Acta Math. Univ. Comenianae
Vol. LX, 1(1991), pp. 133–151

FAITHFUL ENCLOSING OF TRIPLE SYSTEMS: DOUBLING THE INDEX

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ABSTRACT. A triple system of order $v \geq 3$ and index λ is faithfully enclosed in a triple system of order $w \geq v$ and index $\mu \geq \lambda$ when the triples induced on some v elements of the triple system of order w are precisely those from the triple system of order v. When $\lambda = \mu$, faithful enclosing is embedding; when $\lambda = 0$, faithful enclosing asks for an independent set of size v in a triple system of order w. When $\mu = 2\lambda$, we prove that a faithful enclosing of a triple system of order v and index λ into a triple system of order w and index μ exists if and only if $w \geq \lceil \frac{3v-1}{2} \rceil$, $\mu \equiv 0 \pmod{\gcd(w-2, 6)}$, and $(v, w) \notin \{(3, 5), (5, 7)\}$.

1. BACKGROUND AND NECESSARY CONDITIONS

A **triple system** of **order** v and **index** λ, denoted $TS(v, \lambda)$, is a pair $(V, \mathcal{B})$. V is a set of v elements, and $\mathcal{B}$ is a collection of 3-element subsets of V called **triples** or **blocks**. Every 2-subset of V appears in precisely λ of the triples of $\mathcal{B}$. A triple system is **simple** if it has no repeated blocks.

Let $T_1 = (V, \mathcal{B})$ be a $TS(v, \lambda)$ and $T_2 = (W, \mathcal{D})$ be a $TS(w, \mu)$. T_1 is **enclosed** in T_2 if $\mathcal{B} \subseteq \mathcal{D}$ (where $\subseteq$ is multiset inclusion); T_2 is an **enclosing** of T_1. Such an enclosing is **faithful** when the collection of all triples in $\mathcal{D}$ having all three elements from $V \subseteq W$ is precisely $\mathcal{B}$.

Colbourn, Hamm and Rosa [**3**] introduced the notions of enclosing and faithful enclosing as a generalization of embedding, which has been widely studied. Faithful enclosing also generalizes the notion of an independent set: taking $\lambda = 0$, a $TS(v, \lambda)$ has no triples, and of course one exists for every $v \geq 0$. A faithful enclosing of such a $TS(v, 0)$ in a $TS(w, \mu)$ is precisely an independent set of size v in the triple system of order w. Faithful enclosing of triple systems is thus a nontrivial generalization of two apparently different problems: embedding and independent set. In addition, the existence of certain faithful enclosings has applications in the "support size" problem [**5**].

At the outset, we recall from [**2**] the necessary conditions for a $TS(v, \lambda)$ to be faithfully enclosed in a $TS(w, \mu)$. We assume that $w \geq v \geq 3$ to avoid trivial cases; we further require that $\mu \geq \lambda \geq 0$, with $\mu > 0$. A necessary condition for a $TS(w, \mu)$ to exist is that $w \geq 0$, $w \neq 2$ and $\mu \equiv 0 \pmod{gcd(w-2, 6)}$; we call such an integer w μ-**admissible**. This condition is sufficient for the existence of

Received February 11, 1991.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 05B05, 05B07.

a $TS(w,\mu)$; in fact, if in addition $\mu \leq w-2$, a simple $TS(w,\mu)$ exists [**8**]. For a faithful enclosing of a $TS(v,\lambda)$ in a $TS(w,\mu)$ to exist, w must be μ-admissible. The faithful nature of the enclosing underpins two further necessary conditions:

Lemma 1.1. [**2**] *If a $TS(v,\lambda)$ is faithfully enclosed in a $TS(w,\mu)$, $w-v \geq (\mu-\lambda)(v-1)/\mu$, with equality only if $w-v$ is μ-admissible.*

Lemma 1.2. [**2**] *If a $TS(v,\lambda)$ is faithfully enclosed in a $TS(w,\mu)$, $w=v+s$, then if $4(\mu-\lambda)v(v-1) < \mu(v+1)^2$, either*

$$s \leq \frac{v+1}{2} - \frac{\sqrt{\mu^2(v+1)^2 - 4\mu(\mu-\lambda)v(v-1)}}{2\mu}$$

or

$$s \geq \frac{v+1}{2} + \frac{\sqrt{\mu^2(v+1)^2 - 4\mu(\mu-\lambda)v(v-1)}}{2\mu}.$$

When $\mu \geq 2\lambda$, the lemma imposes no further essential condition, except when $v=3$ and $w=5$; in that case, $\mu \geq 3\lambda$ is required. When $\lambda < \mu < 2\lambda$, the lemma imposes a strong condition showing that for a given v, the possible values of w for an enclosing of a $TS(v,\lambda)$ in a $TS(w,\mu)$ miss an interval of μ-admissible values.

In an earlier paper, we proved:

Theorem 1.3. [**2**] *A $TS(v,\lambda)$ can be faithfully enclosed in a $TS(w,\mu)$ whenever $w \geq 2v+1$, $\mu \geq \lambda \geq 0$, $\mu \geq 1$ and w is μ-admissible.*

In this paper, we treat cases when $w \leq 2v$, and prove the following:

Main Theorem 1.4. *For $\lambda \geq 1$, a $TS(v,\lambda)$ can be faithfully enclosed in a $TS(w,2\lambda)$ if and only if $w \geq \lceil \frac{3v-1}{2} \rceil$, w is 2λ-admissible and $(v,w) \notin \{(3,5),(5,7)\}$. When $\lambda=1$ and $w \leq 2v$, the $TS(w,2)$ can be chosen to be simple unless $v=7$ and $w=10$.*

In section 2, we treat the somewhat easier problem of enclosing a $TS(v,\lambda)$ in a $TS(w,\mu)$ in a manner that is not necessarily faithful. In section 3, we prove the Main Theorem when $\lambda=1$; then in section 4, we extend the proof to all λ. In many of the constructions, we require numerous small examples to provide the bases for recursion. For virtually all small examples used, we employed a computational method that is a variant of Stinson's hill-climbing algorithm [**13**]. We outline this algorithm and describe its application to a few thousand small cases in section 5.

2. Enclosing a triple system

Before considering faithful enclosings, we investigate the existence of non-faithful enclosings (or at least enclosings that may or may not be faithful).

A **parallel class** of a $TS(v,\lambda)$, (V,B), is a subcollection of B that partitions V. If $R_1, R_2, \ldots, R_t$ are parallel classes of (V,B) and $m_B(T) = \sum_{i=1}^{t} m_{R_i}(T)$ for every $T \in B$, (V,B) is a **resolvable triple system**. $R_1, R_2, \ldots, R_t$ is a **resolution** into parallel classes. In particular a resolvable Steiner triple system

(V, B) is a Steiner triple system with the property that B can be partitioned into $\frac{v-1}{2}$ classes of triples $R_1, R_2, \ldots, R_{\frac{v-1}{2}}$ such that each R_i partitions the set V. In other words,

1. $R_1, R_2, \ldots, R_{\frac{v-1}{2}} \subseteq B$.
2. $R_i \bigcap R_j = \emptyset$ for $i \neq j$.
3. If $x \in V$ then x appears in exactly one triple of each R_i.

A resolvable Steiner triple system is a **Kirkman triple system**.

Theorem 2.1. [**11**] *There is a Kirkman triple system of order v if and only if $v \equiv 3 \pmod 6$.*

We use this theorem to prove:

Theorem 2.2. *Let (V, B) be a $TS(v, 1)$ with $v \equiv 3 \pmod 6$. Then (V, B) can be enclosed in a $TS(v + s, 2)$ whenever $0 \leq s \leq (v - 1)/2$ and $s \equiv 0, 1 \pmod 3$.*

Proof. Let (V, B') be a $KTS(v)$ with $V = \{1, 2, \ldots, v\}$. Let $R_1, R_2, \ldots, R_{\frac{v-1}{2}}$ denote the parallel classes of (V, B'). $(V, B \bigcup B')$ is a $TS(v, 2)$ that encloses (V, B). If $0 < s \leq (v - 1)/2$ and $s \equiv 0, 1 \pmod 3$ let $W = \{x_1, x_2, \ldots, x_s\}$. For $i = 1, 2, \ldots s$ replace each triple $\{j, k, l\}$ in R_i by the three triples $\{x_i, j, k\}, \{x_i, j, l\}$ and $\{x_i, k, l\}$. Each original edge still occurs in exactly two triples and every new point occurs in exactly two triples with every original point. No new edges have been used. Since $s \equiv 0, 1 \pmod 3$ we define a $TS(s, 2)$ on W. Then every pair of points occurs in exactly two triples and we have constructed a $TS(v + s, 2)$ say $(V \bigcup W, B'')$. □

In general this construction does not create simple enclosings. However, using an algorithm due to Teirlinck [**14**] we can ensure that B and B' are disjoint for $v \neq 3$; then using a simple $TS(s, 3)$ provided $s \neq 3$ gives a simple enclosing. Observe that for $v \equiv 3 \pmod 6$ we have $v + s \equiv 0, 1 \pmod 3$ if and only if $s \equiv 0, 1 \pmod 3$. In other words if (V, B) is a $TS(v, 1)$ where $v \equiv 3 \pmod 6$ and if $0 \leq s \leq (v - 1)/2$ and $v + s$ is 2-admissible then there is a $TS(v + s, 2)$ that encloses (V, B).

The proof of Theorem 2.2 breaks down for $v \equiv 1 \pmod 6$ because there are no Kirkman triple systems of these orders. Fortunately there is a related structure to serve their role. A **nearly Kirkman triple system** of order v, briefly $NKTS(v)$, is a set of v elements, V, and a partitioned family $R = R_0 \bigcup R_1 \bigcup \ldots \bigcup R_t$ of subsets of V that satisfy the following conditions:

1. R_0 is a class of 2-element subsets partitioning V.
2. Every other R_i is a class of 3-element subsets partitioning V.
3. Every pair of points of V is contained in exactly one member of R. That is each pair either appears as a member of R_0 or in exactly one triple of exactly one R_i with $1 \leq i \leq t$.

Theorem 2.3. [**12**] *There is an $NKTS(v)$ if and only if $v \equiv 0 \pmod 6$ and $v \geq 18$.*

Given an $NKTS(v)$ there is a natural way to construct a Steiner triple system of order $v+1$. If $(V, R_0 \bigcup R_1 \bigcup \ldots \bigcup R_t)$ is an $NKTS(v)$ let ∞ be any point **not** in V. Replace R_0 by the collection of triples

$$R_0{}' = \{\{\infty, i, j\} : \{i, j\} \in R_0\}.$$

$(V \bigcup \{\infty\}, R_0{}' \bigcup R_1 \bigcup \ldots \bigcup R_t)$ is a Steiner triple system of order $v+1$. A Steiner triple system that can be constructed in this fashion from an $NKTS(v)$ is a **nearly resolvable Steiner triple system** and the classes $R_1, R_2, \ldots, R_t$ are **near parallel classes**. In general if (V, B) is a $TS(v, \lambda)$ and $R \subseteq B$ partitions $V \setminus \{\infty\}$ for some point $\infty \in V$, R is a **near parallel class** that **misses** ∞. By Theorem 2.3 there is a nearly resolvable Steiner triple system of order v if and only if $v \equiv 1 \pmod 6$ and $v \geq 19$.

We now prove a result similar to Theorem 2.2 in the case $v \equiv 1 \pmod 6$.

Theorem 2.4. *If $v \equiv 1 \pmod 6$ a $TS(v, 1)$ can be enclosed in a $TS(v+s, 2)$ whenever $0 \leq s \leq (v-3)/2$ and $s \equiv 0, 2 \pmod 3$.*

Proof. For $v \in \{7, 13\}$, we adapted a hill-climbing method due to Stinson [**13**] to produce enclosings (and faithful enclosings); this was applied to produce the required solutions, which are given explicitly in [**1**]. If $v \geq 19$ let (V, B) be a $TS(v, 1)$ and let (V, B') be a nearly resolvable Steiner triple system with near parallel classes $R_1, R_2, \ldots, R_{\frac{v-3}{2}}$. Without loss of generality assume that $V = \{1, 2, \ldots, v-1, \infty\}$ where ∞ is the point not contained in any triple of the R_i.

$(V, B \bigcup B')$ is a $TS(v, 2)$ enclosing (V, B). If $0 < s \leq (v-3)/2$ and $s \equiv 0, 2 \pmod 3$ let $W = \{x_1, x_2, \ldots, x_s\}$. For $i = 1, 2, \ldots, s$ replace each triple $\{j, k, l\}$ in R_i by the triples $\{x_i, j, k\}, \{x_i, j, l\}$ and $\{x_i, k, l\}$. As in the proof of Theorem 2.2 every original edge still appears in two triples. Every cross edge appears in two triples except for the cross edges of the form $\{x_i, \infty\}$ which do not appear at all. None of the new edges occur in any triples either. Since $s+1 \equiv 0, 1 \pmod 3$ there is a $TS(s+1, 2)$. Taking all triples of such a $TS(s+1, 2)$ on the set $W \bigcup \{\infty\}$, every new edge and every cross edge of the form $\{\infty, x_i\}$ occurs in exactly two of these triples. Since no other edges appear in any of these triples it follows that we have constructed a $TS(v+s, 2)$, say $(V \bigcup W, B'')$ enclosing (V, B). □

Simplicity can again be ensured here by using Teirlinck's algorithm and using a simple $TS(s+1, 2)$, provided that $v \neq 3$ and $s \neq 2$. We outline some similar results that can be proved using these techniques.

Theorem 2.5. *Let (V, B) be a $TS(v, \lambda)$ with $v \equiv 1 \pmod 6$ and $v \geq 19$. Let $\lambda' \geq \lambda$ be even. If $0 \leq s \leq \frac{\lambda' - \lambda}{\lambda'}(v-3)/2$ and $s+1$ is λ'-admissible there is a $TS(v+s, \lambda')$ enclosing (V, B).*

Proof. Let (V, B') be a nearly resolvable Steiner triple system. Then $(V, B \bigcup (\lambda' - \lambda)B')$ is a $TS(v, \lambda')$ that encloses (V, B). It has $(\lambda' - \lambda)(v-3)/2$ near parallel classes all missing the same point. For $1 \leq i \leq s$, let S_i be a collection of $\lambda'/2$ of these near transversals. (Such a collection exists as $s \leq \frac{(\lambda' - \lambda)}{\lambda}(v-3)$). The proof follows that of Theorem 2.4 except that all triples in S_i are dismantled with each new point x_i in $W = \{x_1, x_2, \ldots, x_s\}$. □

Hanani [**9**] proved that there is a resolvable $TS(v, \lambda)$ for all $v \equiv 0 \pmod 3$ except that no resolvable $TS(6, 2)$ exists.

Theorem 2.6. *Let (V, B) be a $TS(v, \lambda)$ with $v \equiv 0 \pmod 3$, $v \neq 6$. Let $\lambda' \geq \lambda$ be even. If $0 \leq s \leq \frac{\lambda' - \lambda}{\lambda'}(v-1)$ and s is λ'-admissible, there is a $TS(v+s, \lambda')$ enclosing (V, B).*

Proof. The proof is similar to those of Theorem 2.2 and Theorem 2.5. □

3. Simple faithful enclosings for index one

In this section we concentrate on enclosings of $TS(v, 1)$ in $TS(\frac{3v-1}{2} + s, 2)$ where $0 \leq s \leq (v+1)/2$. We show that these enclosings can be both simple and faithful. Combining this with Theorem 1.3 and the results of the previous section, a $TS(v, 1)$ can be enclosed in a $TS(v+s, 2)$ for all $s \geq 0$ with $v + s \equiv 0, 1 \pmod 3$.

If A is a latin rectangle on symbols $1, 2, \ldots, n$, define the latin rectangle $(A+1)$ by

$$(A+1)(i,j) = \begin{cases} A(i,j) + 1 & \text{if } 1 \leq A(i,j) \leq n-1 \\ 1 & \text{if } A(i,j) = n. \end{cases}$$

If A is based on a set such as $\{x_1, x_2, \ldots, x_n\}$ or $\{a, b, c, \ldots\}$ then $(A+1)$ is defined in the analogous way.

Theorem 3.1. *Let (V, B) be a $TS(v, 1)$. If $s \equiv 0, 1 \pmod 3$ and $0 \leq s \leq (v-1)/2$ there is a $TS((3v-1)/2 + s, 2)$ that is a faithful, simple enclosing of (V, B) with the one exception of $v = 7$ and $s = 0$. In this case the enclosing is faithful but cannot be simple.*

Proof. Suppose (V, B) is a $TS(v, 1)$ with $v \notin \{7, 13\}$ and $V = \{1, 2, \ldots, v\}$. Let L and L' be a pair of mutually orthogonal latin squares of order $(v-1)/2$ and suppose L and L' are based on $x_1, x_2, \ldots, x_{\frac{v-1}{2}}$. If $(v-1)/2$ is odd let M be a symmetric idempotent latin square of order $(v-1)/2$. If $(v-1)/2$ is even let M be a symmetric half-idempotent latin square of order $(v-1)/2$. In either case M is also based on $x_1, x_2, \ldots, x_{\frac{v-1}{2}}$. Define A to be the $(v-1)/2 \times (v-1)/2$ square given by

$$A(i,j) = \begin{cases} M(i,j) & \text{if } i \neq j \\ \emptyset & \text{if } i = j. \end{cases}$$

Let $r = 0$ if $v \equiv 1 \pmod 2$ and $r = (v-1)/2$ if $v \equiv 0 \pmod 2$. Define C to be the following $v \times v$ square:

$$C(i,j) = \begin{cases} A(i,j) & \text{if } 1 \le i, j \le (v-1)/2 \\ (A+r)(i - \frac{v-1}{2}, j - \frac{v-1}{2}) & \text{if } (v+1)/2 \le i, j \le v-1 \\ L(i, j - \frac{v-1}{2}) & \text{if } 1 \le i \le (v-1)/2,\ (v+1)/2 \le j \le v-1 \\ L^t(i - \frac{v-1}{2}, j) & \text{if } 1 \le j \le (v-1)/2,\ (v+1)/2 \le i \le v-1 \\ M(i,j) & \text{if } 1 \le i \le (v-1)/2,\ j = v \\ M(i,j) & \text{if } i = v,\ 1 \le j \le (v-1)/2 \\ (M+r)(i - \frac{v-1}{2}, j - \frac{v-1}{2}) & \text{if } (v+1)/2 \le i \le v-1 \text{ and } j = v \\ (M+r)(i,j) & \text{if } i = v,\ (v+1)/2 \le j \le v-1. \end{cases}$$

The diagonal cells of C are left empty.

Every cell of C contains one element from the set $\{x_1, x_2, \dots, x_{\frac{v-1}{2}}\}$ except of course the diagonal cells. Each of these symbols occurs exactly twice in every row and column of C. In other words, C is an "exact $(2,2,1)$-blocked latin square".

We use C to establish the existence of a $TS((3v-1)/2, 2)$ on $V' = V \bigcup \{x_1, x_2, \dots, x_{\frac{v-1}{2}}\}$ that is a faithful, simple enclosing of (V, B). Define a collection of triples B' on V' by the following rules.

1. $B \subseteq B'$.
2. For $1 \le i < j \le v$, $\{i, j, C(i,j)\} \in B'$.
3. $D \subseteq B'$ where $TS(\{x_1, x_2, \dots, x_{\frac{v-1}{2}}\}, D)$ is any simple $TS((v-1)/2, 2)$.

Since each symbol appears twice in every row and column of A and A is symmetric, (V', B') is a $TS((3v-1)/2, 2)$ that encloses (V, B). Every triple in $B' \setminus B$ contains at least one element from $\{x_1, x_2, \dots, x_{\frac{v-1}{2}}\}$ so the enclosing is faithful. The enclosing is also simple.

The latin square L used in the construction of C has an orthogonal mate, and hence L has a decomposition into transversals. Since L has order $(v-1)/2$ there are $(v-1)/2$ transversals in this decomposition. For $1 \le k \le (v-1)/2$ let

$$T_k = \{[({i_1}^{(k)}, {j_1}^{(k)}), L({i_1}^{(k)}, {j_1}^{(k)})], \dots, [({i_t}^{(k)}, {j_t}^{(k)}), L({i_t}^{(k)}, {j_t}^{(k)})]\}$$

be the k^{th} transversal. Here $t = (v-1)/2$. For $t = (v-1)/2$ and $1 \le k \le t$ let

$$\begin{aligned} S_k = \{ & \{{i_1}^{(k)}, {j_1}^{(k)} + (v-1)/2, L({i_1}^{(k)}, {j_1}^{(k)})\}, \dots, \\ & \{{i_t}^{(k)}, {j_t}^{(k)} + (v-1)/2, L({i_t}^{(k)}, {j_t}^{(k)})\} \}. \end{aligned}$$

Each S_k partitions the set $\{1, 2, \dots, v-1, x_1, x_2, \dots, x_{\frac{v-1}{2}}\}$ into triples and S_i and S_j are disjoint whenever $i \ne j$. To see this observe that each transversal T_k contains one element from each row and column of L. Consequently if $1 \le k \le (v-1)/2$,

$$\{{i_1}^{(k)}, {i_2}^{(k)}, \dots, {i_{\frac{v-1}{2}}}^{(k)}\} = \{{j_1}^{(k)}, {j_2}^{(k)}, \dots, {j_{\frac{v-1}{2}}}^{(k)}\} = \{1, 2, \dots, (v-1)/2\}$$

and therefore,

$$\{i_1{}^{(k)}, \ldots, i_{\frac{v-1}{2}}{}^{(k)}, j_1{}^{(k)} + (v-1)/2, \ldots, j_{\frac{v-1}{2}}{}^{(k)} + (v-1)/2\} = \{1, 2 \ldots, v-1\}.$$

Since each symbol $x_1, x_2, \ldots, x_{\frac{v-1}{2}}$ occurs exactly once in each transversal,

$$\{L(i_1{}^{(k)}, j_1{}^{(k)}), \ldots, L(i_{\frac{v-1}{2}}{}^{(k)}, j_{\frac{v-1}{2}}{}^{(k)})\} = \{x_1, x_2, \ldots, x_{\frac{v-1}{2}}\}.$$

Thus every element of $\{1, 2, \ldots, v-1, x_1, x_2, \ldots, x_{\frac{v-1}{2}}\}$ occurs in exactly one triple of S_k so S_k partitions $\{1, 2, \ldots, v-1, x_1, x_2, \ldots, x_{\frac{v-1}{2}}\}$ into triples. None of the transversals contain a common cell, so the S_k are pairwise disjoint.

Since each S_k partitions the set $\{1, 2, \ldots, v-1, x_1, x_2, \ldots, x_{\frac{v-1}{2}}\}$ into triples, each S_k is a near parallel class of (V', B') that misses the point v. We now proceed as in Theorem 2.4. If $0 \leq s \leq (v-1)/2$ and $s \equiv 0, 2 \pmod 3$ let $W_s = \{y_1, y_2, \ldots, y_s\}$. Let $V'' = V' \bigcup W_s$.

Define a collection of triples $B_s{}''$ on V'' as follows:

1. $B' \setminus (S_1 \bigcup S_2 \bigcup \ldots \bigcup S_s) \subseteq B_s{}''$.
2. If $\{a, b, c\} \in S_k$ for $1 \leq k \leq s$, $\{y_k, a, b\}$, $\{y_k, a, c\}$, $\{y_k, b, c\} \in B_s{}''$.
3. $D \subseteq B_s{}''$ where $(W_s \bigcup \{v\}, D)$ is a $TS(s+1, 2)$.

(V'', B'') is a $TS(\frac{3v-1}{2} + s, 2)$ faithfully enclosing (V, B). Provided $s \neq 2$ there is a simple $TS(s+1, 2)$ so the enclosing (V'', B'') can be chosen to be simple.

If $s = 2$ the enclosing contains the triple $\{v, y_1, y_2\}$ twice but no other triples are repeated. Since $v \neq 7$ we remove one of these triples by employing a technique from [**6**].

Notice that $\{x_1, 1, v\} \in B'$. The way we order the transversals $T_1, T_2, \ldots, T_s$ is irrelevant so choose transversals T_1 and T_2 of L such that for some i and j with $1 \leq i, j \leq (v-1)/2$,

$$\{i, x_1, j\} \in T_1 \text{ and } \{1, j, x_2\} \in T_2.$$

Such a choice is possible as L can be decomposed into transversals. T_1 and T_2 are necessarily distinct as a transversal contains exactly one element from each column.

Using these two transversals in the construction of $(V \bigcup W_2, B_2{}'')$ it follows that

$$\{x_1, 1, v\}, \{j, x_1, y_1\}, \{1, j, y_2\} \in B_2{}''$$

and that $\{y_1, y_2, v\}$ occurs twice in $B_2{}''$. Replace $\{x_1, 1, v\}$, $\{j, x_1, y_1\}$, $\{1, j, y_2\}$ and one copy of $\{y_1, y_2, v\}$ by the four triples $\{y_1, x_1, v\}$, $\{y_2, v, 1\}$, $\{y_1, y_2, j\}$ and $\{1, j, x_1\}$. Both of these sets of triples cover the same edges. Thus we still have a $TS(\frac{3v-1}{2} + 2, 2)$ after trading these triples. Since none of the triples $\{x_1, 1, v\}$, $\{j, x_1, y_1\}$, $\{1, j, y_2\}$ or $\{y_1, y_2, v\}$ are members of B, this new triple system encloses (V, B). Since $(V'', B_2{}'')$ is a faithful enclosing and each of the triples

$\{y_1, x_1, v\}$, $\{y_2, v, 1\}$, $\{y_1, y_2, j\}$ and $\{1, j, x_1\}$ contains at least one point from $\{x_1, x_2, \ldots, x_{\frac{v-1}{2}}, y_1, y_2\}$ the new enclosing remains faithful. The triple $\{y_1, y_2, v\}$ now appears only once but to ensure that this enclosing is simple we must verify that the four triples added did not already appear in B_2''. Except for $\{y_1, y_2, v\}$ every triple in B_2'' has one of the following five forms:

1. $\{i, j, k\}$, $1 \leq i, j, \leq v$.
2. $\{i, j, x_k\}$, $1 \leq i, j < v$, $1 \leq k \leq (v-1)/2$.
3. $\{x_i, x_j, x_k\}$, $1 \leq i, j \leq (v-1)/2$.
4. $\{i, x_j, y_k\}$, $1 \leq i < v, 1 \leq j \leq (v-1)/2, k = 1, 2$.
5. $\{i, j, y_k\}$, $1 \leq i, j < v, k = 1, 2$.

None of the triples $\{y_1, x_1, v\}$, $\{y_2, v, 1\}$, $\{y_1, y_2, j\}$ are of this form so they only appear once in the new triple system. Since $\{1, j, x_2\} \in B_2''$ and the pair $\{i, j\}$ occurs in a triple of B, the triple $\{1, j, x_1\} \notin B_2''$. It follows that the $TS(\frac{3v-1}{2} + 2, 2)$ constructed is a faithful, simple enclosing of (V, B).

When $v = 13$ the above proof fails because there is no pair of mutually orthogonal latin squares of order 6. In the preceding proof let C be the square

	a	b	c	d	e	f	c	e	b	d	f	a
a		c	d	e	f	a	b	d	f	c	e	b
b	c		e	f	a	f	b	d	e	a	d	c
c	d	e		a	b	e	a	c	f	f	b	d
d	e	f	a		c	c	f	b	d	b	a	e
e	f	a	b	c		b	d	a	c	e	d	f
f	a	f	e	c	b		a	b	c	d	e	d
c	b	b	a	f	d	a		c	d	e	f	e
e	d	d	c	b	a	b	c		e	f	a	f
b	f	e	f	d	c	c	d	e		a	b	a
d	c	a	f	b	e	d	e	f	a		c	b
f	e	d	b	a	d	e	f	a	b	c		c
a	b	c	d	e	f	d	e	f	a	b	c	

(we have used the symbols a, b, c, d, e, f instead of $x_1, x_2, x_3, x_4, x_5, x_6$). C is used exactly as in the first part of the proof to construct a $TS(19, 2)$, (V', B') that is a faithful, simple enclosing of (V, B). (V', B') contains the following five near parallel classes:

- $S_1 = \{\{1, 8, c\}, \{2, 9, d\}, \{3, 10, e\}, \{4, 11, f\}, \{5, 12, a\}, \{6, 7, b\}\}$,
- $S_2 = \{\{1, 9, e\}, \{2, 10, f\}, \{3, 11, a\}, \{4, 12, b\}, \{5, 7, c\}, \{6, 8, d\}\}$,
- $S_3 = \{\{1, 10, b\}, \{2, 11, c\}, \{3, 12, d\}, \{4, 7, e\}, \{5, 8, f\}, \{6, 9, a\}\}$,
- $S_4 = \{\{1, 11, d\}, \{2, 12, e\}, \{3, 7, f\}, \{4, 8, a\}, \{5, 9, b\}, \{6, 10, c\}\}$,
- $S_5 = \{\{1, 12, f\}, \{2, 7, a\}, \{3, 8, b\}, \{4, 9, c\}, \{5, 10, d\}, \{6, 11, e\}\}$.

Each of these near parallel classes misses the point 13. We can therefore dismantle each of these parallel classes to construct a $TS(22,2)$ and $TS(24,2)$ that form faithful, simple enclosings of (V,B). When $s=2$ we construct a $TS(21,2)$ faithfully enclosing (V,B) with one repeated triple. Our method for removing the repeated triple does not work in this case. However, using hill-climbing [**13**] we found a $TS(21,2)$ and a $TS(25,2)$ that are faithful simple enclosings of (V,B); explicit solutions are given in [**1**].

When $v=7$ our method constructs a $TS(10,2)$, a $TS(12,2)$ and a $TS(13,2)$ that are faithful enclosings of a $TS(7,1)$. These are not simple; however, using hill-climbing, we found a simple $TS(12,2)$ that faithfully encloses a $TS(7,1)$:

$\{0,1,7\},\{2,3,8\},\{4,5,9\},\{6,10,11\},\{0,8,9\},\{1,7,11\},\{2,7,10\},\{3,9,11\},$
$\{4,6,8\},\{5,6,9\},\{0,2,11\},\{2,5,10\},\{3,9,10\},\{3,4,11\},\{0,8,9\},\{7,9,10\},$
$\{0,6,11\},\{3,5,7\},\{0,5,10\},\{1,5,8\},\{1,5,8\},\{1,3,8\},\{5,8,11\},\{8,10,11\},$
$\{2,7,8\},\{0,3,10\},\{1,2,9\},\{1,4,10\},\{5,7,11\},\{4,8,10\},\{0,4,7\},\{6,7,8\},$
$\{1,6,10\},\{2,6,9\},\{4,7,9\},\{1,9,11\},\{2,4,11\},\{3,6,7\}.$

This contains a parallel class that allows us to construct a faithful, simple enclosing of order 13.

We complete the proof by noticing that no $TS(10,2)$ can be a faithful, simple enclosing of a $TS(7,1)$. Let $V=\{1,2,\ldots,7\}$ and $W=\{a,b,c\}$. There are 21 original edges that must appear in triples of the form $\{i,j,x\}$ where $i,j\in V$ and $x\in W$. There are 42 cross-edges and consequently all cross-edges must appear in triples of the form described above. Therefore the triple $\{a,b,c\}$ must occur twice so this enclosing cannot be simple. $\square$

Theorem 3.1 handles enclosings of a $TS(v,1)$ into all orders $w\leq 2v-1$, and Theorem 1.3 handles all $w\geq 2v+1$. It remains to handle $w=2v$. We develop a general technique that can be applied here.

A **group divisible design** with block size 3, or 3-*GDD*, of type ${g_1}^{t_1}\ldots{g_m}^{t_m}$ is a triple $(X,\mathcal{B},\mathcal{G})$; $\mathcal{G}$ is a partition of the elements of G into t_i classes of size g_i for $i=1,\ldots,m$, where the classes of the partition are called **groups**. $\mathcal{B}$ is a set of 3-subsets (triples) of X, with the property that every 2-subset of elements from different groups appears in exactly one triple of $\mathcal{B}$, and no 2-subset of a group appears in a triple of $\mathcal{B}$. We employ 3-*GDD*s of type g^tu^1; Colbourn, Hoffman and Rees [**4**] settled existence for such *GDD*s:

Lemma 3.2. *A 3-GDD of type g^tu^1 exists if and only if*

1. *if $g>0$ then $t\geq 3$, or $t=2$ and $u=g$, or $t=1$ and $u=0$, or $t=0$;*
2. *$u\leq g(t-1)$ or $gt=0$;*
3. *$g(t-1)+u\equiv 0 \pmod 2$ or $gt=0$;*
4. *$gt\equiv 0 \pmod 2$ or $u=0$;*
5. *$g^2\binom{t}{2}+gtu\equiv 0 \pmod 3$.*

Theorem 3.3. *Suppose that a 3-GDD of type ${g_1}^{t_1}\ldots{g_s}^{t_s}$ exists. Let $x\geq y\geq 0$. Suppose that for each $1\leq i\leq s$ there is a $TS(x+2g_i,2\lambda)$ faithfully enclosing a $TS(y+g_i,\lambda)$, and having a sub-$TS(x,2\lambda)$ that faithfully encloses a* *sub-$TS(y,\lambda)$ of the TS of order $y+g_i$. Then there exists a $TS(x+\sum_{i=1}^{s} 2g_it_i, 2\lambda)$ faithfully enclosing a $TS(y+\sum_{i=1}^{s} g_it_i, \lambda)$.*

Proof. Let the GDD be on element set T, having groups $G_1,\dots,G_m$. The TS constructed is on elements $(X \times \{0,1\}) \cup \{e_1,\dots,e_x\}$. To form the blocks of the TS, for each triple $\{a,b,c\}$ of the GDD, place triples $\{(a,i),(b,j),(c,k)\}$ for all eight combinations with $i,j,k \in \{0,1\}$ in the TS. Then for one group G_i, place on $(G_i \times \{0,1\}) \cup \{e_1,\dots,e_x\}$ the triples of a $TS(x+2g_i, 2\lambda)$ meeting the requirements of the theorem, so that the enclosed $TS(y+g_i,\lambda)$ is on $(G_i \times \{0\}) \cup \{e_1,\dots,e_y\}$ and the sub-$TS(y,\lambda)$ is on the elements $\{e_1,\dots,e_s\}$. For the remaining groups, first remove all triples of the sub-$TS(x,2\lambda)$ from the $TS(x+2g_i,2\lambda)$ and then proceed as above. It is easy to verify that this is a faithful enclosing. □

Corollary 3.4. *A $TS(v,1)$ with $v \equiv 3 \pmod 6$ has a simple faithful enclosing in a $TS(2v,2)$.*

Proof. Apply Theorem 3.3 using a 3-GDD of type $3^{v/3}$, with $x=y=0$, and use **simple** $TS(6,2)$'s faithfully enclosing a $TS(3,1)$. □

4. Faithful enclosings for all indices

We now examine faithful enclosings of $TS(v,\lambda)$ in $TS(v',2\lambda)$ for $\lambda > 1$.

Lemma 4.1. *Suppose there is a $TS(v,\lambda)$ faithfully enclosed in a $TS(v',\lambda')$. Then for any $\gamma \geq 1$ a $TS(v,\gamma\lambda)$ can be faithfully enclosed in a $TS(v',\gamma\lambda')$.*

Proof. Let (V',B') be a $TS(v',\lambda')$ faithfully enclosing (V,B) a $TS(v,\lambda)$. Let D be a collection of triples on V given by taking each occurrence of a triple in B γ times. Then (V,D) is a $TS(v,\gamma\lambda)$. Taking each occurrence of a triple in B' γ times, we form (V',D'), a $TS(v',\gamma\lambda')$ that faithfully encloses (V,D). □

It follows from Lemma 4.1, Theorem 3.1 and Corollary 3.4 that if $v \equiv 1,3 \pmod 6$ then any $TS(v,\lambda)$ can be faithfully enclosed in a $TS(\frac{3v-1}{2}+s, 2\lambda)$ whenever $0 \leq s \leq (v+1)/2$ and $\frac{3v-1}{2}+s \equiv 0,1 \pmod 3$. We extend the method of Theorem 3.1 to handle other cases for odd orders.

Theorem 4.2. *Let (V,B) be a $TS(v,\lambda)$ with $v \equiv 1 \pmod 2$, $v \neq 5$ and let $s+1$ be 2λ-admissible. Suppose that $(v-1)/2 = t\lambda + r$ where $0 \leq r < \lambda$. If $0 \leq s \leq \lambda(t-1)$ there is a $TS(\frac{3v-1}{2}+s, 2\lambda)$ that is a faithful enclosing of (V,B).*

Proof. If $v=13$, apply Theorem 3.1 and Lemma 4.1. So we assume that $v \notin \{5,13\}$. Let C be the array constructed in the proof of Theorem 3.1. (extending the definition in the obvious way for $v \equiv 5 \pmod 6$). For $i=0,1,\dots,\lambda-1$, let

$$C_i(l,m) = (C+i)(l,m), 1 \leq l,m \leq v$$

where $(C+i)$ is defined in the obvious way. Define a symmetric, exact $(2\lambda,2\lambda,\lambda)$-blocked latin square C' by setting

$$C'(l,m) = \{C_0(l,m), C_1(l,m), \dots, C_{\lambda-1}(l,m)\}$$

for $1 \le l, m \le v$. We use C' to construct an enclosing.

Let $V' = V \bigcup \{x_1, x_2, \dots, x_{\frac{v-1}{2}}\}$. The following three rules define a collection of triples B'.

1. $B \subseteq B'$.
2. If $x_i \in C(l, m)$, for $1 \le l < m \le v$, $\{l, m, x_i\} \in B'$.
3. $D \subseteq B'$ where $(\{x_1, x_2, \dots, x_{\frac{v-1}{2}}\}, D)$ is some $TS((v-1)/2, 2\lambda)$.

As in the proof of Theorem 3.1, (V', B') is a faithful enclosing of (V, B).

Let ${T_1}^{(0)}, {T_2}^{(0)}, \dots, {T_{\frac{v-1}{2}}}^{(0)}$ be the triples arising from $(v-1)/2$ disjoint transversals of L and for $i = 1, 2, \dots, \lambda-1$ let ${T_1}^{(i)}, {T_2}^{(i)}, \dots, {T_\lambda}^{(i)}$ be the corresponding triples from $L_i = (L + i)$. (Recall that L is the latin square with orthogonal mate used in the definition of C.) Define

- ${R_1}^{(0)} = {T_1}^{(0)} \bigcup {T_2}^{(0)} \bigcup \dots \bigcup {T_\lambda}^{(0)}$,
- ${R_2}^{(0)} = {T_{\lambda+1}}^{(0)} \bigcup {T_{\lambda+2}}^{(0)} \bigcup \dots \bigcup {T_{2\lambda}}^{(0)}$,
- $\vdots$
- ${R_{t-1}}^{(0)} = {T_{(t-1)\lambda+1}}^{(0)} \bigcup {T_{(t-1)\lambda+2}}^{(0)} \bigcup \dots \bigcup {T_{t\lambda}}^{(0)}$,
- ${R_1}^{(1)} = {T_1}^{(1)} \bigcup {T_2}^{(1)} \bigcup \dots \bigcup {T_\lambda}^{(1)}$,
- $\vdots$
- ${R_{t-1}}^{(\lambda-1)} = {T_{(t-1)\lambda+1}}^{(\lambda-1)} \bigcup {T_{(t-1)\lambda+2}}^{(\lambda-1)} \bigcup \dots \bigcup {T_{t\lambda}}^{(\lambda-1)}$.

Rename these as $R_1, R_2, \dots, R_{\lambda(t-1)}$.

Let $V'' = V' \bigcup \{y_1, y_2, \dots, y_s\}$ and define a collection of triples B'' on V'' by the following three rules:

1. $B' \setminus (R_1 \bigcup R_2 \bigcup \dots \bigcup R_s) \subseteq B''$.
2. If $[(i, j), x_k] \in R_l$ for $1 \le l \le s$, let $\{i, j, y_l\}$, $\{i, x_k, y_l\}$, $\{j, x_k, y_l\} \in B''$.
3. $D \subseteq B''$ where $(\{y_1, y_2, \dots, y_s, v\}, D)$ is a $TS(s+1, 2\lambda)$.

(V'', B'') is a $TS((3v-1)/2 + s, 2\lambda)$ faithfully enclosing (V, B). □

The case when $v = 5$ is not considered in the proof, because the possibility of enclosing a $TS(5, 3\lambda)$ in a $TS(7, 6\lambda)$ is eliminated by Lemma 1.2.

Now suppose that $w = \frac{3v-1}{2} + s \le 2v$, and that w is 2λ-admissible (and that v is λ-admissible). When does Theorem 4.2 produce the faithful enclosing of a $TS(v, \lambda)$ in a $TS(w, 2\lambda)$? First, the 2λ-admissibility of w ensures that $s + 1$ is 2λ-admissible, **except** when $\lambda \equiv 0 \pmod 3$ and $s = 1$. Moreover, we require all values of s for which $s + 1$ is 2λ-admissible having $0 \le s \le \frac{v+1}{2}$, and the Theorem applies when $0 \le s \le \lambda\left(\lfloor \frac{v-1}{2\lambda} \rfloor\right)$. Corollary 3.4 handles the remaining case when $v \equiv 3 \pmod 6$, but when $v \equiv 5 \pmod 6$, we must treat the cases when $s \in \{\frac{v-3}{2}, \frac{v-1}{2}, \frac{v+1}{2}\}$, and when $v \equiv 1 \pmod 6$ we must handle $s = \frac{v+1}{2}$. This leaves two problems to settle which we treat in turn.

Lemma 4.3. *A* $TS(v,3)$ *with* $v \equiv 1 \pmod 2$*,* $v > 3$ *has a faithful enclosing in a* $TS(v + \frac{v-1}{2} + 1, 6)$*.*

Proof. Write $v = 2x + 1$. We form the $TS(3x + 2, 6)$ on $(Z_x \times Z_3) \cup \{\alpha, \beta\}$. First we treat the case when x is odd, $x = 3$ or $x \geq 7$. For $j \in \{1, 2\}$, place a $TS(2x + 1, 3)$ on $(Z_x \times \{0, j\}) \cup \{\alpha\}$ having an independent set of size $x + 1$ on $(Z_x \times \{0\}) \cup \{\alpha\}$. Next form a collection of triples on $(Z_x \times \{1, 2\}) \cup \{\alpha\}$ in which

1. there is an independent set on $(Z_6 \times \{1\}) \cup \{\alpha\}$;
2. $\{\alpha, x_1, x_2\}$ is a triple for $x = 0, 1, 2$;
3. the pairs $\{x_1, x_2\}$ appear in five triples; and
4. all other pairs appear in three triples each.

To form this collection of triples, we first choose a $TS(x, 3)$ on $Z_x \times \{2\}$ with a triply repeated triple on $\{0_2, 1_2, 2_2\}$; include all triples **except** the three copies of this triply repeated triple. Now choose a one-factorization on $(Z_x \times \{1\}) \cup \{\alpha\}$ having a sub-one-factorization on $\{\alpha, 0_1, 1_1, 2_1\}$. Call the x factors of this one factorization $F_0, \ldots, F_{x-1}$, so that the sub-one-factorization is contained in $\{F_0, f_1, F_2\}$. Now for $a \in Z_x$ and $\{b, c\} \in F_a$ with $\{b, c\} \not\subset \{\alpha, 0_1, 1_1, 2_1\}$, add the triple $\{a_2, b, c\}$ three times to the system being constructed. In this way, note that no additional triples are placed on $(Z_x \times \{1\}) \cup \{\alpha\}$; in effect, we have constructed all of a $TS(2x + 1, 3)$ **but** a hole on 7 points, and having an independent set on $x + 1$ points. Then add the following:

- the triples $\{\alpha, z_1, z_2\}$ for $z \in \{0, 1, 2\}$ three times each;
- the triples $\{z_1, (z + 1)_1, (z + y)_2\}$ for $y \in \{0, 1, 2\}$, computing mod 3;
- the triples $\{(z + 2)_1, z_2, (z + 1)_2\}$, computing mod 3; and
- the triple $\{0_2, 1_2, 2_2\}$ twice.

Now take a $TD(3, x)$ with a parallel class. Remove the parallel class and place the result on $Z_x \times Z_3$ with the groups aligned on $Z_x \times \{i\}$, $i = 0, 1, 2$, and the triples from the (missing) parallel class on $\{i\} \times Z_3$ for $i \in Z_x$. Now add all triples of the form $\{beta, z_i, z_j\}$ for $0 \leq i < j \leq 2$ and $z \in Z_x$ three times each. The result has every pair in six triples **except** that $\{\alpha, \beta\}$ appears in no triples, and $\{0_1, 0_2\}$, $\{1_1, 1_2\}$ and $\{2_1, 2_2\}$ each appear in eight triples. Moreover, the triples on $(Z_x \times \{0, 1\}) \cup \{\alpha\}$ form a $TS(2x + 1, 3)$. To complete the construction then, remove one each of the triples $\{\alpha, z_1, z_2\}$ and $\{\beta, z_1, z_2\}$ for $z \in \{0, 1, 2\}$, and add instead the triples $\{\alpha, \beta, z_i\}$ for each $z \in \{0, 1, 2\}$, $i \in \{1, 2\}$. This is the required $TS(3x + 2, 6)$. The remaining case (when $x = 5$) is handled using hill-climbing.

Now we turn to the case when v is even. Write $v = 2x$. Suppose that there is a 3-GDD of type $g^t u^1$ with $gt + u = x$ on Z_x; further suppose that there exists a faithful enclosing of a $TS(2u + 1, 3)$ in a $TS(3u + 2, 6)$. Now form the $TS(3x + 2, 6)$ as follows. For each triple $\{a, b, c\}$ of the GDD, add the triples $\{a_i, b_i, c_j\}$, $\{a_i, b_j, c_i\}$, $\{a_j, b_i, c_i\}$, and $\{a_j, b_j, c_j\}$ three times each for $0 \leq i < j \leq 2$. Then add the triples $\{p_0, q_1, r_2\}$ three times each for $(p, q, r) \in \{(a, b, c), (a, c, b), (b, a, c), (b, c, a), (c, a, b), (c, b, a)\}$.

Now if G is the group of size u, add the triples of the $TS(3u+2,6)$ on $(G\times Z_3)\cup\{\alpha,\beta\}$, with the $TS(2u+1,3)$ on $(G\times\{0,1\})\cup\{\alpha,\beta\}$ faithfully enclosed. Finally, for every other group G, place on $(G\times Z_3)\cup\{\alpha,\beta\}$ a collection of triples faithfully enclosing a $TS(2g+1,3)$ on $(G\times\{0,1\})\cup\{\alpha\}$ that covers every pair exactly six times, except for the pair $\{\alpha,\beta\}$ which is uncovered. This can be constructed by the method in the proof of Theorem 4.1 **except** when $x=2$.

To apply this, use a 3-GDD of type $6^t u^1$ with $6t+u=x$ and $u\in\{2,4,6\}$ (we require solutions for $2x=4,8,12$ here). This handles all cases for $x\geq 18$ given the solutions for $x\in\{2,4,6\}$. Using instead a 3-GDD of type $4^{x/4}$ handles $x\in\{12,16\}$ given the solution for $x=4$. Using a 3-GDD of type $4^{(x-2)/4}2^1$ handles $x=14$ given the solution for $x=2$. To complete the proof then, we employed hill-climbing to find the required enclosings when $x\in\{2,4,6,8,10\}$. □

Lemma 4.4. *A $TS(v,3)$ with $v\equiv 5 \pmod 6$ has faithful enclosings in $TS(2v-c,6)$ for $c\in\{0,1,2\}$. A $TS(v,3)$ with $v\equiv 1 \pmod 6$ has a faithful enclosing in a $TS(2v,6)$.*

Proof. We apply Theorem 3.3 in each case. First take $v\equiv 3 \pmod 6$. Apply Theorem 3.3 to a 3-GDD of type $3^{(v-5)/3}5^1$, using $x=y=0$; use a faithful enclosing of a $TS(3,3)$ in a $TS(6,6)$ and a faithful enclosing of a $TS(5,3)$ in a $TS(10,6)$. This gives a faithful enclosing of a $TS(v,3)$ in a $TS(2v,6)$ provided $v\geq 17$ (see Lemma 3.2). The faithful enclosing of a $TS(11,3)$ in a $TS(22,6)$ was found using hill-climbing.

For enclosing in a $TS(2v-1,6)$, use a 3-GDD of type $4^{(v-1)/4}$ or one of type $4^{(v-7)/4}6^1$ and set $x=y=1$; then use faithful enclosings of a $TS(5,3)$ in a $TS(9,6)$ and of a $TS(7,3)$ in a $TS(13,6)$ (the latter exists by Theorem 4.2; the first is found using hill-climbing). This handles all cases for $v\geq 17$; the remaining example, a faithful enclosing of a $TS(11,3)$ in a $TS(21,6)$, was found using hill-climbing.

For enclosing in a $TS(2v-2,6)$, we first construct a useful ingredient: a $TS(20,6)$ having a sub-$TS(8,6)$, and faithfully enclosing a $TS(11,3)$ that has a sub-$TS(5,3)$, for which the $TS(5,3)$ is faithfully enclosed in the $TS(8,6)$. We use element set $(Z_6\times Z_2)\cup\{e_1,e_2,\ldots,e_8\}$. On $\{e_1,\ldots,e_8\}$ we place a $TS(8,6)$ faithfully enclosing a $TS(5,3)$ on $\{e_1,\ldots,e_5\}$. First, choose a 1-factorization $A_1,\ldots A_5$ on Z_6 arbitrarily. Now form a 1-factorization on $Z_6\times Z_2$ by taking

1. for each factor A_i, form a factor F_i which is A_i on each of $Z_6\times\{0\}$ and $Z_6\times\{1\}$;
2. for each factor A_i, form a factor H_i containing all edges of the form $\{x_0,y_1\}$ for each edge $\{x,y\}\in A_i$;
3. form a factor C containing those edges of the form $\{x_0,x_1\}$ for $x\in Z_6$.

Now for $i=1,2,3$, $F_i\cup H_i\cup C$ consists of three vertex disjoint complete graphs on four vertices each; in the system being constructed, place the triples of a $TS(4,2)$ on each. There remain six copies of factors F_4,F_5,H_4,H_5 and four copies each of factors F_i,H_i, $1\leq i\leq 3$. Partition these 48 factors arbitrarily into eight sets $S_1,\ldots,S_8$ of six factors each, so that each set contains three F factors and three H factors (one can choose the partition so that no set contains a factor twice).

Now for $1 \leq i \leq 8$, for $\{x, y\} \in F \in S_i$, form a triple $\{e_i, x, y\}$. It is easy to verify that the result is a (simple) $TS(20, 6)$. The enclosed $TS(11, 3)$ is on $\{e_1, \dots e_5\} \cup (Z_6 \times \{0\})$.

Using this $TS(20, 6)$, apply Theorem 3.3 with $x = 8$, $y = 5$ to a GDD of type $6^{(v-5)/6}5^1$ to obtain the result for $v \geq 23$. The faithful enclosings of $TS(5, 3)$ into $TS(8, 6)$ and of $TS(11, 3)$ into $TS(20, 6)$ are given as part of the construction of the $TS(20, 6)$. The remaining case, $TS(17, 3)$ in $TS(32, 6)$, was found using hill-climbing.

Now consider $v \equiv 1 \pmod 6$. Apply Theorem 3.3 to a 3-GDD of type $3^{(v-7)/3}7^1$ with $x = y = 0$; use faithful enclosings of a $TS(3, 3)$ in a $TS(6, 6)$, and of a $TS(7, 3)$ in a $TS(14, 6)$ (the second is given in [**2**]). By Lemma 3.2, this handles all cases for $v \geq 19$. To complete the proof, we need a faithful enclosing of a $TS(13, 3)$ in a $TS(26, 6)$; again, this is easily produced with hill-climbing. □

This completes the proof of the Main Theorem when v is odd:

Theorem 4.5. *For v odd, a $TS(v, \lambda)$ can be faithfully enclosed in a $TS(\frac{3v-1}{2} + s, 2\lambda)$ whenever $\frac{3v-1}{2} + s$ is 2λ-admissible, and $s \geq 0$ except when $v = 3$ and $s = 1$, or $v = 5$ and $s = 0$.*

Proof. Apply Theorem 1.3 to handle all cases with $s \geq \frac{v+1}{2} + 1$. Apply Theorem 4.2 and Lemma 4.1 if $0 \leq s \leq \lambda\lfloor\frac{v-1}{2}\rfloor$, $s \neq 1$. Apply Lemmas 4.3 and 4.1 if $s = 1$. Lemmas 4.4 and 4.1 handle the remaining cases with $v \equiv 1, 5 \pmod 6$, and Corollary 3.4 and Lemma 4.1 handles the final case ($s = \frac{v+1}{2}$) when $v \equiv 3 \pmod 6$. □

A $TS(v, 2)$ exists if and only if $v \equiv 0, 1 \pmod 3$ and Theorem 4.5 covers half these cases. To construct enclosings of $TS(v, 2)$ with $v \equiv 0, 4 \pmod 6$ we modify our approach.

Recall that if a $TS(v + s, 4)$ is a faithful enclosing of a $TS(v, 2)$ and $v \equiv 0, 4 \pmod 6$ then $s \geq v/2$. We show that this necessary condition is also sufficient.

Suppose that (V, B) is a $TS(v, \lambda)$. The **neighbourhood** $N(x)$ of an element $x \in V$ is the multigraph with vertex set $V \setminus \{x\}$ and edge multiset

$$\{\{y, z\} : \{x, y, z\} \in B\}.$$

Theorem 4.6. [**6**] *Every 2-regular multigraph on $v - 1$ points with $v - 1 \equiv 0, 2 \pmod 3$ points is the neighbourhood graph of an element in a $TS(v, 2)$ with two exceptions: $C_2 \bigcup C_3$ and $C_3 \bigcup C_3$.*

We use this result to prove

Theorem 4.7. *Let (V, B) be a $TS(v, 2)$ with $v \equiv 0, 4 \pmod 6$ and $v \neq 4, 12$. Then there is a $TS((3v/2)+s, 4)$ faithfully enclosing (V, B) for all $s \equiv 0, 1 \pmod 3$ with $0 \leq s \leq v/2$.*

Proof. Let L be a latin square of order $v/2$ with orthogonal mate and based on $\{x_1, x_2, \dots, x_{\frac{v}{2}}\}$. Let $L_1 = (L + 1)$.

Let A_1 be the back circulant matrix with first row $(x_{\frac{v}{2}}, x_1, x_2, \ldots, x_{\frac{v-2}{2}})$ and let $A_2 = (A_1 + 1)$. Let A be the $v/2 \times v/2$ square based on $\{x_1, x_2, \ldots, x_{\frac{v}{2}}\}$ given by

$$A(i,j) = \begin{cases} A_1(i,j) & \text{if } 1 \leq i < j \leq v/2 \\ A_2(i,j) & \text{if } 1 \leq j < i \leq v/2. \end{cases}$$

The diagonal of A is left empty.

A symbol can appear once, twice or not at all in a row or column of A. If we consider row i and column i together, every element occurs exactly twice except for the symbols $A_1(i,i)$ and $A_2(i,i)$ which only occur once.

Define a $v \times v$ square D by setting

$$D(i,j) = \begin{cases} A(i,j) & \text{if } 1 \leq i,j \leq v/2 \\ A(i-(v/2), j-(v/2)) & \text{if } (v/2)+1 \leq i,j \leq v \\ L(i, j-(v/2)) & \text{if } 1 \leq i \leq v/2,\ (v/2)+1 \leq j \leq v \\ L_1(i-(v/2), j) & \text{if } (v/2)+1 \leq i \leq v,\ 1 \leq j \leq v/2. \end{cases}$$

The diagonal of D is left empty. Together in row i and column i, every symbol occurs four times except the elements $A_1(j,j)$ and $A_2(j,j)$ where $i = j$ or $i = j + (v/2)$. These symbols occur three times.

Let $V' = V \bigcup \{x_1, x_2, \ldots, x_{\frac{v}{2}}\}$ and define a collection of triples B' as follows:

- $B \subseteq B'$.
- If $D(i,j) = x_k$ then $\{i, j, x_k\} \in B'$.

For every $i = 1, 2, \ldots, v/2$, row i and column i of D together contain elements $A_1(i,i)$ and $A_2(i,i)$ exactly three times. Thus

- For $1 \leq i \leq v/2$, $\{i, A_1(i,i), A_2(i,i)\} \in B'$ and
  $\{i + (v/2), A_1(i,i), A_2(i,i)\} \in B'$.

All original edges have occurred in exactly four triples as have all cross edges. The set of new edges that have occurred in triples forms a 4-regular graph. If $v/2$ is odd it is two copies of a Hamilton cycle and if $v/2$ is even it is four copies of a perfect matching. Depending on whether $v/2$ is odd or even, let $(\{x_1, x_2, \ldots, x_{\frac{v}{2}}, \infty\}, C)$ be a $TS((v/2)+1, 2)$ such that $N(\infty)$ is the Hamilton cycle or two copies of the perfect matching. Such a system exists by Theorem 4.6. Let $C' = \{T \in C : \infty \notin T\}$ and

- Every triple in C' occurs twice in B'.

Every pair of elements of $\{x_1, x_2, \ldots, x_{\frac{v}{2}}\}$ occurs in exactly two triples of C' except for those pairs that appear as edges of the Hamilton cycle or matching. These pairs occur in exactly one triple if $v/2$ is odd and in no triple if $v/2$ is even. It follows that (V', B') is a $TS(3v/2, 2)$ faithfully enclosing (V, B).

The latin square L has a decomposition into transversals and this decomposition gives rise to v parallel classes in (V', B'). Employing a familiar argument these parallel classes are dismantled constructing $TS(\frac{3v}{2} + s, 4)$ that faithfully enclose (V, B) for $0 \leq s \leq v/2$ and $s \equiv 0, 1 \pmod 3$. With each new point added, **two** parallel classes are dismantled. $\square$

Several comments are needed. First, this proof does not construct simple enclosings as the fourth rule defining B' condemns this construction to have repeated blocks.

The proof of Theorem 4.7 fails for $v \in \{4, 12\}$ because pairs of mutually orthogonal latin squares don't exist for orders 2 or 6. Using hill-climbing again, we have found faithful enclosings of a $TS(4,2)$ in a $TS(6,4)$ and a $TS(7,4)$, and of a $TS(12,2)$ in a $TS(w,4)$ for $w \in \{18,19,21,22,24\}$; see [**1**] for explicit constructions. We describe the construction for $w = 18$ here, since numerous attempts using hill-climbing failed to produce an example. Take element set $Z_6 \times Z_3$. For $0 \leq i < j \leq 2$, place on $Z_6 \times \{i,j\}$ the triples of a $TS(12,2)$ having an independent set on $Z_6 \times \{i\}$; such a TS exists [**2**]. Then add the triples of a $TD(3,6)$ twice each, with groups aligned on $Z_6 \times \{i\}$, $i = 0,1,2$. The $TS(12,2)$ on $Z_6 \times \{0,1\}$ is then faithfully enclosed.

Hence we have:

Theorem 4.8. *A $TS(v,2)$ can be faithfully enclosed in a $TS(v+s,4)$ for $\lceil \frac{v-1}{2} \rceil \leq s \leq v$ whenever $v+s$ is 4-admissible.*

To prove the existence of $TS(v+s, 2\lambda)$ that faithfully enclose $TS(v,\lambda)$ for all values of λ we require one further result:

Lemma 4.9. *Let $v \equiv 0 \pmod 2$ and let (V,B) be a $TS(v,6)$. Then (V,B) can be faithfully enclosed in a $TS(\frac{3v}{2}+s, 12)$ for $0 \leq s \leq v/2$, $s \neq 2$.*

Proof. Let D be the $v \times v$ square constructed in the proof of Theorem 4.7. (There is no problem in constructing D when $v \equiv 2 \pmod 6$). Let D' be formed from D by repeating the entry in each cell three times. Let $V' = \{x_1, x_2, \ldots, x_{\frac{v}{2}}\}$ and define B' as follows:

- $B \subseteq B'$.
- If $D'(i,j) = \{x_k, x_k, x_k\}$ the triple $\{i, j, x_k\}$ is taken three times in B'.

Symbols $A_1(i,i)$ and $A_2(i,i)$ appear in row i and column i together exactly 9 times.

- For $1 \leq i \leq v/2$, $\{i, A_1(i,i), A_2(i,i)\}$ and $\{i+(v/2), A_1(i,i), A_2(i,i)\}$ are taken three times in B'.

As in the proof of Theorem 4.7 let $(\{x_1, x_2, \ldots, x_{\frac{v}{2}}, \infty\}, C)$ be a $TS((v/2)+1, 2)$ such that $N(\infty)$ is a Hamilton cycle or two copies of a perfect matching. Set $C' = \{T \in C : \infty \notin T\}$. Then

- Every triple in C' is taken six times in B'.

(V', B') is a $TS((3v)/2, 12)$ that faithfully encloses (V,B). By dismantling the parallel classes, we construct $TS(\frac{3v}{2}+s, 12)$ for $0 \leq s \leq v/2$, $s \neq 2$ that faithfully encloses (V,B). $\square$

Lemma 4.10. *For v even, a $TS(v,6)$ has a faithful enclosing in a $TS(v+\frac{v}{2}+2, 12)$.*

Proof. Let $v = 2x$, $x = 7$ or $x \geq 9$. We describe the solution on $(Z_x \times Z_3) \cup \{\alpha, \beta\}$. There is a $TS(2x, 6)$ having an independent set of size x [**2**]. For

$0 \le i < j \le 2$, place a copy of this TS on $Z_x \times \{i, j\}$ with the independent set on $Z_x \times \{i\}$. There is a collection of triples on $Z_x \times Z_3$ having the properties that (1) all pairs of the form $\{x_i, y_j\}$ appear in a triple exactly once unless $\{x, y\} \subseteq \{0, 1\}$, in which case the pair appears in no triple, and (2) there are two disjoint sets S_1, S_2 of triples that each partition $(Z_x \setminus \{0, 1\}) \times Z_3$ (this arises from incomplete self-orthogonal latin squares [**10**]).

Now add all triples of the partition **except** those in S_1, S_2 to the TS six times each. For each edge $\{x, y\}$ in a triple of S_1, add the triple $\{\alpha, x, y\}$ six times; and similarly for S_2, add $\{\beta, x, y\}$ six times. Now for each $x \in \{0, 1\} \times Z_3$, include the triple $\{x, \alpha, \beta\}$ twice. Then include the triples of a GDD of type 2^3 with groups on $\{0, 1\} \times \{i\}$, $i \in Z_3$, twice each. Finally include the triples of a GDD of type 2^4 with groups as above in addition to $\{\alpha, \beta\}$ ten times each.

It remains to treat the cases when $x \in \{2, 3, 4, 5, 6, 8\}$, and all are easily handled using hill-climbing. □

We have considered enclosings of $TS(v, \lambda)$ for $\lambda \in \{1, 2, 3, 6\}$; using Lemma 4.1, we extend the results to arbitrary λ, completing the proof of the Main Theorem.

5. Results for small orders

In this section, we give a small improvement in the necessary conditions (Lemmas 1.1 and 1.2). We then outline the hill-climbing method used to produce faithful enclosings, and describe our results for small orders.

First, we tighten Lemma 1.1. If a $TS(v, \lambda)$ is faithfully enclosed in a $TS(w, \mu)$, suppose that V is the element set of the enclosed TS, and S contains those $s = w - v$ points added in the enclosing. In an enclosing, there are μvs pairs occurring in triples that contain one element of V and one of S. Hence there are $\frac{\mu s v}{2}$ triples containing two of these cross edges. Of these triples, exactly $(\mu - \lambda)\binom{v}{2}$ are accounted for by the triples containing a pair of V (the enclosing is faithful). In addition, the largest number of triples that can appear entirely on S is determined by the number of triples in a maximum partial triple system of order s and index μ. In general, this does not exhaust all pairs on S, and some number $\psi_{s,\mu}$ of pairs remains. This function ψ is easily determined using the characterization of maximum partial triple systems (see, e.g., [**7**]):

Lemma 5.1. *If $s = 2$, $\psi_{s,\mu} = \mu$, and if $s = 1$, $\psi_{s,\mu} = 0$. Otherwise, if $\mu(s-1) \equiv 0 \pmod 2$,*

$$\psi_{s,\mu} = \begin{cases} 0 & \text{if } \mu s(s-1) \equiv 0 \pmod 6 \\ 4 & \text{if } \mu s(s-1) \equiv 2 \pmod 6 \\ 2 & \text{if } \mu s(s-1) \equiv 4 \pmod 6 \end{cases}$$

and if $\mu(s-1) \equiv 1 \pmod 2$, writing $\mu s(s-2) = 6x + 2r$ for $r \in \{0, 1, 2\}$, we have $\psi_{s,\mu} = \frac{s}{2} + r$.

Our earlier arguments establish the following improvement upon Lemma 1.1:

Lemma 5.2. *If a $TS(v, \lambda)$ is faithfully enclosed in a $TS(w, \mu)$, $\mu v(w - v) \geq (\mu - \lambda)v(v - 1) + 2\psi_{w-v,\mu}$.*

Lemma 5.2 eliminates some cases not found inadmissible by Lemma 1.1; for example, faithful enclosings of a $TS(7, 1)$ in a $TS(11, 3)$, or of a $TS(11, 3)$ in a $TS(15, 5)$, do not exist. It is interesting to remark in contrast that faithful enclosings of $TS(7, 2)$ in $TS(11, 6)$, and of $TS(11, 6)$ in $TS(15, 10)$, **do** exist.

In the proofs of the preceding three sections, we required a very large number of small enclosings. Hence we adopted a strategy of completeness and attempted to determine for all $1 \leq \lambda < \mu \leq 12$, $3 \leq v \leq 20$ and $v < w \leq 2v$ meeting the conditions of admissibility and Lemmas 1.2 and 5.1, whether a $TS(v, \lambda)$ can be faithfully enclosed in a $TS(w, \mu)$.

There are 1795 cases to consider. We therefore adapted Stinson's hill-climbing algorithm for generating triple systems (referred to frequently for small cases in the preceding sections). The modification of his basic algorithm to permit a different λ value for each edge is routine; we therefore set it to $\mu - \lambda$ for each pair of elements from $\{0, \ldots, v - 1\}$, and to μ for every other pair on $\{0, \ldots, w - 1\}$. Stinson's basic method constructs the triple system one triple at a time, by either adding a new triple (chosen randomly), or replacing a triple already present with another (randomly chosen) triple. It never allows the number of triples chosen thus far to decrease. We made a trivial modification, forcing any triple on the set $\{0, \ldots, v-1\}$ to be rejected out of hand as a candidate, and thereby forcing a new triple to be generated. This has the (very undesirable) effect of causing many more rejected triples at each step in the hill-climbing. We found, however, that except when λ is small compared with μ, the effect of asking for a faithful enclosing rather than an arbitrary enclosing was not observable for our small examples.

Remarkably, in each of the 1795 cases, hill-climbing produced the required enclosing, establishing that in the range examined the necessary conditions are in fact sufficient. In addition to the remarkable fact that such simple necessary conditions are sufficient for small orders with no exceptions, the result is a testament to the hill-climbing strategy itself.

On the basis of these computational results, it appears plausible that the necessary conditions in Lemmas 1.2 and 5.1 are sufficient; at the present time, however, our techniques do not suggest how to treat cases in general where $\mu >> \mu - \lambda$.

Acknowledgements. Research of the second author is supported by NSERC Canada under grant number A0579. The authors thank Alex Rosa and Paul Schellenberg for useful suggestions in the course of this research.

References

1. Bigelow D. C., *Enclosings of latin squares and triple systems*, Ph.D. thesis, Pure Mathematics, University of Waterloo, 1990.

2. Bigelow D. C. and Colbourn C. J., *Faithful enclosings of triple systems: A generalization of Stern's theorem*, Discrete Math. (to appear).

3. Colbourn C. J., Hamm R. C. and Rosa A., *Embedding, immersing and enclosing*, Congressus Numerantium **47** (1985), 229–236.

4. Colbourn C. J., Hoffman D. G. and Rees R.,, *A new class of group divisible designs with block size three*, J. Comb. Theory A (to appear).
5. Colbourn C. J. and Milici S., *Support sizes of triple systems with small index*, J. Comb. Math. Comb. Comput. **6** (1989), 155–161.
6. Colbourn C. J. and Rosa A., *Element neighbourhoods in twofold triple systems*, J. Geometry **30** (1987), 36–41.
7. ———, *Leaves, Excesses and Neighbourhoods in Triple Systems*, preprint, 1990.
8. Dehon M., *On the existence of* 2*-designs* $S_\lambda(2,3,v)$ *without repeated blocks*, Discrete Math. **43** (1983), 155–171.
9. Hanani H., *On resolvable balanced incomplete block designs*, J. Comb. Theory Ser. A **17** (1974), 275–289.
10. Heinrich K. and Zhu L., *Incomplete self-orthogonal latin squares*, J. Austral. Math. Soc. (A) **42** (1987), 365–384.
11. Ray-Chaudhuri D. K. and Wilson R. M., *Solution of Kirkman's school-girl problem*, Proc. Symp. Pure Math. 19, Amer. Math. Soc. Providence, pp. 187–203.
12. Rees R. and Stinson D. R., *On resolvable group divisible designs with block size three*, Ars Combinatoria **23** (1987), 107–120.
13. Stinson D. R., *Hill-climbing algorithms for the construction of combinatorial configurations*, Annals of Discrete Math. **26** (1985), 321–334.
14. Teirlinck L., *On making two Steiner triple systems disjoint*, J. Comb. Theory Ser. A **23** (1977), 349–350.

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