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Spanning Trees With Leaves Bounded By Independence Number*

Ling-li Sun†

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Abstract

In [2], Rahman and Kaykobad present an algorithm to construct a spanning tree with degree bounded by independence number. In this paper, we prove that every connected graph G has a spanning tree with at most $\alpha(G)(\alpha(G) \geq 2)$ leaves (where $\alpha(G)$ denotes the independence number of G), which yields an algorithm to obtain a spanning tree with not only degree but also leaves bounded by independence number.

The graphs considered in this paper are finite, undirected, and simple (no loops or multiple edges). The sets of vertices and edges of a graph G are denoted by $V(G)$ and $E(G)$, respectively. We denote the maximum degree of vertices of G by $\Delta(G)$. A connected acyclic graph is called a *tree*. A vertex with degree one in a tree is called a *leaf*. A vertex with degree at least 3 in a tree is called a *branch vertex*. If T is a tree, we denote the set of all leaves of T by $L(T)$. If vertices u and v are connected in G, the *distance* between u and v in G, denoted by $d_G(u, v)$, is the length of a shortest (u, v)-path in G; if there is no path connecting u and v we define $d_G(u, v)$ to be infinite. Let C be a cycle and x, y, z be different vertices in C, we denote the path from x to z passing through y on C by $C[x, y, z]$. For terminology and notation not defined in this paper the reader is referred to [1].

Rahman and Kaykobad proved the following theorem in [2].

THEOREM 1. Every connected graph has a spanning tree with degree bounded by its independence number. Furthermore, the corresponding spanning tree can be computed in $O(n^2)$ worst case time where n is the number of vertices of the graph.

In this paper, we present a similar result like THEOREM 1. In particular, we prove the existence of spanning trees in every connected graph with number of leaves bounded by the independence number. We also prove that existence of spanning tree with bounded number of leaves would imply the existence of spanning tree with bounded degrees. In this sense, our result is stronger than that of [2]. Finally we devise an algorithm to find the corresponding spanning tree.

We first prove the core theorem of this paper.

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†Institute of Systems Science, Chinese Academy of Sciences, Beijing 100080, P. R. China and Department of Mathematics and Information Sciences, Huazhong Agricultural University, Wuhan 430070, P. R. China

THEOREM 2. Every connected graph G has a spanning tree with at most $\alpha(G)$ ($\alpha(G) \geq 2$) leaves.

PROOF. Let T be a spanning tree of G with fewest leaves. We may assume that T has t leaves $v_1, v_2, \ldots, v_t (t \geq 2)$. Since T is a spanning tree with fewest leaves, we have that $\{v_1, v_2, \ldots, v_t\}$ is an independent set. Otherwise if there exists $v_i v_j \in E(G) (1 \leq i, j \leq t)$, let u_i be the branch vertex of T such that $d_T(u_i, v_i)$ is minimum, P_i be the path from u_i to v_i in T and e_i be the edge incident with u_i in P_i, then $T' = T + \{v_i v_j\} - e_i$ is a spanning tree with fewer leaves than T, contradiction. Therefore $t \leq \alpha(G)$. This completes the proof.

The following proposition is Exercise 2.1.6 in [1].

PROPOSITION 3. If G is a tree with maximum degree $\Delta(G)$, then G has at least $\Delta(G)$ vertices of degree one.

THEOREM 4. Every connected graph G has a spanning tree with both degree and leaves bounded by $\alpha(G)$ ($\alpha(G) \geq 2$).

PROOF. By THEOREM 2, we have that G has a spanning tree T with at most $\alpha(G) (\alpha(G) \geq 2)$ leaves, i.e., $|L(T)| \leq \alpha(G)$. Furthermore, we have that $\Delta(T) \leq \alpha(G)$. Otherwise if there exists a vertex $v \in T$ such that $d_T(v) > \alpha(G)$, then by PROPOSITION 3, $|L(T)| \geq \Delta(T) \geq d_T(v) > \alpha(G)$, contradiction. This completes the proof.

By Theorem 2 and the proof of Theorem 4, we only need to find a spanning tree of G with at most $\alpha(G)$ leaves. In this section we outline a simple algorithm for constructing a spanning tree with at most d leaves from an input graph G with $\alpha(G) = d$. We also present a simple worst case analysis of the algorithm and compare our algorithm with the algorithm in [2].

ALGORITHM.
// Input: A connected graph $G = (V, E)$ with $\alpha(G) = d$.
// Output: A spanning tree T with at most d leaves.
Begin
1. Construct an arbitrary spanning tree T of G.
2. Form the set $L(T)$.
3. *if* $|L(T)| \leq d$, *then*
4. *return* T
5. *else*
6. *while* $|L(T)| > d$ *do*
7. Find a pair of vertices $x, y \in L(T)$ such that $(x, y) \in E(G)$
8. Let C be the cycle yielded in $T + (x, y)$ and $(u, v) \in E(C[u, x, y])$(where u is the branch vertex of T such that $d_T(x, u)$ is minimum)
9. $T = T + (x, y) - (u, v)$
10. $L(T) = L(T) - x - y + v$
11. *end while*
12. *end if*
13. *return* T

End

ANALYSIS: Let $|V| = n$ and $|E| = m$. Constructing an arbitrary spanning tree takes $O(m)$ computational effort. Forming the set $L(T)$ at most takes $O(n)$. To find out how many operations are actually done in the worst case in the while loop of line #6 to line #11, we just need to realize that there can be $O(n^2)$ pairs of x, y to be checked in line #7 in the worst case. So the overall running time is $O(n^2)$.

Our algorithm comparing with the algorithm in [2] reduces the steps forming the set of vertices with $d_T(v) > d$ and finds a spanning tree with both degree and leaves bounded by d. Thus we get the following theorem.

THEOREM 5. Given a connected graph $G = (V, E)$ with $\alpha(G) = d(d \geq 2)$, we can find a spanning tree with both degree and leaves bounded by d in $O(n^2)$ computational effort, where $n = |V(G)|$.

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