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# ON GARCIA NUMBERS

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Abstract. We study real algebraic integers larger than 1, whose norm is $\pm 2$ and all of whose conjugates have modulus larger than 1. These numbers are characterized in some easy cases.

## 1. Introduction

Some important kinds of algebraic integers are usually characterized by specifying the size of their conjugates, e.g., Pisot, Salem or Perron numbers. Here we are concerned with a similarly defined class of algebraic integers. Let us fix $d, m \in \mathbb{N}, m \geq 2$. We denote by $\mathcal{G}_d(m)$ the (finite) set of all positive real algebraic integers $\alpha$ of degree $d$ with $|\alpha_i| > 1$ and $\prod_{i=1}^{d} |\alpha_i| = m$, where $\alpha_1 = \alpha, \alpha_2, \ldots, \alpha_d$ are the conjugates of $\alpha$. The interest in this class of algebraic numbers primarily stems from the case $m = 2$ which seems to have first appeared in a publication of D.-J. Feng [9], where the following terminology occurred.

**Definition 1.1** ([9])**.** An algebraic integer is called a Garcia number if it is real and larger than 1, if all conjugates have modulus larger than 1 and if its norm has modulus 2.

We denote by $\mathcal{P}_d(m)$ the set of minimal polynomials $P_\alpha$ of $\alpha \in \mathcal{G}_d(m)$. In view of the main goal of this note we shortly write $\mathcal{P}_d$ instead of $\mathcal{P}_d(2)$. Let us start with some obvious examples.

*Example* 1.2.
- (i) If $\alpha \in \mathcal{G}_d(m)$ is larger than the modulus of each of its conjugates then $\alpha$ is a Perron number.
- (ii) $\mathcal{P}_1(m) = \{X - m\}$.
- (iii) Let $d \geq 2$ and assume that $m$ is not a $p$-th power in $\mathbb{Z}$ for all primes $p$ which divide $d$. Then $X^d - m$ is irreducible (see e.g. [11, Ch. VIII, Theorem 16]), and therefore $X^d - m \in \mathcal{P}_d(m)$. In particular, the positive root of $X^d - 2$ is a Garcia number (see [9]).

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(iv) If $\alpha$ is a real CNS basis (see [7] for the definition and further details) then
$$-\alpha \in \mathcal{G}_{\deg(P_\alpha)}(P_\alpha(0)).$$

We mention that the case $P(0) = 2$ which is of interest here has extensively been studied by A. Kovács [10] and by P. Burcsi and A. Kovács [8].

In this note we determine the sets $\mathcal{P}_d$ for $d \leq 4$ and characterize totally real Garcia numbers and Garcia numbers whose minimal polynomials are trinomials.

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## 2. Basic properties of $\mathcal{G}_d(m)$

We start with a description of $\mathcal{P}_d(m)$ which allows its calculation in simple cases. Recall that a polynomial with complex coefficients is called expansive if each of its roots has modulus larger than 1 (see e.g. [2, 8]).

**Theorem 2.1.** *Let $P \in \mathbb{Z}[X]$ be a monic irreducible polynomial of degree $d$ with $|P(0)| = m$. Then the following statements are equivalent.*

(i) $P \in \mathcal{P}_d(m)$

(ii) *$P$ is expansive and has a positive root.*

(iii) *$P = X^d + \sum_{i=0}^{d-1} a_{d-i}X^i$ has a positive root, and for $i = 1, \dots, d$ we have $\operatorname{sgn}(\det(M_i)) = (-1)^i$, where $M_d$ is the $2d \times 2d$ matrix given by*
$$m_{ij} = \begin{cases} a_{j-i}, & \text{for } 1 \leq i \leq d \text{ and } i \leq j \leq i+d, \\ a_{i-j}, & \text{for } d+1 \leq i \leq 2d \text{ and } i-d \leq j \leq i, \\ 0, & \text{otherwise.} \end{cases}$$
*$M_{d-1}$ is created from $M_d$ by deleting the $d$-th and $2d$-th rows and columns and so on.*

(iv) *For $i = 1, \dots, d$ we have $\operatorname{sgn}(\det(M_i)) = (-1)^i$, and the inequality $v(m) < v(1)$ holds true where for $a \in \mathbb{R}$ we denote by $v(a)$ the number of changes of sign in the sequence $P(a), P'(a), R_1(a), \dots, R_k(a)$ given by the Sturmian sequence $P, P', R_1, \dots, R_k$ of $P$.*

*Proof.* The first equivalence follows directly from the definition of $\mathcal{P}_d(m)$. The second equivalence is an immediate consequence of [2, Theorem 3.1]. The third equivalence is derived from Sturm's theorem (see e.g., [12, Satz 18.1]). ☐

The specialization to Garcia numbers yields the following corollary.

**Corollary 2.2.** *Let $P \in \mathbb{Z}[X]$ be a monic polynomial with $|P(0)| = 2$. Then $P$ is the minimal polynomial of a Garcia number if and only if $P$ is expansive and has a positive root.*

*Proof.* Clear by Theorem 2.1 and the following Lemma 2.3. ☐

**Lemma 2.3.** *Let $f \in \mathbb{Z}[X]$ be a monic expansive polynomial. If $|f(0)|$ is a prime then $f$ is irreducible.*

*Proof.* Observe that the modulus of the constant term of every nonconstant factor of $f$ would exceed 1. $\square$

Now, we combine some easy consequences of the definition of $\mathcal{G}_d(m)$ with well-known results. Lemma 2.4 deals with the moduli of the conjugates of $\alpha \in \mathcal{G}_d(m)$, and Lemma 2.6 collects some results on the coefficients of the minimal polynomial of $\alpha$.

**Lemma 2.4.** *Let $d \geq 2$, $\alpha \in \mathcal{G}_d(m)$ with conjugates $\alpha_1 = \alpha, \alpha_2, \dots, \alpha_d$. Then we have the following inequalities.*

(i) $m^{1/d} \leq \overline{|\alpha|} < m$

(ii)
$$\sum_{i=1}^{d} |\alpha_j| < m(d-1+\frac{1}{m^{1/d}})$$

*Proof.* (i) Trivial.
(ii) By [14, Lemma 1] we have
$$\sum_{i=1}^{d} |\alpha_j| \leq (d-1)\overline{|\alpha|} + \frac{1}{\overline{|\alpha|}^{d-1}},$$
hence (i) yields the assertion. $\square$

The following theorem rephrases classical facts in the present terminology.

**Theorem 2.5.** (i) *Every totally real Garcia number is of the form $2\cos(\pi r)$ with $r \in \mathbb{Q}$.*

(ii) *Let $\alpha \in \mathcal{G}_d(m), \beta \in \mathcal{G}_k(n) \quad (n \geq 2)$ and assume $\alpha = \overline{|\alpha|}$ and $\beta = \overline{|\beta|}$. Then*
$$\mathbb{Q}(\alpha+\beta) = \mathbb{Q}(\alpha, \beta).$$
*Further, if $\alpha\beta \in \mathbb{N}$ then there is some $e \in \mathbb{N}$ with $\alpha^e \in \mathbb{N}$ and $\beta^e \in \mathbb{N}$.*

*Proof.* (i) Clear by Lemma 2.4 (i) and [13, VIII. Abschn., Nr. 21].
(ii) See [15, Corollary 2, Corollary 3]. $\square$

**Lemma 2.6.** *The coefficients of the polynomial $P = X^d + \sum_{j=0}^{d-1} a_j X^j \in \mathcal{P}_d(m)$ $(d \geq 2)$ satisfy the following statements.*

(i) $|a_0| = m$, *and $a_i < 0$ for some $i \in \{0, \dots, d-1\}$.*

(ii) $|a_{d-1}| \leq d+m-1$ *and*
$$|a_i| \leq \binom{d-1}{i} m + \binom{d-1}{i-1} \qquad (i = 1, \dots, d-2).$$

(iii) $\left|a_{d-1}^2 - 2a_{d-2}\right| < dm^2$

(iv) *For $i = 1, \dots, d-1$ we have*
$$|a_i| \leq 1 + m + \sum_{j=1, j\neq i}^{d-1} |a_j|.$$

(v) *If* $d \geq 3, \quad 1 + m + \sum_{i=1}^{d-1} a_i \geq 0$ *and* $a_i \geq 0$ *for all* $i \in \{0, \ldots, d-1\} \setminus \{1, 2\}$ *then*

$$a_1 + 2a_2 < -d - \sum_{i=3}^{d-1} i a_i.$$

(vi) *If* $a_0 < 0$ *then*

$$\sum_{j=0}^{d-1} (-1)^j a_j \leq (-1)^{d-1} - 1 \qquad \text{and} \qquad \sum_{j=0}^{d-1} a_j \leq -2.$$

(vii) *If* $a_0 > 0$ *then*

$$\sum_{j=0}^{d-1} (-1)^j a_j \geq (-1)^{d-1} + 1 \qquad \text{and} \qquad \sum_{j=0}^{d-1} a_j \geq 0.$$

*Further,* $d$ *is odd or* $P$ *does not have a negative root.*

*Proof.* (i) Trivial.
(ii) The bounds follow from [5, Lemma 3].
(iii) This is clear by [1, Proof of Theorem 1, p. 103] and Lemma 2.4 (i).
(iv) This is an easy consequence of Rouché's theorem.
(v) Observe that $X^d + \sum_{j=1}^{d-1} a_j X^j + m$ has a root in the interval $(1, m)$.
(vi) Clearly, $P(-1) < 0$ and $P(1) < 0$.
(vii) The first statement follows from the observations $P(-1) > 0$ and $P(1) > 0$ analogously as in (vi). The number $n$ of negative roots of $P$ is zero or odd. The sign of $P(0)$ is given by $(-1)^d(-1)^n$, hence $d$ must be odd if $n \neq 0$. □

Lemma 2.3 and the following Proposition 2.8 are useful for constructing examples.

*Example* 2.7. Let $a_1, \ldots, a_{d-1} \in \mathbb{Z}$ and put $s = 1 + \sum_{i=1}^{d-1} |a_i|$.

(i) The polynomial $X^d + \sum_{i=1}^{d-1} a_i X^i - m$ with $m > s$ is expansive by [3, Lemma 1]. Therefore it belongs to $\mathcal{P}_d(m)$ provided it is irreducible.
(ii) Let $p$ be a prime and assume $p > s$. Then by (i) and Lemma 2.3 we have

$$X^d + \sum_{i=1}^{d-1} a_i X^i - p \in \mathcal{P}_d(p).$$

**Proposition 2.8.** *Let* $t \in \mathbb{N}_{>0}$.

(i) *If* $t$ *divides* $d$ *and* $f(X^t) \in \mathcal{P}_d(m)$ *then* $f(X) \in \mathcal{P}_{\frac{d}{t}}(m)$.
(ii) *Let* $p$ *be a prime. If* $f(X) \in \mathcal{P}_d(p)$ *then* $f(X^t) \in \mathcal{P}_{td}(p)$.

*Proof.* (i) This is clear by Theorem 2.1 (ii).
(ii) This follows from Lemma 2.3. □

In the next section we will apply Proposition 2.8 to exhibit trinomials in $\mathcal{P}_d(p)$ for prime $p$.

## 3. Trinomial Garcia numbers

We are now aiming at the description of quadratic and trinomial Garcia numbers. Furthermore, we show that $X^d - X^q - p \in \mathcal{P}_d(p)$ for $0 < q < d$ and every odd prime $p$.

**Lemma 3.1.** *Let $0 < q < d$ and $X^d + bX^q + c \in \mathcal{P}_d(m)$.*

(i) $|b| \le |c| + 1$.

(ii) *If $c = m$ we have $-m \le b \le -2$ and $d \le -qb - 1$.*

*Proof.* (i) This follows from Rouché's theorem.
(ii) By (i) and Lemma 2.6 (i) we find $-(m+1) \le b \le -1$. As $P = X^d + bX^q + c$ cannot vanish at 1 we must have $b \ne -(m+1)$. The assumption $d + qb \ge 0$ implies that the derivative of $P$ is nonnegative in $[1, \infty)$ which contradicts the fact that $P$ must have a root in this interval. Therefore, $d + qb < 0$ and $b \ne -1$. ☐

Now, we are able to determine the set $\mathcal{P}_2(m)$.

**Theorem 3.2.** *For $b, c \in \mathbb{Z}$ the following statements are equivalent.*

(i) $X^2 + bX + c \in \mathcal{P}_2(m)$.

(ii) $b^2 - 4c$ *is not a square, and either*

$$c = -m, \qquad -m + 2 \le b \le 0$$

*or*

$$c = m \ge 5, \qquad -m \le b \le -\sqrt{4m+1}.$$

*Proof.* Let $P = X^2 + bX + c \in \mathcal{P}_2(m)$. Then clearly $|c| = m$, $b^2 - 4c$ is not a square, $b^2 > 4c$ and $|b| < |c + 1|$ by Theorem 2.1. In the case of $c = -m$, this yields $|b| \le m - 2$ and $b \le 0$ because otherwise the roots of $P$ would not exceed 1. If $c = m$ Lemma 3.1 yields $b \le -3$, and we conclude $4m + 1 \le b^2 \le m^2$.

Finally, if (ii) is satisfied then one easily checks that $P$ is irreducible and that its roots fulfill the required conditions. ☐

In particular, we have shown that there is only a single quadratic Garcia number.

**Corollary 3.3.** $\mathcal{P}_2 = \{X^2 - 2\}$.

**Proposition 3.4.** *Let $0 < q < d$ and $p$ be a prime. In case $p = 2$ we require that $d/(d, q)$ be odd or $q/(d, q)$ be even. Then $X^d - X^q - p \in \mathcal{P}_d(p)$.*

*Proof.* Obviously, $P = X^d - X^q - p$ has a real root $> 1$ because $P(1) < 0$. Therefore, using Lemma 2.3 it suffices to show that all roots of $P$ lie outside the closed unit circle. In view of Proposition 2.8 we may assume that $(d, q) = 1$. By [4] $P$ does not vanish inside the unit circle, hence we are left with the boundary of the unit circle. Assume on the contrary that there is some $\zeta \in \mathbb{C}$ with $|\zeta| = 1$ and $P(\zeta) = 0$. Then $w = \zeta^q$ lies on the boundaries of the unit circle and of the circle of radius 1 centered at $-p$. This is only possible for

$p = 2, w = -1$, thus $\zeta^d = 1$. Pick $a, b \in \mathbb{Z}$ with $ad + bq = 1$. Then $\zeta^d = (-1)^b$ and $(-1)^{bd} - (-1)^{bq} = 2$ which contradicts our assumptions on $d$ and $q$. □

Now, we are in a position to complete the characterization of those Garcia numbers whose minimal polynomials are trinomials. Our result slightly extends an example given in [9].

**Theorem 3.5.** *Let $d \geq 3$. The trinomials inside $\mathcal{P}_d$ are given by*

$$\{X^d - 2\} \cup \{X^d - X^q - 2 \,:\, 0 < q < d,\ d/(d,q) \text{ odd or } q/(d,q) \text{ even }\} \cup \mathcal{P}_d^\star,$$

*where*

$$\mathcal{P}_d^\star = \{X^d - 2X^{d/3} - 2\}$$

*if $d \equiv 0 \pmod 3$, and $\mathcal{P}_d^\star = \emptyset$ otherwise.*

*Proof.* We easily check $X^3 - 2X - 2 \in \mathcal{P}_3$, hence by Example 1.2, Proposition 3.4 and Proposition 2.8 all trinomials mentioned above belong to $\mathcal{P}_d$.

Let $P = X^d + bX^q + c \in \mathcal{P}_d$, $t = (d,q), n = d/t$, and $k = q/t$. Then $f = X^n + bX^k + c \in \mathcal{P}_n$ by Proposition 2.8. By Lemma 3.1 we have $c = -2$ and $|b| \leq 3$. Clearly, $b = 1$ is impossible, and if $b = -1$ then $d$ must be odd or $q$ must be even, because otherwise $f(-1) = 0$. Using Bohl's theorem [4] it can be checked that $f$ has a root inside the unit circle for $|b| \in \{2, 3\}$. □

*Remark* 3.6. If $\alpha$ is a Garcia number then the minimal polynomial of $-\alpha$ is not necessarily a CNS polynomial. For instance, let $\alpha$ be a root of $X^5 - X^3 - 2$. Then the minimal polynomial of $-\alpha$ is $X^5 - X^3 + 2$ which is not a CNS polynomial by [6, Theorem 3]. Thus, Theorem 3.5 shows that the converse of Example 1.2 (iii) does not hold.

## 4. Cubic and quartic Garcia numbers

In this section we determine the minimal polynomials of Garcia numbers of the third and fourth degrees.

**Theorem 4.1.** *The minimal polynomials of the cubic and quartic Garcia numbers are given by*

$$\begin{aligned}\mathcal{P}_3 = \{&X^3 - 2, X^3 - X^2 - 2, X^3 - X - 2, X^3 - 2X - 2,\\ &X^3 - X^2 + X - 2, X^3 + X^2 - X - 2, X^3 - 2X^2 + 2X - 2\}.\end{aligned}$$

*and*

$$\begin{aligned}\mathcal{P}_4 = \{&X^4 - 2, X^4 - X^2 - X - 2, X^4 - X^2 + X - 2,\\ &X^4 - 2X^3 + 2X - 2, X^4 + 2X^3 - 2X - 2,\\ &X^4 - X^3 + X - 2, X^4 + X^3 - X - 2\}.\end{aligned}$$

*Proof.* (i) Theorem 3.5 shows that $X^3-2, X^3-X^2-2, X^3-X-2, X^3-2X-2 \in \mathcal{P}_3$ and $X^4 - 2 \in \mathcal{P}_4$. By checking the roots and using Lemma 2.3 one verifies that the other polynomials actually belong to $\mathcal{P}_3$ and $\mathcal{P}_4$, respectively.

(ii) Let $X^3+aX^2+bX+c \in \mathcal{P}_3$. By Theorem 2.1 and [2, Lemma 3.1] we have

$$|b-ac| \leq 2 \tag{4.0.1}$$

and

$$|b+1| < |a+c|\,, \tag{4.0.2}$$

and Lemma 2.6 yields $|a| \leq 4$. If $ab=0$ we are done by Theorem 3.5. Therefore we assume $ab \neq 0$ for the remainder of the proof for the case $d=3$.

The assumption $c=2$ yields $b<0$ by (4.0.2) and Lemma 2.6 (i). Now, $a \geq 0$ is impossible because then $-(b-2a) \leq 2$ which implies $-b \leq 0$ which is absurd. In view of (4.0.2) we are left with $a \in \{-1,-3,-4\}$. The polynomial $X^3-X^2-X+2$ does not have a positive root, $a=-3$ yields $b=-1$ by (4.0.2) contradicting (4.0.1), and analogously $a=-4$ yields $b \in \{-1,-2\}$ by (4.0.2) contradicting (4.0.1).

Now, let $c=-2$. If $b \geq 0$ then $a \in \{-4,\ldots,-1\}$ by (4.0.2), and we exclude $a=-4,-3$ similarly as above. Finally, let $b<0$, hence $a \in \{0,1,3,4\}$ by (4.0.1). In view of (4.0.1) and (4.0.2) the cases $a=3,4$ are impossible.

(iii) Let $P=X^4+aX^3+bX^2+cX+d \in \mathcal{P}_4$. Lemma 2.6 (ii) yields

$$|a| \leq 5, \qquad |b| \leq 9, \qquad |c| \leq 7.$$

If two of the coefficients $a,b,c$ vanish then $P$ is a trinomial, and we are done by Theorem 3.5. Therefore in the sequel we may assume that at most one of the quantities $a,b,c$ is zero.

In case $d=2$ Theorem 2.1 and [2, Lemma 3.2] yield

$$|c-2a| \leq 2, \qquad |a+c| \leq 2+b, \qquad c^2+2a^2-3ac \leq 2-b.$$

Similarly as above these inequalities and elementary analysis show that none of the cases $a \in \{-5,\ldots,5\}$ can occur.

Finally, in case $d=-2$ [2, Lemma 3.2] yields

$$|c+2a| \leq 2, \qquad |a+c| \leq -b, \qquad c^2-2a^2+ac \geq -8-9b,$$

hence $b \leq 0$. A long, but straightforward calculation shows that these inequalities yield the polynomials mentioned above. This completes the proof. □

*Remark* 4.2. Note that a Garcia number need not be maximal among the moduli of its conjugates. For instance, the modulus of the complex roots of the polynomial $X^3+X^2-X-2$ is larger than its positive root, and the modulus of the negative root of the polynomial $X^4-X^2+X-2$ exceeds its positive root. Further, if a Garcia number $\alpha$ is the maximal real conjugate with $\alpha \neq |\alpha_i|$ for all nonreal conjugates $\alpha_i$ then $\alpha$ need not be a Perron number. E.g., take $\alpha$ the (unique) real root of the polynomial $X^3+X^2-X-2$. Then $\alpha$ is smaller than the moduli of its nonreal conjugates.

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