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A Common Fixed Point Theorem For Weakly Compatible Mappings*

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Abstract

The aim of this paper is to prove the existence of common fixed points for a pair of weakly compatible selfmaps satisfying weakly contractive condition and property (E. A).

1 Introduction

In 2002, Aamri and Moutawakil [1] introduced the notion of property (E. A). There are a number of results (Aliouche [2], Imdad *et al.* [5], Liu *et al.* [8], Pathak *et al.* [9]) that use this concept to prove existence results in common fixed point theory.

Throughout this paper, (X, d) denotes a metric space; and f and T are selfmaps of X.

DEFINITION 1.1. The pair (f, T) is said to

(i) be *compatible* (Jungck [6]) if $\lim_{n\to\infty} d(fTx_n, Tfx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} Tx_n = t$ for some t in X;

(ii) be *noncompatible* if there is at least one sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} Tx_n = t$, for some t in X, but $\lim_{n\to\infty} d(fTx_n, Tfx_n)$ is either non-zero or non-existent;

(iii) *satisfy property* (E. A) (Aamri and El Moutawakil [1]) if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} Tx_n = t$, for some t in X;

(iv) be *weakly compatible* (Jungck [7]) if $Tfx = fTx$ whenever $fx = Tx,\ x \in X$.

REMARK 1.2. Every pair of noncompatible selfmaps of a metric space (X, d) satisfies property (E. A), but its converse need not be true as shown by the following example.

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EXAMPLE 1.3. Let $X = [0, 1)$ with the usual metric. We define mappings f and T on X by

$$f(x) = \begin{cases} \frac{2}{3} & \text{if } 0 \le x < \frac{2}{3} \\ 1 - \frac{1}{2}x & \text{if } \frac{2}{3} \le x < 1 \end{cases} \quad \text{and} \quad T(x) = \begin{cases} \frac{2}{3} & \text{if } 0 \le x < \frac{2}{3} \\ \frac{4}{3} - x & \text{if } \frac{2}{3} \le x < 1 \end{cases}.$$

Then the pair (f, T) is compatible on X, for if $\{x_n\}$ is a sequence in $[0, 1)$ with $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} Tx_n = z \in X$, then $z = \frac{2}{3}$ and $\lim_{n\to\infty} Tfx_n = \lim_{n\to\infty} fTx_n = \frac{2}{3}$ so that $\lim_{n\to\infty} d(Tfx_n, fTx_n) = 0$. Hence the pair (f, T) is not noncompatible on X. We observe that (f, T) satisfies property (E, A).

REMARK 1.4. (i) Weak compatibility and property (E. A) are independent to each other (Pathak *et al.* [9]). (ii) Every compatible pair is weakly compatible but its converse need not be true (Jungck *et al.* [7]).

Throughout this paper, we denote $R_+ = [0, \infty)$; and N, the set of all natural numbers, and

$$\Phi = \{\varphi |\ \varphi : R_+ \to R_+ \text{ is continuous, } \varphi(0) = 0,\ \varphi(t) > 0 \text{ for } t > 0\}.$$

In 2007, Beg and Abbas [3] established the following existence theorem of common fixed points of a pair of selfmaps.

THEOREM 1.5. (Beg and Abbas [3], Theorem 2.5). Let (X, d) be a metric space and let $T,\ f : X \to X$ be weakly compatible selfmaps. Assume that there exists a monotone increasing $\varphi \in \Phi$ with $\lim_{t\to\infty} \varphi(t) = \infty$ such that for all $x, y \in X$,

$$d(Tx, Ty) \le d(fx, fy) - \varphi(d(fx, fy)).$$

If $T(X) \subseteq f(X)$ and $f(X)$ is a complete subspace of X, then f and T have a unique fixed point in X.

DEFINITION 1.6. A selfmap $T : X \to X$ is said to be *weakly contractive with respect to a selfmap* $f : X \to X$ if there exists a $\varphi \in \Phi$ such that for all $x, y \in X$,

$$d(Tx, Ty) \le d(fx, fy) - \varphi(d(fx, fy)). \tag{1}$$

The aim of this paper is to give a modified version of Theorem 1.5 by relaxing the conditions 'φ is monotonically increasing and $\lim_{t\to\infty} \varphi(t) = \infty$'. Further we prove the existence of common fixed points for a pair of weakly compatible selfmaps satisfying weakly contractive condition and property (E. A).

2 A Modified Version Of Beg and Abbas Theorem

The following theorem suggests that the conditions 'φ is monotone increasing and $\lim_{t\to\infty} \varphi(t) = \infty$' of Theorem 1.5 are redundant.

THEOREM 2.1. Let $T,\ f : X \to X$ be weakly compatible selfmaps. If T is weakly contractive with respect to f such that $T(X) \subseteq f(X)$ and $f(X)$ is a complete subspace of X, then f and T have a unique common fixed point in X.

PROOF. Let $x_0 \in X$. Since $T(X) \subseteq f(X)$, there exists $x_1 \in X$ such that $Tx_0 = fx_1$. On continuing this process, inductively we get a sequence $\{x_n\}$ in X such that $y_n = T(x_n) = f(x_{n+1})$.

We now show that the sequence $\{d(fx_n, fx_{n+1})\}$ is a decreasing sequence. Now consider,

$$d(fx_{n+1}, fx_{n+2}) = d(T(x_n), T(x_{n+1})) \leq d(f(x_n), f(x_{n+1})) - \varphi(d(f(x_n), f(x_{n+1})). \tag{2}$$

Hence,

$$d(fx_{n+1}, fx_{n+2}) \leq d(f(x_n), f(x_{n+1})) \text{ for all } n = 0, 1, 2, ... \tag{3}$$

Hence the sequence $\{d(f(x_n), f(x_{n+1}))\}$ is a decreasing sequence of non-negative reals and converges to a limit l (say) and $l \geq 0$.

We claim that $l = 0$. Suppose $l > 0$. Letting $n \to \infty$ in (2), by the continuity of φ, we get $l \leq l - \varphi(l)$, a contradiction. Hence, $l = 0$. i.e.,

$$\lim_{n\to\infty} d(f(x_n), f(x_{n+1})) = 0. \tag{4}$$

We now claim that $\{y_n\}$ is Cauchy. By (3) and (4), it is sufficient to show that $\{y_{2n}\}$ is Cauchy. Otherwise, there exists an $\varepsilon > 0$ and there exist sequences $\{m_k\}$ and $\{n_k\}$ with $m_k > n_k > k$ such that

$$d(y_{2m_k}, y_{2n_k}) \geq \varepsilon \quad \text{and} \quad d(y_{2m_k-2}, y_{2n_k}) < \varepsilon. \tag{5}$$

Hence,

$$\varepsilon \leq \liminf_{k\to\infty} d(y_{2m_k}, y_{2n_k}). \tag{6}$$

For each positive integer k, by the triangle inequality we have,

$$d(y_{2m_k}, y_{2n_k}) \leq d(y_{2m_k}, y_{2m_k-1}) + d(y_{2m_k-1}, y_{2m_k-2}) + d(y_{2m_k-2}, y_{2n_k}).$$

On taking limit supremum of both sides, as $k \to \infty$, we get

$$\limsup_{k\to\infty} d(y_{2m_k}, y_{2n_k}) \leq \varepsilon. \tag{7}$$

Hence, from (6) and (7), we have

$$\lim_{k\to\infty} d(y_{2m_k}, y_{2n_k}) = \varepsilon. \tag{8}$$

Now

$$d(y_{2m_k}, y_{2n_k+1}) \leq d(y_{2m_k}, y_{2n_k}) + d(y_{2n_k}, y_{2n_k+1}).$$

On taking limit supremum, as $k \to \infty$, we get

$$\limsup_{k\to\infty} d(y_{2m_k}, y_{2n_k+1}) \leq \varepsilon. \tag{9}$$

Again we have

$$d(y_{2m_k}, y_{2n_k}) \leq d(y_{2m_k}, y_{2n_k+1}) + d(y_{2n_k+1}, y_{2n_k}).$$

On taking limit infimum, as $k \to \infty$, we get

$$\varepsilon \leq \liminf_{k\to\infty} d(y_{2m_k}, y_{2n_k+1}). \tag{10}$$

From (9) and (10), we have

$$\lim_{k\to\infty} d(y_{2m_k}, y_{2n_k+1}) = \varepsilon. \tag{11}$$

Similarly, we can show that

$$\lim_{k\to\infty} d(y_{2m_k-1}, y_{2n_k}) = \varepsilon. \tag{12}$$

Now consider,

$$\begin{aligned} d(y_{2m_k}, y_{2n_k+1}) &= d(Tx_{2m_k}, Tx_{2n_k+1}) \\ &\leq d(fx_{2m_k}, fx_{2n_k+1}) - \varphi(d(fx_{2m_k}, fx_{2n_k+1})) \\ &\leq d(Tx_{2m_k-1}, Tx_{2n_k}) - \varphi(d(Tx_{2m_k-1}, Tx_{2n_k})) \\ &\leq d(y_{2m_k-1}, y_{2n_k}) - \varphi(d(y_{2m_k-1}, y_{2n_k})). \end{aligned} \tag{13}$$

Letting $k \to \infty$ in (13), using (11), (12) and the continuity of φ, we get $\varepsilon \leq \varepsilon - \varphi(\varepsilon)$, a contradiction. Hence, $\{y_{2n}\}$ is Cauchy so that $\{y_n\}$ is a Cauchy sequence in X. Thus $\{fx_{n+1}\}$ is a Cauchy sequence in X. Since $f(X)$ is complete and $\{fx_{n+1}\} \subset f(X)$, we have

$$\lim_{n\to\infty} fx_{n+1} = fu, \text{ for some } u \in X. \tag{14}$$

Next we claim that $T(u) = f(u)$. Now consider,

$$d(fx_{n+1}, Tu) = d(Tx_n, Tu) \leq d(fx_n, fu) - \varphi(d(fx_n, fu)). \tag{15}$$

Letting $n \to \infty$, from (14) using (15) and the continuity of φ, we get

$$d(fu, Tu) \leq d(fu, fu) - \varphi(d(fu, fu)) = 0.$$

Hence, $Tu = fu = z$ (say). Since the pair of maps (f, T) is weakly compatible, we have $Tfu = fTu$ and hence $Tz = fz$.

We now claim that $Tz = z$. Suppose $Tz \neq z$. Consider

$$d(Tz, z) = d(Tz, Tu) \leq d(fz, fu) - \varphi(d(fz, fu)) = d(Tz, z) - \varphi(d(Tz, z)),$$

a contradiction. Hence, $Tz = z$. Hence, $fz = Tz = z$. The uniqueness of z follows from the weakly contractive nature of T. Hence, the theorem follows.

The following is an example in support of our Theorem 2.1.

EXAMPLE 2.2. Let $X = R_+$ with the usual metric. We define mappings f and T on X by

$$f(x) = \begin{cases} \frac{1}{3} & \text{if } 0 \leq x < \frac{2}{3} \\ \frac{2}{3} & \text{if } x = \frac{2}{3} \\ \frac{5}{6} & \text{if } x > \frac{2}{3} \end{cases} \quad \text{and} \quad T(x) = \begin{cases} \frac{5}{6} & \text{if } 0 \leq x < \frac{2}{3} \\ \frac{2}{3} & \text{if } x \geq \frac{2}{3} \end{cases}.$$

Clearly, the pairs (f, T) is weakly compatible, $T(X) \subseteq f(X)$ and $f(X) = \{\frac{1}{3}, \frac{2}{3}, \frac{5}{6}\}$ is complete.

We define $\varphi : R_+ \to R_+$ by

$$\varphi(t) = \begin{cases} \frac{3}{2}t^2 & \text{if } 0 \leq t \leq \frac{1}{3} \\ \frac{2}{9(1+t)} & \text{if } t \geq \frac{1}{3} \end{cases} .$$

Clearly $\varphi \in \Phi$. With this φ, T is weakly contractive with respect to f. Hence, f and T satisfy all the conditions of Theorem 2.1 and $\frac{2}{3}$ is the unique common fixed point of f and T. Here φ is not monotonically increasing on R_+, and $\lim\limits_{n\to\infty} \varphi(t) \neq \infty$ so that Theorem 1.5 is not applicable.

3 Main Result

Our main result is the following.

THEOREM 3.1. Let (X, d) be a metric space and let $T,\ f : X \to X$ be weakly compatible selfmaps satisfying property (E. A). Assume that T is weakly contractive with respect to f. If $f(X)$ is closed, then f and T have a unique common fixed point in X.

PROOF. Since the pair (f, T) satisfies property (E. A), there exists a sequence $\{x_n\}$ in X such that $\lim\limits_{n\to\infty} fx_n = \lim\limits_{n\to\infty} Tx_n = z$, for some z in X. Since $f(X)$ is closed, $z = f(u)$ for some $u \in X$. Now replacing u for x and x_n for y in (1), we get

$$d(Tu, Tx_n) \leq d(fu, fx_n) - \varphi(d(fu, fx_n)). \tag{16}$$

Letting $n \to \infty$ in (16), by the continuity of φ, we get

$$d(Tu, z) \leq d(fu, z) - \varphi(d(fu, z)) = 0.$$

Hence, $Tu = z$, and

$$fu = Tu = z. \tag{17}$$

Since f and T are weakly compatible, from (17), we have $fz = Tz$. If $z \neq Tz$, then from the inequality (1), we have

$$d(Tz, z) = d(Tz, Tu) \leq d(fz, fu) - \varphi(d(fz, fu)) = d(Tz, Tu) - \varphi(d(Tz, Tu)),$$

a contradiction. Hence, $Tz = z$, and $fz = Tz = z$. The uniqueness of z follows from the inequality (1). This complete the proof the Theorem.

Since two noncompatible selfmaps of a metric space (X, d) satisfy the property (E. A), we get the following corollary.

COROLLARY 3.2. Let (X, d) be a metric space and let $f, T : X \to X$ be noncompatible and weakly compatible selfmaps. Assume that T is weakly contractive with respect to f. If $f(X)$ is closed, then f and T have a unique common fixed point in X.

In Theorem 3.1, if we choose $\varphi \in \Phi$ such that $\varphi(t) = t - \psi(t)$, where $\psi \in \Phi$ with $\psi(t) < t$ for $t > 0$, we get the following Corollary.

COROLLARY 3.3. Let (X, d) be a metric space and let $T,\ f : X \to X$ be weakly compatible selfmaps satisfying property (E. A). Assume that there exists a $\psi \in \Phi$ with $\psi(t) < t$ for $t > 0$ such that

$$d(Tx, Ty) \leq \psi(d(fx, fy)) \text{ for all } x, y \in X. \tag{18}$$

If $f(X)$ is closed, then f and T have a unique common fixed point in X.

In this case, when T and f satisfy the inequality (18), we say that T *is a Boyd-Wong type contraction with respect to* f [4].

The following is an example in support of Theorem 3.1.

EXAMPLE 3.4. Let $X = [\frac{1}{2}, 1]$ with the usual metric. We define mappings f and T on X by

$$f(x) = \begin{cases} 1 & \text{if } \frac{1}{2} \leq x < \frac{2}{3} \\ x & \text{if } \frac{2}{3} \leq x \leq 1 \end{cases} \quad \text{and} \quad T(x) = \begin{cases} \frac{1}{2} & \text{if } \frac{1}{2} \leq x < \frac{2}{3} \\ 1 - \frac{1}{2}x & \text{if } \frac{2}{3} \leq x \leq 1 \end{cases}.$$

Since the sequence $\{x_n\}$, $x_n = \frac{2}{3} + \frac{1}{n}$, $n \geq 4$, in X with $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} Tx_n = \frac{2}{3}$, the pair (f, T) satisfy property (E. A). Clearly, the pairs (f, T) is weakly compatible and $f(X) = [\frac{2}{3}, 1]$ is closed. We define $\varphi : R_+ \to R_+$ by

$$\varphi(t) = \begin{cases} \frac{3}{2}t^2 & \text{if } 0 \leq t \leq \frac{2}{3} \\ \frac{10}{9(1+t)} & \text{if } t \geq \frac{2}{3} \end{cases}.$$

Clearly $\varphi \in \Phi$. With this φ, T is weakly contractive with respect to f. Hence, f and T satisfy all conditions of Theorem 3.1 and $\frac{2}{3}$ is the unique common fixed point of f and T. Further we mention that the pair (f, T) is not compatible, for $\lim_{n\to\infty} d(fTx_n, Tfx_n) = \frac{1}{6} \neq 0$.

Here we observe that neither $T(X) \subseteq f(X)$ nor $f(X) \subseteq T(X)$, and φ is not monotonically increasing on R_+, so that Theorem 1.5 and Theorem 2.1 are not applicable.

EXAMPLE 3.5. Let l_∞ be the set of all bounded nonnegative real numerical sequences $\{x_n\}$. We define metric d on l_∞ by $d(x, y) = \sup\{|x_n - y_n| : n \in N\}$, where $x = \{x_n\}$ and $y = \{y_n\}$ in l_∞. Then (l_∞, d) is a complete metric space. We define $T : l_\infty \to l_\infty$ by $T(\{x_n\}) = \{\frac{x_n}{1+x_n}\}$ and $f = I$, the identity map on l_∞. Let $x = \{x_n\}, y = \{y_n\} \in l_\infty$. Then

$$\begin{aligned} d(Tx, Ty) &= d\left(\left\{\frac{x_n}{1+x_n}\right\}, \left\{\frac{y_n}{1+y_n}\right\}\right) \\ &= \sup\left\{\left|\frac{x_n}{1+x_n} - \frac{y_n}{1+y_n}\right| : n \in N\right\} \\ &\leq \sup\left\{\frac{|x_n - y_n|}{1 + |x_n - y_n|} : n \in N\right\} \\ &\leq \frac{\sup\{|x_n - y_n| : n \in N\}}{1 + \sup\{|x_n - y_n| : n \in N\}} \\ &= d(\{x_n\}, \{y_n\}) - \varphi(d(\{x_n\}, \{y_n\})) \\ &= d(x, y) - \varphi(d(x, y)), \end{aligned}$$

where $\varphi(t) = \frac{t^2}{1+t}$ for $t \geq 0$. Thus T is a weakly contractive map with respect to f.

Clearly, f and T satisfy property (E. A), by choosing the sequence $\{x_n\}, x_n = (0, 0, ...) \in l_\infty$ for all $n = 1, 2, ...$ Since the null vector is the only coincidence point of f and T, we have f and T are weakly compatible on l_∞. Further, $f(l_\infty) = l_\infty$ is closed. Hence, f and T satisfy all conditions of Theorem 3.1 and the null vector is the unique common fixed point of f and T in l_∞.

We observe that T is not a contraction on l_∞, for taking $x = (0, 0, ...)$, for all $k \in (0, 1)$ there exists $y = (t, 0, 0, ...) \in l_\infty$, where $t \in (0, \frac{1-k}{k})$, such that $d(Tx, Ty) = \frac{t}{1+t} > k\ t = k\ d(x, y)$.

COROLLARY 3.6. Let K be a nonempty closed subset of a metric space (X, d) and $T : X \to X$. Assume that there exists a $\varphi \in \Phi$ such that

$$d(Tx, Ty) \leq d(x, y) - \varphi(d(x, y)),$$

for all $x, y \in X$. If there exists a sequence $\{x_n\}$ in K such that $\lim\limits_{n\to\infty} Tx_n = \lim\limits_{n\to\infty} x_n = z,\ z \in K$, then z is the fixed point of T in K.

The following is an example in support of Corollary 3.6.

EXAMPLE 3.7. Let $X = R_+$ with the usual metric and $K = \{0\} \cup [\frac{1}{3}, 1]$. We define a mapping T on K by

$$Tx = \begin{cases} \frac{1}{3} & \text{if} \quad x = 0 \\ \frac{1}{3} + \frac{1}{2}x & \text{if } \frac{1}{3} \leq x \leq 1 \end{cases}.$$

Then T satisfies all the conditions of Corollary 3.6 with $\varphi : R_+ \to R_+$ defined by $\varphi(t) = \frac{t^2}{1+t},\ t \geq 0$ and $\frac{2}{3}$ is the unique fixed point of T.

We observe that the sequence $\{x_n\}$, $x_n = \frac{2}{3} + \frac{1}{n},\ n \geq 4$, is in K with $\lim\limits_{n\to\infty} Tx_n = \lim\limits_{n\to\infty} x_n = \frac{2}{3}$.

REMARK 3.8. If we delete the condition '$f(X)$ is closed' from Theorem 2.1, then the maps f and T may have no common fixed points, which is shown by the following example.

EXAMPLE 3.9. Let $X = [\frac{1}{2}, 1]$ with the usual metric. We define mappings f and T on X by

$$f(x) = \begin{cases} 1 & \text{if } \frac{1}{2} \leq x \leq \frac{2}{3} \\ x & \text{if } \frac{2}{3} < x \leq 1 \end{cases} \quad \text{and} \quad T(x) = \begin{cases} \frac{1}{2} & \text{if } \frac{1}{2} \leq x \leq \frac{2}{3} \\ 1 - \frac{1}{2}x & \text{if } \frac{2}{3} < x \leq 1 \end{cases}.$$

Since the sequence $\{x_n\}$, $x_n = \frac{2}{3} + \frac{1}{n},\ n \geq 4$, in X with $\lim\limits_{n\to\infty} fx_n = \lim\limits_{n\to\infty} Tx_n = \frac{2}{3}$, the pair (f, T) satisfies property (E. A). Clearly, the pair (f, T) is weakly compatible. We define $\varphi : R_+ \to R_+$ by $\varphi(t) = \frac{3}{2}t^2,\ t \geq 0$. Clearly $\varphi \in \Phi$. With this φ, T is weakly contractive with respect to f. But $f(X) = (\frac{2}{3}, 1]$ is not closed. We observe that f and T have no common fixed points.

REMARK 3.10. In Theorem 3.1, if we relax the condition 'f and T satisfy property (E. A)' then they may have no common fixed points.

EXAMPLE 3.11. Let $X = [\frac{1}{2}, 1]$ with the usual metric. We define mappings f and T on X by

$$f(x) = \begin{cases} 1 & \text{if } \frac{1}{2} \le x \le \frac{2}{3} \\ \frac{2}{3} & \text{if } \frac{2}{3} < x \le 1 \end{cases} \quad \text{and} \quad T(x) = \begin{cases} \frac{2}{3} & \text{if } \frac{1}{2} \le x \le \frac{2}{3} \\ \frac{1}{2} & \text{if } \frac{2}{3} < x \le 1 \end{cases}.$$

Here f and T are trivially weakly compatible on X and $f(X) = \{\frac{2}{3}, 1\}$ is closed. Further, T is weakly contractive with respect to f with $\varphi(t) = \frac{1}{2}t$, $t \ge 0$. But f and T do not satisfy property (E. A), since for any sequence $\{x_n\}$ in X we have $\lim_{n\to\infty} fx_n \neq \lim_{n\to\infty} Tx_n$ in X. We observe that f and T have no common fixed points.

REMARK 3.12. In Theorem 1.5, the authors assumed the condition $T(X) \subseteq f(X)$, where as in the results of this paper, this condition is relaxed by imposing the condition property (E. A).

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