*Acta Mathematica Academiae Paedagogicae Nyíregyháziensis*
**24** (2008), 5–14
www.emis.de/journals
ISSN 1786-0091

# ZERMELO PROBLEM OF NAVIGATION ON HERMITIAN MANIFOLDS

NICOLETA ALDEA

Abstract. In this paper we describe the complex Randers metrics as the solutions of Zermelo problem of navigation on Hermitian manifolds. Based on it, we construct such examples of complex Randers metrics and we study some of their geometrical properties.

## 1. Introduction

Zermelo navigation problem was first discussed by E. Zermelo in [16]. Together with its variants, it has served as a rich example of various problems in the Calculus of Variations and Optimal Control ([8]).

The aim of Zermelo navigation problem on a Riemannian manifold $(M, h)$ is to find the paths of shortest travel time in $M$, under the influence of a current which is represented by a vector field $W$ on $M$. Recently, in the paper [7] it was shown that the real Randers metrics are solutions to Zermelo navigation problem on a Riemannian manifold and the classification of real Randers metrics of constant flag curvature was finally completed.

In the previous paper, [5], we initiated the study of complex Finsler spaces with $(\alpha, \beta)-$ metric, i.e. complex metrics constructed from just two pieces of familiar data: a purely Hermitian metric and a differential 1-form both globally defined on an underlying complex manifold. A special approach was dedicated to the complex Randers metrics $F := \alpha + |\beta|$ in [4]. By this class of complex Finsler spaces we extended the examples number of complex Finsler metrics known: Kobayashi and Caratheodory metrics (see [1]), which quickened the study of such Finsler geometry, and two rather trivial classes of complex Finsler metrics (the complex Finsler metrics which come from Hermitian metrics on the

2000 *Mathematics Subject Classification.* 53C56, 53C60.
*Key words and phrases.* complex Finsler spaces, Randers spaces, Zermelo navigation.

base manifold, so-called the purely Hermitian metrics in [12], and the locally Minkowski complex metrics).

In the present paper we describe the complex Randers metrics as the solutions of Zermelo problem of navigation on Hermitian manifolds (section 3). This technique permits us to construct such examples of complex Randers metrics and we study some of their geometrical properties. Namely, in section 4 we perturb the purely Hermitian metrics: Euclidean metric on $\mathbf{C}^2$, $\mathbf{C}^n$ and Bergman metric on the unit disk $\mathbf{\Delta}^n \subset \mathbf{C}^n$ by the vector field $W$ which satisfy some conditions. These perturbations generate the complex Randers metrics which are not of holomorphic constant curvature.

## 2. Complex Randers metrics

We recall here only the basic notions which are needed; for more information see [1, 12]. For the beginning, we shall make an introduction to the complex Finsler geometry and then we present the complex Randers metrics, [4].

Let $M$ be a complex manifold, $\dim_C M = n$, with $(z^k)_{k=\overline{1,n}}$ complex coordinates in a local chart. The complexified of the real tangent bundle $T_C M$ splits into the sum of holomorphic tangent bundle $T'M$ and its conjugate $T''M$. The bundle $T'M$ is in its turn a complex manifold, the local coordinates in a chart will be denoted by $u = (z^k, \eta^k)$ and these are changed by the rules: $z'^k = z'^k(z)$, $\eta'^k = \frac{\partial z'^k}{\partial z^j}\eta^j$. The complexified tangent bundle of $T'M$ is decomposed as $T_C(T'M) = T'(T'M) \oplus T''(T'M)$. A natural local frame for $T'_u(T'M)$ is $\{\frac{\partial}{\partial z^k}, \frac{\partial}{\partial \eta^k}\}$, which have changes by the rules obtained with Jacobi matrix of above transformations. Note that the change rule of $\frac{\partial}{\partial z^k}$ contains the second order partial derivatives.

Let $V(T'M) = \ker \pi_* \subset T'(T'M)$ be the vertical bundle, spanned locally by $\{\frac{\partial}{\partial \eta^k}\}$. A complex nonlinear connection, briefly (*c.n.c.*), determines a supplementary complex subbundle to $V(T'M)$ in $T'(T'M)$, i.e. $T'(T'M) = H(T'M) \oplus V(T'M)$. It determines an adapted frame $\{\frac{\delta}{\delta z^k} = \frac{\partial}{\partial z^k} - N_k^j \frac{\partial}{\partial \eta^j}\}$, where $N_k^j(z,\eta)$ are the coefficients of the (*c.n.c.*), ([1], [2], [12]).

A continuous function $F : T'M \to \mathbb{R}^+$ is called complex Finsler metric on $M$ if it satisfies the conditions:

*i)* $L := F^2$ is smooth on $\widetilde{T'M} := T'M \backslash \{0\}$;

*ii)* $F(z,\eta) \geq 0$, the equality holds if and only if $\eta = 0$;

*iii)* $F(z, \lambda\eta) = |\lambda| F(z,\eta)$ for $\forall \lambda \in \mathbb{C}$;

*iv)* the Hermitian matrix $\big(g_{i\bar{j}}(z,\eta)\big)$, with $g_{i\bar{j}} = \frac{\partial^2 L}{\partial \eta^i \partial \bar{\eta}^j}$, called the fundamental metric tensor, is positive definite.

The pair $(M, F)$ is called a *complex Finsler space.* The iv)-th assumption involves the strongly pseudoconvexity of the Finsler metric $F$ on complex indicatrix $I_{F,z} = \{\eta \in T'_z M \ \mid \ F(z,\eta) < 1\}$.

Further, in a complex Finsler space a Hermitian connection of $(1,0)-$type has a special meaning, named in [1] the Chern-Finsler connection. In notations from [12] it is $D\Gamma N = (L^i_{jk}, 0, C^i_{jk}, 0)$, where

$$\overset{CF}{N^i_j} = g^{\bar{m}i}\frac{\partial g_{l\bar{m}}}{\partial z^j}\eta^l \ \ ; \ \ L^i_{jk} = g^{\bar{m}i}\frac{\delta g_{j\bar{m}}}{\delta z^k} = \frac{\partial N^i_k}{\partial \eta^j} \ \ ; \ \ C^i_{jk} = g^{\bar{m}i}\frac{\partial g_{j\bar{m}}}{\partial \eta^k}. \tag{2.1}$$

Recall that a complex Finsler space is weakly Kähler iff $g_{i\bar{l}}(L^i_{jk} - L^i_{kj})\eta^j\overline{\eta}^l = 0$ and the holomorphic curvature of $F$ in direction $\eta$, with respect to the Chern-Finsler $(c.l.c.)$, briefly *holomorphic curvature* is

$$\mathcal{K}_F(z,\eta) := \frac{2\overline{\eta}^j\eta^k R_{\bar{j}k}}{L^2(z,\eta)}, \tag{2.2}$$

where $R_{\bar{j}k} := R_{i\bar{j}k\bar{h}}\eta^i\overline{\eta}^h = -g_{l\bar{j}}\delta_{\bar{h}}(N^l_k)\overline{\eta}^h$ and $R_{i\bar{j}k\bar{h}} = g_{l\bar{j}}R^l_{i\bar{h}k}$ is $h\bar{h}$ Riemann tensor associated to the Chern-Finsler $(c.l.c.)$ (see [1, 12, 3]). It depends both on the position $z \in M$ and the direction $\eta$.

We consider $z \in M$, $\eta \in T'_zM$, $\eta = \eta^i\frac{\partial}{\partial z^i}$, $a := a_{i\bar{j}}(z)dz^i \otimes d\bar{z}^j$ a purely Hermitian positive metric and $b = b_i(z)dz^i$ a differential $1-$form. By these objects in [4] we defined the *complex Randers metric* on $T'M$ by

$$F(z,\eta) := \alpha(z,\eta) + |\beta(z,\eta)|, \tag{2.3}$$

where

$$\begin{aligned} &\alpha(z,\eta) := \sqrt{a_{i\bar{j}}(z)\eta^i\overline{\eta}^j}; \\ &|\beta(z,\eta)| = \sqrt{\beta(z,\eta)\overline{\beta(z,\eta)}} \ \text{ with } \ \beta(z,\eta) = b_i(z)\eta^i. \end{aligned} \tag{2.4}$$

A natural question: when is the function (2.3) a complex Finsler metric? Some remarks are immediate. Due the presence of $|\beta|$ the complex Randers metric $F := \alpha + |\beta|$ is positive and smooth on $T'M\backslash\{0\}$. The complex Randers metric is purely Hermitian if and only if $\beta$ vanishes identically. The function $L := F^2 = (\alpha + |\beta|)^2$ depends on $z$ and $\eta$ by means of $\alpha := \alpha(z,\eta) \in \mathbb{R}$ and $\beta := \beta(z,\eta) \in \mathbb{C}$. Moreover $\alpha$ and $\beta$ are homogeneous with respect to $\eta$, i.e. $\alpha(z,\lambda\eta) = |\lambda|\alpha(z,\eta)$, $\beta(z,\lambda\eta) = \lambda\beta(z,\eta)$ for any $\lambda \in \mathbb{C}$, thus $L(z,\lambda\eta) = \lambda\bar{\lambda}L(z,\eta)$ for any $\lambda \in \mathbb{C}$.

So, the main issue that needs to be checked is the strongly pseudoconvexity of the complex Randers function. For this in [4], we considered the settings

$$\begin{aligned} &\frac{\partial\alpha}{\partial\eta^i} = \frac{1}{2\alpha}l_i; \quad \frac{\partial|\beta|}{\partial\eta^i} = \frac{\bar{\beta}}{2|\beta|}b_i; \\ &b^i := a^{\bar{j}i}b_{\bar{j}}; \quad ||b||^2 := a^{\bar{j}i}b_ib_{\bar{j}}; \quad \gamma := L + \alpha^2(||b||^2 - 1); \\ &\eta_i := \frac{\partial L}{\partial\eta^i} = L_\alpha\frac{\partial\alpha}{\partial\eta^i} + L_{|\beta|}\frac{\partial|\beta|}{\partial\eta^i} = \frac{F}{\alpha}l_i + \frac{F\bar{\beta}}{|\beta|}b_i, \end{aligned} \tag{2.5}$$

where $l_i := a_{i\bar{j}}\bar{\eta}^j$ and $\left(a^{\bar{j}i}\right)$ is the Hermitian inverse of $\left(a_{i\bar{j}}\right)$ matrix, and we proved

**Proposition 1.** [4]*For the complex Randers metric $F := \alpha + |\beta|$ we have*

*i) The fundamental metric tensor*

$$g_{i\bar{j}} = \frac{\partial^2(\alpha+|\beta|)^2}{\partial\eta^i\partial\bar{\eta}^j} = \frac{F}{\alpha}h_{i\bar{j}} + \frac{F}{2|\beta|}b_i b_{\bar{j}} + \frac{1}{2L}\eta_i\eta_{\bar{j}},$$

*where* $h_{i\bar{j}} := a_{i\bar{j}} - \frac{1}{2\alpha^2}l_i l_{\bar{j}}$.

*ii)* $g^{\bar{j}i} = \dfrac{\alpha}{F}a^{\bar{j}i} + \dfrac{|\beta|(\alpha||b||^2+|\beta|)}{L\gamma}\eta^i\bar{\eta}^j - \dfrac{\alpha^3}{F\gamma}b^i\bar{b}^j - \dfrac{\alpha}{F\gamma}\left(\bar{\beta}\eta^i\bar{b}^j + \beta b^i\bar{\eta}^j\right).$

*iii)* $\det\left(g_{i\bar{j}}\right) = \left(\dfrac{F}{\alpha}\right)^n \dfrac{\gamma}{2\alpha|\beta|}\det\left(a_{i\bar{j}}\right).$

Having the formula for $\det\left(g_{i\bar{j}}\right)$, we can say that $g_{i\bar{j}}(z,\eta)$ is positive definite if and only if $\gamma > 0$ at each nonzero $\eta$ in $T'_zM$. So we have proved

**Theorem 1** ([4])**.** *A complex Randers metric with $\gamma > 0$ is a complex Finsler metric.*

If the quadratic form $h(z,\eta) := (a_{i\bar{j}} - b_i b_{\bar{j}})\eta^i\bar{\eta}^j$ is positive definite, then substituting $\eta^i\bar{\eta}^j$ with $b^i\bar{b}^j$ it follows that $||b||^2(1-||b||^2) > 0$, which says that $||b||^2 \in (0,1)$ and then $\gamma > 0$, since $\gamma = 2\alpha|\beta| + |\beta|^2 + \alpha^2||b||^2$. Equivalently the positive definite of the quadratic form means that $\alpha^2 > |\beta|^2$, or in other words $\sup\frac{|\beta|}{\alpha} < 1$, for all $(z,\eta) \in T'M\backslash\{0\}$.

From [4], we have the expression of the weakly Kähler condition for a complex Randers space

$$\frac{\alpha^2|\beta|}{\gamma\delta}\left[\beta\frac{\alpha||b||^2+|\beta|}{|\beta|}\frac{\partial b_{\bar{m}}}{\partial z^r}\bar{\eta}^m + \bar{\beta}\left(\frac{\partial b_r}{\partial z^l} - b^{\bar{m}}\frac{\partial a_{l\bar{m}}}{\partial z^r}\right)\eta^l - \alpha|\beta|b^{\bar{m}}\frac{\partial b_{\bar{m}}}{\partial z^r}\right]\eta^r C_k$$
$$-\left(\alpha\bar{\beta}F_{kl} + \alpha b_l\frac{\partial b_{\bar{r}}}{\partial z^k}\bar{\eta}^r + 2|\beta|a_{l\bar{r}}\Gamma^{\bar{r}}_{\bar{j}k}\bar{\eta}^j\right)\eta^l + \alpha b_k\frac{\partial b_{\bar{m}}}{\partial z^r}\bar{\eta}^m\eta^r = 0,$$

where $C_j := \delta\left(\frac{1}{\alpha^2}l_j - \frac{\bar{\beta}}{|\beta|^2}b_j\right)$, $\delta := \frac{\alpha^2||b||^2-|\beta|^2}{2\gamma} - \frac{n|\beta|}{2F}$, $\Gamma^{\bar{r}}_{\bar{j}i} = \frac{1}{2}a^{\bar{r}k}\{\frac{\partial a_{k\bar{j}}}{\partial z^i} - \frac{\partial a_{i\bar{j}}}{\partial z^k}\}$ are the coefficients of Levi-Civita connection of $a_{i\bar{j}}$ and $F_{il} := \frac{\partial b_l}{\partial z^i} - \frac{\partial bi}{\partial z^l}$.

## 3. Zermelo navigation

Following the ideas and the same interpretation as in Zermelo navigation on Riemannian manifolds, [7], we shall describe Zermelo problem of navigation on Hermitian manifolds.

Let $M$ be a complex manifold, $\dim_C M = n$, and $z \in M$ be the base point of the tangent vectors $\eta \in T'_zM$, $\eta = \eta^i\frac{\partial}{\partial z^i}$. Considering the purely Hermitian metric $h := h_{i\bar{j}}(z)dz^i \otimes d\bar{z}^j$ on $M$, the norm of $\eta$ is $||\eta|| := \sqrt{h(\eta,\bar{\eta})} = \sqrt{h_{i\bar{j}}(z)\eta^i\bar{\eta}^j}$.

As in [7], $||\eta||$ is the necessary time to travel from the base point $z$ of the tangent vector $\eta$ to its tip, using an engine with a fixed power output.

We set a tangent vector $u \in T'_zM$, $u = u^i \frac{\partial}{\partial z^i}$ with $||u|| = 1$, i.e. a travel from $z$ to the tip of $u$ would take one unit of time and another tangent vector $W \in T'_zM$, $W = W^i \frac{\partial}{\partial z^i}$, where $||W||$ is the time to travel from $z$ to the tip of $W$, under the influence of a current on the Hermitian manifold $(M, h)$. We suppose that $||W|| < 1$. The current has to produce a deviation from the path such that the travel time is not equal with one unit time. Note that, in the travel from $z$ to the tip of $u$, the current is absent. Now, we navigate along the tangent vector $v := u - W$, which starts from $z$, too. The norm of $v$ is not 1. For example, if $u = 3W$, then $||W|| = \frac{1}{3}$ and $||v|| = \frac{2}{3}$. Else, if $u = -3W$, then $||W|| = \frac{1}{3}$ and $||v|| = \frac{4}{3}$. So, by the presence of the current, the purely Hermitian metric $h$ does not give the travel time along the tangent vectors. Therefore, we have to build a function $F : T'M \to \mathbb{R}^+$ which keeps the path of travel under the influence of a current on $(M, h)$, such that the travel time is one unit time, along tangent vector $v = u - W$. So, $F(z, v) = 1$.

Next, we determine the formula of the function $F$, taking into account the assumption $F(z, v) = 1$. We have

$$1 = ||u||^2 = h(u, \bar{u}) = h(v + W, \bar{v} + \bar{W}) = ||v||^2 + 2 \operatorname{Re} h(v, \bar{W}) + ||W||^2.$$

Denoting by $\theta$ the angle between directions of $v$ and $W$, we can write $\cos\theta = \frac{\operatorname{Re} h(v,\bar{W})}{||v|| \cdot ||W||}$, thus above relation became

$$||v||^2 + 2||v||\ ||W|| \cos\theta - \varepsilon = 0, \text{ where } \varepsilon := 1 - ||W||^2.$$

Because $||W|| < 1$, and so $\varepsilon > 0$, we obtain only the solution

$$||v|| = -||W|| \cos\theta + \sqrt{||W||^2 \cos^2\theta + \varepsilon}.$$

Multiplying the last relation by $||v||$, we obtain

$$||v||^2 = -\operatorname{Re} h(v, \bar{W}) + \sqrt{\left[\operatorname{Re} h(v, \bar{W})\right]^2 + \varepsilon||v||^2},$$

equivalently with

$$\varepsilon = \operatorname{Re} h(v, \bar{W}) + \sqrt{\left[\operatorname{Re} h(v, \bar{W})\right]^2 + \varepsilon||v||^2}.$$

Since $\operatorname{Re} h(v, \bar{W}) = \left|h(v, \bar{W})\right| \cos\varphi$, where $\varphi = \arg h(v, \bar{W})$ is a real valued function which depends on $z$ and $\left|h(v, \bar{W})\right| := \sqrt{h(v, \bar{W})\overline{h(v, \bar{W})}}$, we write

$$\varepsilon = \sqrt{\left|h(v, \bar{W})\right|^2 \cos^2\varphi + \varepsilon||v||^2} + \left|h(v, \bar{W})\right| \cos\varphi.$$

So,

$$F(z, v) = \frac{\sqrt{\varepsilon||v||^2 + \left|h(v, \bar{W})\right|^2 \cos^2\varphi}}{\varepsilon} + \frac{\left|h(v, \bar{W})\right| \cos\varphi}{\varepsilon}. \tag{3.1}$$

Note that every $\eta \in T'M \backslash \{0\}$ is collinear with some tangent vector $v$ with property $F(z, v) = 1$. So, $\eta = cv$, $c \in \mathbf{R}$, and making use of $\varphi = \arg h(\eta, \bar{W}) =$

$\arg h(cv, \bar{W}) = \arg ch(v, \bar{W}) = \arg h(v, \bar{W})$, for any $c \in \mathbf{R}$, we deduce $F(z, \eta) = F(z, cv) = |c|\, F(z, v) = |c|$. This means that the travel time along the tangent vector $\eta$, under the influence of a current, is $|c|$ unit of time and

$$F(z,\eta) = \frac{\sqrt{\varepsilon||\eta||^2 + \left|h(\eta,\bar{W})\right|^2 \cos^2\varphi}}{\varepsilon} + \frac{\left|h(\eta,\bar{W})\right|\cos\varphi}{\varepsilon}, \tag{3.2}$$

where $\varphi = \varphi(z)$.

Some remarks are necessary. First, note that $F$ is purely Hermitian if and only if $W = 0$. We have $W = 0$ if and only if $\cos\varphi = 0$. Without restricting the generality, we can suppose that $\cos\varphi > 0$. This assumption assure that the function $F(z, \eta)$ is positive on $T'M\backslash\{0\}$. If $\cos\varphi < 0$, then we choose the direction $v = u + W$, such that $F(z, v) = 1$. As above, similar computation leads to a positive function $F(z, \eta)$ on $T'M\backslash\{0\}$.

An obviously remark is that if $\varphi = 0$, then $\cos\varphi = 1$ and $h(v, \bar{W})$ is real valued. Indeed, a natural question is when does this fact occur? It immediately results that if $v$ and $W$ are collinearity, then $h(v, \bar{W})$ is real. The converse is not true. For example, if we consider the Euclidean metric $h_{i\bar{j}} = \delta_{i\bar{j}}$ on $\mathbf{C}^2$ and we choose $W = \left(\frac{1}{2} + i\frac{1}{2}, \frac{1}{4} + i\frac{1}{4}\right)$, $||W|| = \frac{5}{8} < 1$, by computation it results $v = \left(\frac{\sqrt{11}-5}{8} + i\frac{\sqrt{11}-3}{8}, \frac{1}{2}\right)$, which is not collinear with $W$, when $h(v, \bar{W}) \in \mathbf{R}$ and $||u|| = 1$. Moreover, we find $h(v, \bar{W}) = \frac{\sqrt{11}-3}{8}$ and $u = \left(\frac{\sqrt{11}-1}{8} + i\frac{\sqrt{11}+1}{8}, \frac{3}{4} + i\frac{1}{4}\right)$.

On the other hand, the formula of the function $F(z, \eta)$ permit us to write $F(z, \eta) := \alpha + |\beta|$, where

$$\begin{aligned} &\alpha(z,\eta) = \sqrt{a_{i\bar{j}}(z)\eta^i\bar{\eta}^j}; \quad \beta(z,\eta) = b_i(z)\eta^i; \\ &a_{i\bar{j}}(z) := \frac{h_{i\bar{j}}}{\varepsilon} + b_i b_{\bar{j}}; \quad b_i(z) := \frac{W_i\cos\varphi}{\varepsilon}; \\ &W_i := h_{i\bar{j}}\bar{W}^j; \quad \varepsilon = 1 - W_i W^i. \end{aligned} \tag{3.3}$$

The first term of $F$ defines the norm of $\eta$ with respect to a new purely Hermitian metric $a := a_{i\bar{j}}(z)dz^i \otimes d\bar{z}^j$ and the second term of $F$ is the value on $\eta$ of a differential $1-$ form $b = b_i(z)dz^i$. Therefore, the solution of Zermelo navigation problem on some Hermitian manifold $(M, h)$, under the influence of a current $W$ with $||W||^2 < 1$ is a function which has the form of a complex Randers metric. Moreover, the function $F(z, \eta)$ given by (3.3) is positive, complex homogeneous with respect to $\eta$, i.e. $F(z, \lambda\eta) = |\lambda| F(z, \eta)$, for any $\lambda \in \mathbb{C}$, and smooth on $T'M\backslash\{0\}$.

The inverse of $\left(a_{i\bar{j}}(z)\right)$ from (3.3) is $a^{\bar{j}i} = \varepsilon\left(h^{\bar{j}i} - \frac{\cos^2\varphi}{1-||W||^2\sin^2\varphi}W^i\bar{W}^j\right)$ and so,

$$b^i := a^{\bar{j}i}b_{\bar{j}} = \varepsilon\left(h^{\bar{j}i} - \frac{\cos^2\varphi}{1-||W||^2\sin^2\varphi}W^i\bar{W}^j\right)\frac{W_{\bar{j}}\cos\varphi}{\varepsilon} = \varepsilon W^i\frac{\cos\varphi}{1-||W||^2\sin^2\varphi},$$

$||b||^2 := b_i b^i = \frac{||W||^2 \cos^2 \varphi}{1-||W||^2 \sin^2 \varphi} < 1$, because $||W||^2 < 1$.

Taking into account Theorem 2.1, the obtained condition $||b||^2 < 1$ guarantees the strongly pseudoconvexity of the function $F(z,\eta)$ from (3.2) on its complex indicatrix $I_{F,z}$. Indeed the function $F(z,\eta)$ from (3.2) defines a complex Finsler metric on $M$ of complex Randers type.

Now, it is natural for us to inquire about: can any complex metric $F(z,\eta) = \alpha + |\beta|$ be obtained through by perturbation of some purely Hermitian metric $h$ with some vector field $W$ satisfying $||W|| < 1$? The answers come below.

Considering the complex metric $F(z,\eta) = \alpha + |\beta|$, with $\alpha = \sqrt{a_{i\bar{j}}(z)\eta^i\bar{\eta}^j}$, $\beta = b_i(z)\eta^i$, $b^i := a^{\bar{j}i}b_{\bar{j}}$, $||b||^2 := b^i b_i < 1$ and $\omega := 1 - ||b||^2$, we construct $h$ and $W$ as

$$h_{i\bar{j}}(z) = \omega(a_{i\bar{j}} - b_i b_{\bar{j}});\ W^i(z) = \frac{b^i}{\omega}. \tag{3.4}$$

By direct computation we obtain:

$$||W||^2 := h_{i\bar{j}} W^i \bar{W}^j = \omega(a_{i\bar{j}} - b_i b_{\bar{j}})\frac{b^i}{\omega}\frac{\bar{b}^j}{\omega} = ||b||^2 < 1.$$

So $\varepsilon = \omega$ and $W_i := h_{i\bar{j}}\bar{W}^j = \omega(a_{i\bar{j}} - b_i b_{\bar{j}})\frac{\bar{b}^j}{\omega} = \omega b_i$.

It results that perturbing the above $h$ by $W$, we obtain a function

$$\tilde{F}(z,\eta) = \tilde{\alpha} + |\tilde{\beta}|$$

with

$$\tilde{\alpha}(z,\eta) = \sqrt{\tilde{a}_{i\bar{j}}(z)\eta^i\bar{\eta}^j}, \quad \tilde{\beta}(z,\eta) = \tilde{b}_i(z)\eta^i,$$

where

$$\tilde{a}_{i\bar{j}}(z) = \frac{h_{i\bar{j}}}{\varepsilon} + \tilde{b}_i\tilde{b}_{\bar{j}} = \frac{h_{i\bar{j}}}{\varepsilon} + \frac{W_i\cos\varphi}{\varepsilon}\frac{\bar{W}_j\cos\varphi}{\varepsilon} = a_{i\bar{j}} + (1-\cos^2\varphi)b_i b_{\bar{j}}.$$

We have $\tilde{a}_{i\bar{j}}(z) = a_{i\bar{j}}(z)$ if and only if $\cos\varphi = 1$. Therefore, by perturbation (3.4) we obtain again the complex metric $F(z,\eta) = \alpha + |\beta|$ if and only if $\cos\varphi = 1$. So, the answer of the question is yes, iff $\cos\varphi = 1$.

Corroborating above results, we have proved:

**Theorem 2.** *A complex Finsler metric is of complex Randers type, i.e. it has the form $F(z,\eta) = \alpha + |\beta|$, if and only if it solves the Zermelo navigation problem on some Hermitian manifold $(M,h)$, under the influence of a current $W$ with $||W||^2 < 1$ and $\cos\varphi = 1$.*

## 4. Examples

In the sequel, we construct some examples of complex Randers metrics by above technique. We perturb a purely Hermitian metric $h$ by any vector field $W$, satisfying $||W|| < 1$ and $\cos\varphi = 1$.

**I. The perturbation of Euclidean metric.** $h_{i\bar{j}} = \delta_{i\bar{j}}$

**I.1.** We perturb the Euclidean metric on $\mathbf{C}^2$ by $W := c_1 \frac{\partial}{\partial z^1} + c_2 \frac{\partial}{\partial z^2}$, where $c_i \in \mathbf{C}$, $i = 1, 2$, such that $||W||^2 = |c_1|^2 + |c_2|^2 < 1$. To reduce the clutter, we denote $(z^1, z^2) := (z, w)$ and $(\eta^1, \eta^2) := (\eta, \theta)$. The outcome is a complex Randers metric with

$$\begin{aligned} \alpha^2(\eta, \theta) &= \frac{(|\eta|^2 + |\theta|^2)(1 - |c_1|^2 - |c_2|^2) + (\bar{c}_1\eta + \bar{c}_2\theta)(c_1\bar{\eta} + c_2\bar{\theta})}{(1 - |c_1|^2 - |c_2|^2)^2}; \\ |\beta(\eta, \theta)|^2 &= \frac{(\bar{c}_1\eta + \bar{c}_2\theta)(c_1\bar{\eta} + c_2\bar{\theta})}{(1 - |c_1|^2 - |c_2|^2)^2}. \end{aligned} \tag{4.1}$$

It is a locally Minkowski metric and so its holomorphic curvature is zero.

**I.2.** With same notations as above, the perturbing vector field is $W := zf(z, w)\frac{\partial}{\partial z} + 0\frac{\partial}{\partial w}$, where $f := f(z, w)$ is a positively real valued function, and $||W||^2 = f^2|z|^2 < 1$. Therefore, the complex Randers metric has the form $F_f := \alpha + |\beta|$, with

$$\begin{aligned} \alpha^2(z, w, \eta, \theta) &= \frac{(|\eta|^2 + |\theta|^2)(1 - f^2|z|^2) + f^2|z|^2|\eta|^2}{(1 - f^2|z|^2)^2}; \\ \beta(z, w, \eta, \theta) &= \frac{f\bar{z}\eta}{1 - f^2|z|^2}. \end{aligned} \tag{4.2}$$

There is not a function $f$ such that the metrics $F_f$ should be at least weakly Kähler.

**I.3.** Considering the metric $h_{i\bar{j}} = \delta_{i\bar{j}}$ on $\mathbf{C}^n$, we perturb it by $W := z^i f(z)\frac{\partial}{\partial z^i}$, where $f := f(z)$ is a positively real valued function, such that $||W||^2 = f^2|z|^2 < 1$. It results the complex Randers metrics $F_f := \alpha + |\beta|$, where

$$\begin{aligned} \alpha^2(z, \eta) &= \frac{|\eta|^2}{1 - f^2|z|^2} + \frac{f^2 \left|< z, \eta >\right|^2}{(1 - f^2|z|^2)^2}; \\ \beta(z, \eta) &= \frac{f \left|< z, \eta >\right|}{1 - f^2|z|^2}, \end{aligned} \tag{4.3}$$

where $|z|^2 := \sum_{k=1}^n z^k \bar{z}^k$, $< z, \eta > := \sum_{k=1}^n z^k \bar{\eta}^k$, $\left|< z, \eta >\right|^2 = < z, \eta > \overline{< z, \eta >}$. Even in this example there is not $f$ that $F_f$ should be weakly Kähler. In particular, for $f = 1$, $\alpha^2$ is the Bergman metric on the unit disk $\mathbf{\Delta}^n$ and

$$F_1 := \frac{\sqrt{|\eta|^2 (1 - |z|^2) + \left|< z, \eta >\right|^2}}{1 - |z|^2} + \frac{\left|< z, \eta >\right|}{1 - |z|^2} \tag{4.4}$$

is a complex Randers metric on $\mathbf{\Delta}^n$ of negatively holomorphic curvature

$$\mathcal{K}_{F_1} = \frac{-2\alpha F_1}{\gamma}, \quad \gamma := F_1^2 - \alpha^2(1 - |z|^2).$$

**II. The perturbation of Bergman metric.**

$$h_{i\bar{j}} = \frac{1}{1-|z|^2}\left(\delta_{i\bar{j}} + \frac{\overline{z}^i z^j}{1-|z|^2}\right) \tag{4.5}$$

on the unit disk $\mathbf{\Delta}^n$ by the vector field $W := z^i f(z)\frac{\partial}{\partial z^i}$, where $f := f(z)$ is a positively real valued function, with $||W||^2 = \frac{f^2|z|^2}{(1-|z|^2)^2} < 1$, leads to the complex Randers metric $F_f := \alpha + |\beta|$, where

$$\begin{aligned} \alpha^2(z,\eta) &= \frac{(1-|z^2)|\eta|^2}{(1-(2+f^2)|z|^2+|z|^4)} \\ &\quad + \frac{(1-|z|^2)(1-|z|^2+f^2)\,|< z,\eta >|^2}{(1-(2+f^2)|z|^2+|z|^4)^2}; \\ |\beta(z,\eta)| &= \frac{f\,|< z,\eta >|}{1-(2+f^2)|z|^2+|z|^4}. \end{aligned} \tag{4.6}$$

Though the Bergman metric is Kähler of holomorphic curvature $-4$, by this perturbation the complex Randers metric (4.6) is not weakly Kähler, for any $f$. Only if $f = \sqrt{1-|z|^2}$, the purely Hermitian metric $\alpha^2(z,\eta)$ from (4.6) is Kähler.

**Conclusions.** Zermelo navigation on Riemannian manifold produces examples of real Randers metrics of constant flag curvature. In contrast with the real case, above examples show that by perturbation of some purely Hermitian metrics of constant holomorphic sectional curvature, the obtained complex Randers metrics are not of constant holomorphic curvature.

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Faculty of Mathematics and Informatics,
Transilvania Univ.,
Iuliu Maniu 50, Braşov 500091, Romania
*E-mail address*: nicoleta.aldea@lycos.com