (jki\dots lm\dots)$ and $D_2(y) = (jki\dots ml\dots)$. For consecutive $x$ and $y$ in $S$ we therefore have the following "local picture" of the graph $H_{n+1}$ (we put $C_1(x)$ instead of $C_1$, etc., see Fig. 1).

Let us now do the same procedure with **each** of the $\binom{n}{2}$ consecutive pairs in our cyclic sequence $S$. Then, if $x$ is an arbitrary edge of $K_n$, $y$ is the successor of $x$, and $z$ is the predecessor of $x$ in $S$, the local picture of $H_{n+1}$ extends to the one shown in Fig. 2.

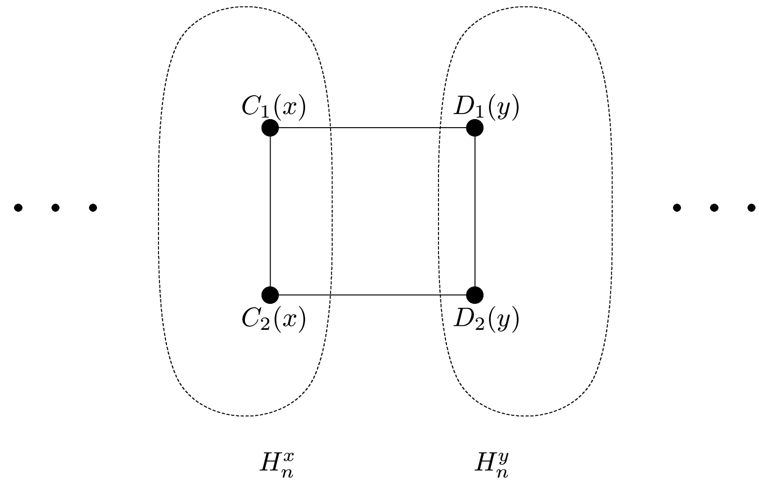


**Figure 1**. The local picture of $H_{n+1}$.

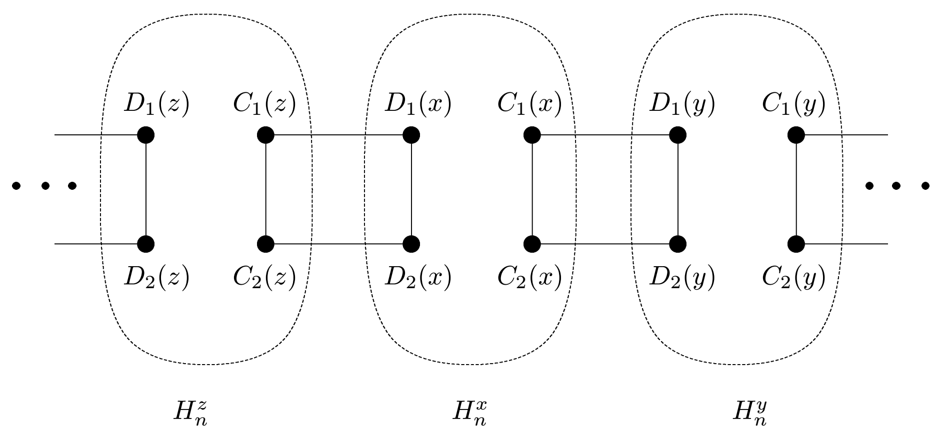


**Figure 2**.

By the Corollary, for each $x$ in $S$ there is a hamiltonian cycle $B(x)$ in $H_n^x$ passing through the edges $C_1(x)C_2(x)$ and $D_1(x)D_2(x)$. For consecutive $x$, $y$ in $S$ let $Q(x,y)$ denote the quadrangle $(C_1(x)C_2(x)D_2(y)D_1(y))$. Consider the symmetric difference

$$F = (\bigcup_{x \in S} E(B(x))) \triangle (\bigcup_{xy} E(Q(x,y)))$$

where $xy$ in the second union runs through all consecutive pairs in $S$ except $1n$ and $n-1n$. Clearly, the edge set $F$ induces a hamiltonian cycle in $H_{n+1}$.

It remains to show that for every given edge $e$ of $H_{n+1}$ there exists a hamiltonian cycle in $H_{n+1}$ containing $e$. Consider first the case when $e$ lies in $H_n^z$ for some $z \in S$. Without loss of generality we may assume that $z = 1n$. Moreover, re-labelling the vertices of $K_{n+1}$ (if necessary) we clearly may achieve that $e$ is different from the edge $f = C_1(z)C_2(z)$. Then we proceed as above, with the only exception that for $B(z)$ we take a hamiltonian cycle in $H_n^z$ passing through both $e$ and $f$. The resulting hamiltonian cycle of $H_n$ will contain $e$. Finally, let $e$ be an edge traversing from $H_n^x$ to $H_n^y$ for some $x \neq y \in E(K_n)$, say, $e$ joins the vertices $(j0i\ldots)$ and $(l0k\ldots)$. Then, without loss of generality, $j = l$ but $i \neq k$. Again, using a suitable re-labelling if necessary we may consider $x = ij$ and $y = kj$ to be consecutive in the ordering given by $S$. Therefore, we may identify $e$ with, say, the edge $C_1(x)D_1(y)$. This completes the proof. □

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