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THE BOUNDARY ELEMENT METHOD APPLIED TO THE DETERMINATION OF THE GLOBAL QUASIGEOID

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Abstract. The main task of geodesy is the determination of the size and shape and the gravity field of the Earth from terrestrial observations and also from satellite measurements in the last decades. The determination of the exterior gravity field of the Earth is usually formulated in terms of a boundary value problem (BVP) for the Laplace equation. Thanks to satellite geodesy (especially to the Global Positioning System (GPS)) it is possible to determine the geocentric position on the Earth's surface that represents the boundary surface for the geodetic BVP considered. The combination of gravimetric and satellite measurements on the Earth's surface allows to formulate Neumann's boundary condition that corresponds to gravity disturbances that are known on the Earth's surface in this case. This paper discusses an application of the boundary element method (BEM) to the numerical solution of the geodetic BVP with the boundary conditions of the Neumann type that originates in the global quasigeoid modelling. The result of the numerical solution of the global quasigeoid determination is presented.

1. Introduction. As is well known the fundamental differential equation that is satisfied by the disturbing potential T outside the Earth (neglecting the atmosphere) is the Laplace equation

$$\Delta T(\mathbf{x}) = 0 , \qquad \mathbf{x} \in R^3 - \Omega, \tag{1}$$

where Ω is a domain representing the body of the Earth. The disturbing potential is defined as the difference between the actual and the normal gravity potential

$$T(\mathbf{x}) = W(\mathbf{x}) - U(\mathbf{x}) , \qquad \mathbf{x} \in R^3, \tag{2}$$

where W is the actual and U the normal gravity potential. The normal gravity potential is generated by a normal body. To keep its average difference from the actual gravity potential as small as possible the normal body is defined as a geocentric equipotential ellipsoid of rotation that is completely determined by four parameters derived from the actual Earth:

- semimajor axis a
- geometrical flattening f
- geocentric gravitational constant GM
- spin angular velocity ω

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The normal potential U_o on the ellipsoidal surface is equal to the actual potential W_o on the geoid that approximates the mean sea level (Geodetic Reference System GRS-80). The same angular velocity of rotation ω generates also the same centrifugal potential for both bodies, the Earth and the equipotential ellipsoid. Therefore disturbing potential (2) is only the difference of gravitational components

$$T(\boldsymbol{x}) = V_g(\boldsymbol{x}) - U_g(\boldsymbol{x}) , \quad \boldsymbol{x} \in R^3, \tag{3}$$

where V_g is the actual and U_g the normal gravitational potential, and the potential equation (1) is valid.

In general the boundary conditions (BC) associated with the geodetic BVP has usually the form of the so-called fundamental equation of physical geodesy (see [7])

$$- \frac{\partial T(\mathbf{x})}{\partial \tau} + \frac{1}{\gamma_o(\mathbf{x})} \frac{\partial \gamma(\mathbf{x})}{\partial \tau} T(\boldsymbol{x}) = \Delta g(\boldsymbol{x}) , \qquad \boldsymbol{x} \in \Gamma, \tag{4}$$

where γ is the normal gravity, $\partial\tau$ is a partial derivative with respect to the isozenithal, Δg is the gravity anomaly and Γ is a boundary of the domain. In Stokes's concept the gravity anomalies and BC are defined on the geoid. In Molodenskij's concept the surface gravity anomalies and BC are defined on the telluroid. In both definitions the boundary surfaces are not Earth's surface, only its approximations.

Thanks to satellite measurements it is possible to determine geocentric position on the Earth's surface and we can define BC of the Neumann type in form of the gravity disturbances. Applying the gradient operator to (2) and neglecting the deflection of vertical (spatial angle smaller than 1' in mountains and 20" in lowlands (see [10])), we have

$$grad\ T(\boldsymbol{x}) \cong \boldsymbol{g}(\boldsymbol{x}) - \boldsymbol{\gamma}(\boldsymbol{x}) = \delta\boldsymbol{g}(\boldsymbol{x}),\ \boldsymbol{x} \in \Gamma, \tag{5}$$

where $\boldsymbol{g}$ and γ are the vectors of actual and normal gravity and $\delta\boldsymbol{g}$ is the gravity disturbance defined on the actual Earth's surface Γ. We can formulate the geodetic BVP

$$\Delta T(\boldsymbol{x}) = 0 , \qquad \boldsymbol{x} \in R^3 - \Omega , \tag{6a}$$

$$\langle \nabla T(\boldsymbol{x}) , \boldsymbol{n}_e(\boldsymbol{x}) \rangle = \langle \delta\boldsymbol{g}(\boldsymbol{x}) , \boldsymbol{n}_e(\boldsymbol{x}) \rangle , \qquad \boldsymbol{x} \in \Gamma, \tag{6b}$$

$$T(\boldsymbol{x}) \to 0 \ \ for\ \ \boldsymbol{x} \to \infty . \tag{6c}$$

where $\boldsymbol{n}_e$ is the ellipsoidal normal and $\langle\ ,\ \rangle$ represents the scalar product of vectors. Equations (6) represent the exterior oblique derivative BVP for the Laplace equation with Neumann's BC. The boundary surface Γ is the Earth's surface. The normal to the Earth's surface Γ doesn't coincide with the ellipsoidal normal $\boldsymbol{n}_e$.

We use boundary element method (BEM) for numerical solution of our problem because it is suitable for the exterior BVP. BEM is based on the weak formulation (see [2,9]) of the differential equation (6a) and on the discretization of the boundary surface by a triangulation. As a result we obtain an approximate solution of the formulated geodetic BVP.

2. The BEM application. The BEM variational formulation of our BVP for the Laplace equation is expressed in terms of a boundary integral equation for an unknown function u representing the disturbing potential T (see [1])

$$2\pi\alpha\, u(\mathbf{x}) + \int_\Gamma u(\mathbf{y})\, q^*(\mathbf{x},\mathbf{y})\, d\mathbf{y} = \int_\Gamma q(\mathbf{y})\, u^*(\mathbf{x},\mathbf{y})\, d\mathbf{y}\,, \quad \mathbf{x}\in\Gamma, \tag{7}$$

where α=1 for 2D and α=2 for 3D cases. The function q is the normal derivative of u and represents the gravity disturbances. Let us note that in a current implementation we consider the normal derivative to the Earth's surface instead of the oblique derivative given in (6b). The kernel function u^* is the fundamental solution of the Laplace equation, q^* is its normal derivative:

$$u^*(\mathbf{x},\mathbf{y}) = (4\pi\cdot l(\mathbf{x},\mathbf{y}))^{-1}\,,\ \mathbf{x},\mathbf{y}\in\Gamma, \tag{8}$$

where l is the distance. The boundary given by the Earth's surface is approximated by the triangulation of the topography. The vertices of the triangles represent nodes in which input boundary data are generated (the gravity disturbance δg and the node position in form of the ellipsoidal coordinates B, L, H). Assuming elements with constant basis functions we discretize the boundary integral equation (7) for the node i in the following form

$$c_i u_i + \sum_{j=1}^{N} \{\int_{\Gamma_j} (4\pi)^{-1}\cdot l_{ij}^{-2} d\Gamma\}\, u_j = \sum_{j=1}^{N} \{\int_{\Gamma_j} (4\pi\cdot l_{ij})^{-1} d\Gamma\}\, q_j\,, \qquad i = 1, \ldots, N, \tag{9}$$

where N is the number of nodes and Γ_j is the area of element j. The function c_i depends on the shape of the boundary surface. There is the interior angle in the node i delimited by neighbouring elements for 2D cases, analogously for 3D cases (see [1]). Computing integrals in (9) approximately using

$$M_{ij} = \frac{1}{4\pi} A_j \sum_{k=1}^{K} \frac{1}{l_{ik}^2} w_k\,, \qquad for\ i \neq j\,, \tag{10a}$$

$$M_{ij} = \frac{1}{4\pi} A_j \sum_{k=1}^{K} \frac{1}{l_{ik}^2} w_k + c_i\,, \quad for\ i = j\,, \tag{10b}$$

$$G_{ij} = \frac{1}{4\pi} A_j \sum_{k=1}^{K} \frac{1}{l_{ik}} w_k \,, \tag{10c}$$

where A_j is the area of element j and w_k are weights (see [1]), we get a system of linear equations

$$\boldsymbol{M} \cdot \boldsymbol{u} = \boldsymbol{f} \,, \tag{11}$$

where

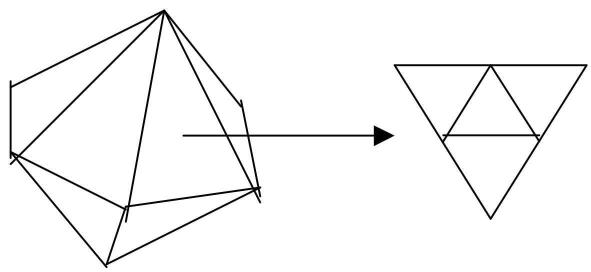
$$\boldsymbol{f} = \boldsymbol{G} \cdot \boldsymbol{q} \,. \tag{12}$$

Solving the system (11) we obtain values of the disturbing potential in the nodes on the Earth's surface. Using Bruns's formula we can transform disturbing potential in the nodes to the height anomalies, respectively to the quasigeoidal heights above the ellipsoid (see [7]). However there is a problem of unknown sea level heights. Thanks to a small value of gravity gradients we can overcome this problem in an iterative way writing symbolically

$$\zeta^{i+1}(B,L) = \frac{T(B,L,H)}{\gamma\left(B, H - \zeta^{i}\right)} \,, \tag{13}$$

where ζ is the quasigeoidal height above ellipsoid, T is the disturbing potential on the Earth's surface and γ is the normal gravity on the "iterative" telluroid. The normal gravity is computed by means of formulas of GRS-80. In the first iteration all the quasigeoidal heights are equal to zero. To obtain convergence it is sufficiently to use two or three iterations.

3. Numerical experiments. An application of BEM to the linearized Molodenskij's problem in 2D case has been given in [3]. This paper discusses a 3D BEM application to the BVP defined by (6). In our experiment we use the Earth geopotential model EGM96 to generate gravity disturbances in the nodes. The ellipsoidal coordinates determine the node positions. The ellipsoidal heights are combinations of sea level heights from terrain model and geoidal heights from EGM96. The geopotential model is formulated as the spherical harmonics series expansion of the geopotential with Stokes's geopotential coefficients (see [10]). Low frequency components are accurately obtained by satellite measurements while higher ones from gravimetry and altimetry. The definition of Earth geopotential models with known Stokes's geopotential coefficients allow to compute several geodetic quantities, e.g. disturbing potential, geoidal heights, gravity anomalies, gravity disturbances, etc. In our experiment we use program F477B (see [8]) for computing gravity disturbances and geoidal heights in the nodes.



a.) b.)

FIG.1

The triangulation of the topography is based on a subdivision of the triangular faces of a “12-hedron” (fig.1a). Each of the 12 faces is subdivided into 4 congruent subtriangles by halving the sides (fig.1b). This process is repeated until required level.

In our experiment the Earth’s surface is approximated by 1946 nodes and 3888 triangles. The computations have been done on PC with 250 MB internal memory using the software Mathematica 3.0. The results of the formulated BEM application compared with EGM96 are in the appendix (the profiles along the parallels of latitude).

4. Perspectives for physical geodesy. The results of the BEM application to the geodetic BVP show the evident correlation and agreement with geopotential model EGM96. Increasing number of nodes and local grid refinement, e.g. in areas of interest, as well as input of real measured data, we can achieve a more precise solution of the global quasigeoid and also its local determination. The main consequence of the new BC definition in form of the gravity disturbance is a simplification of the necessary geodetic measurements. That means the combination of GPS and gravimetry without economical demanding levelling. This contribution is evident in mountainous areas.

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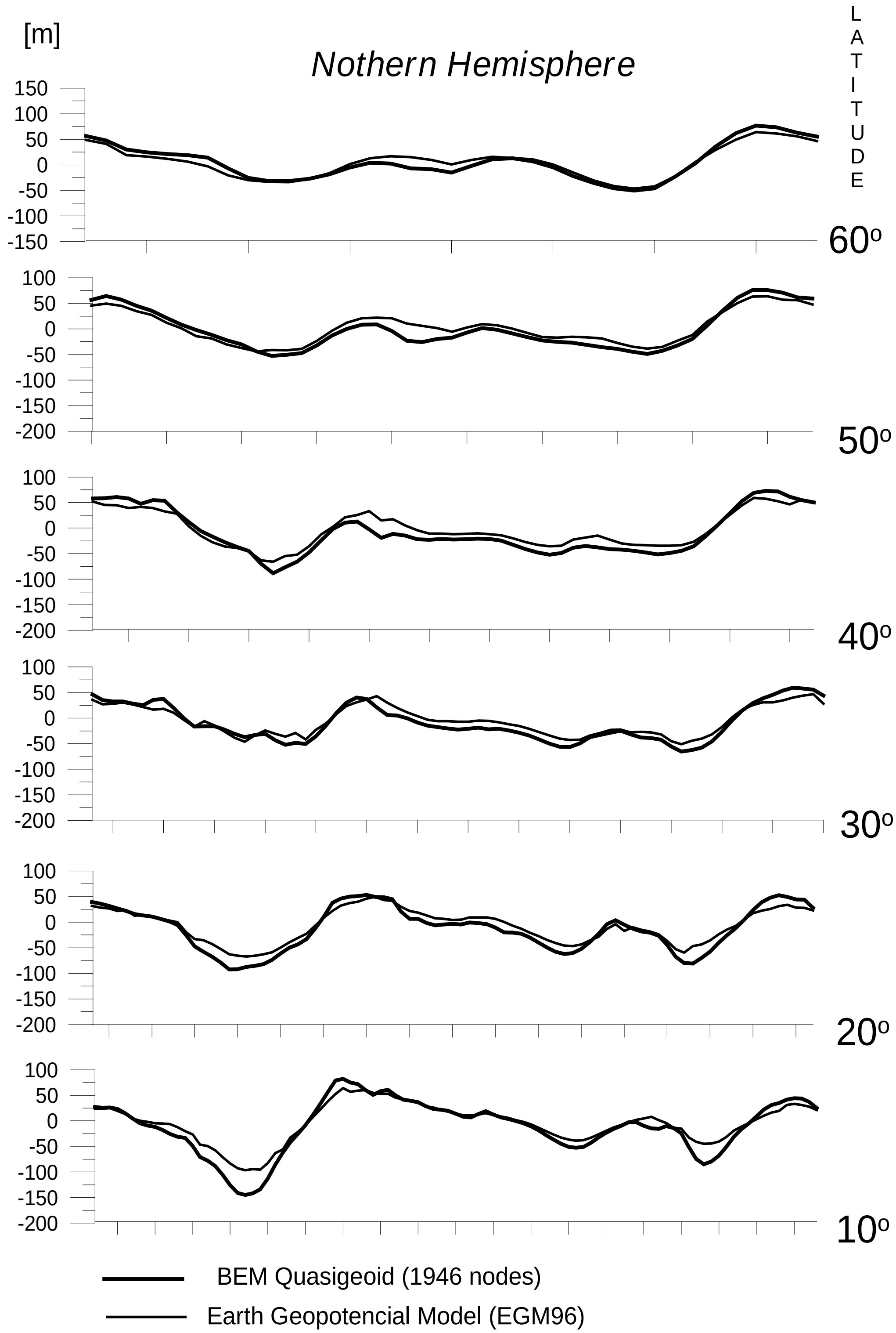
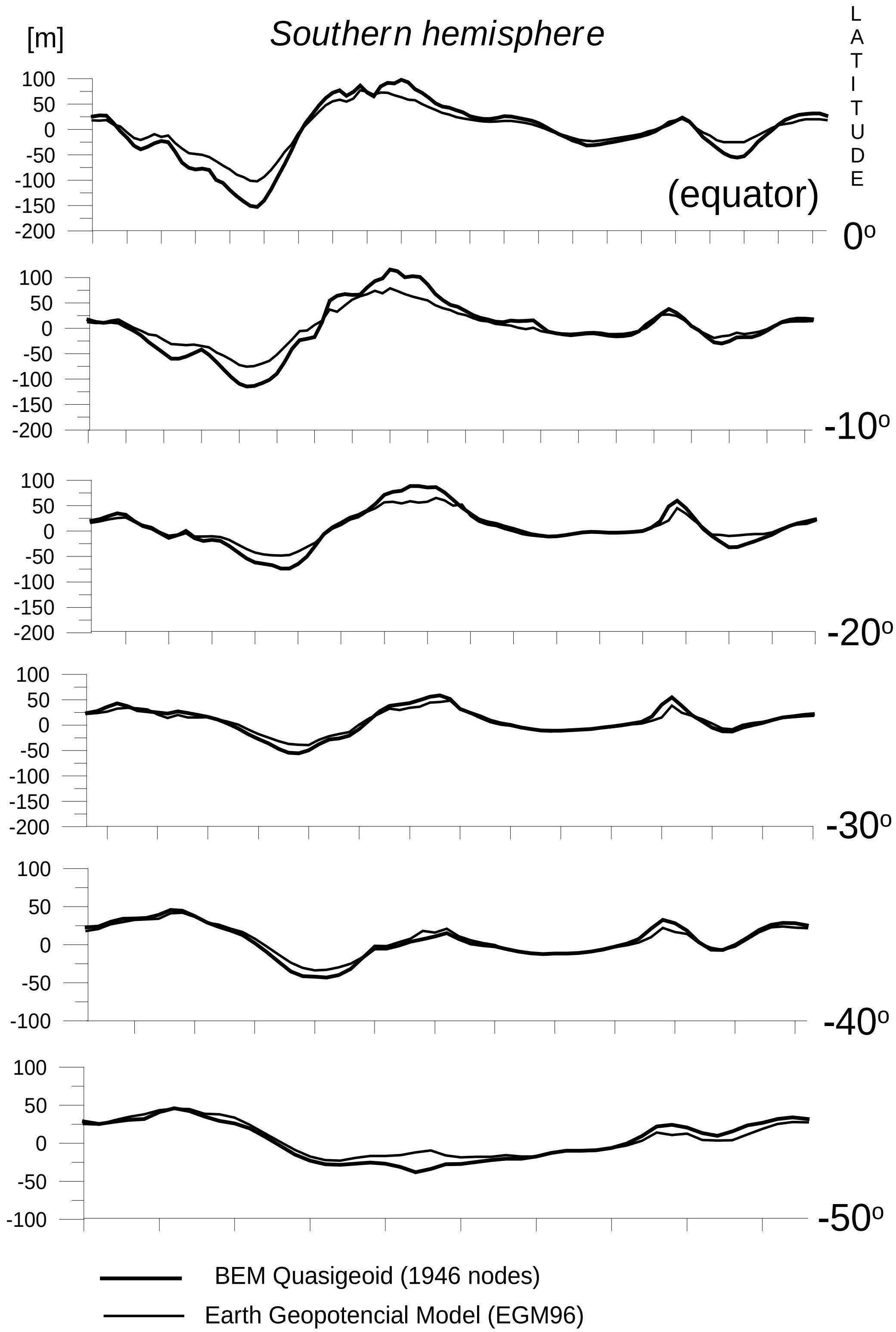
Appendix:

The BEM application to the geodetic BVP with boundary conditions of the Neumann type. The BEM quasigeoid (1946 nodes) is compared with the Earth geopotential model EGM96 (the profiles along the parallels).


[m]
Nothern Hemisphere
LATITUDE
150
100
50
0
-50
-100
-150
60º
100
50
0
-50
-100
-150
-200
50º
40º
30º
20º
10º
BEM Quasigeoid (1946 nodes)
Earth Geopotencial Model (EGM96)


Southern hemisphere
[m]
LATITUDE
(equator)
0o
-10o
-20o
-30o
-40o
-50o
100
50
0
-50
-100
-150
-200
BEM Quasigeoid (1946 nodes)
Earth Geopotencial Model (EGM96)