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TANGENT BUNDLE OF THE HYPERSURFACES IN A EUCLIDEAN SPACE

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Abstract. We consider an immersed orientable hypersurface $f\colon M \to R^{n+1}$ of the Euclidean space (f an immersion), and observe that the tangent bundle TM of the hypersurface M is an immersed submanifold of the Euclidean space R^{2n+2}. Then we show that in general the induced metric on TM is not a natural metric and obtain expressions for the horizontal and vertical lifts of the vector fields on M. We also study the special case in which the induced metric on TM becomes a natural metric and show that in this case the tangent bundle TM is trivial.

1. Introduction

The geometry of the tangent bundle TM of a Riemannian manifold is an interesting field in differential geometry. The first attempt to define a Riemannian metric on TM was made by Sasaki [8], and since then the tangent bundle has become focus of study with this metric. Specially after the work of Dombrowoski [2], who has introduced a nice theory of linking the geometry of the tangent bundle with Sasaki metric to the geometry of the base manifold, many mathematicians have studied the geometry of the tangent bundle through various aspects (cf. the survey article [3] and references therein). Since there is a naturally associated almost complex structure J to the tangent bundle TM of a Riemannian manifold M, one naturally expects fairly good properties associated to this almost complex structure vis-a-vis the complex geometry. However, the Sasaki metric on TM offers a significant obstruction on the almost complex structure and does not even allow it to be a complex unless the base manifold is flat. This deficiency in the Sasaki metric lead mathematicians to search for other metrics on the tangent bundle other than Sasaki metric, for instance Cheeger-Gromoll metric, Oproiu metric (cf. [1], [3], [7],

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[9]). This lead to the class of metrics on TM which make the natural submersion $\pi\colon TM \to M$ into a Riemannian submersion and this class of metrics is known as natural metrics. In this paper we are interested in the tangent bundle TM of an immersed orientable hypersurface M in the Euclidean space R^{n+1}. If $f\colon M \to R^{n+1}$is the smooth immersion which makes M as an immersed hypersurface of R^{n+1}, then we show that the smooth map $F = df\colon TM \to R^{2n+2}$ is also an immersion, thereby making TM a submanifold of R^{2n+2} and consequently has an induced metric $\overline{g}$. We study the Riemannian manifold $(TM, \overline{g})$ as submanifold of the Euclidean space $(R^{2n+2}, \langle , \rangle)$ and first show that in general the induced metric $\overline{g}$ is not a natural metric by calculating the horizontal and vertical lifts of vector fields on M to TM. Then we consider a special case, in which the metric $\overline{g}$ becomes a natural metric and observe that in this case the tangent bundle TM is trivial.

2. Preliminaries

Let (M, g) be a Riemannian manifold and TM be its tangent bundle with projection map $\pi\colon TM \to M$. Then for each $(p, u) \in TM$, the tangent space $T_{(p,u)}TM = \mathfrak{H}_{(p,u)} \oplus \mathfrak{V}_{(p,u)}$, where $\mathfrak{V}_{(p,u)}$ is kernel of $d\pi_{(p,u)}\colon T_{(p,u)}TM \to T_pM$ and $\mathfrak{H}_{(p,u)}$ is the kernel of the connection map $K_{(p,u)}\colon T_{(p,u)}TM \to T_pM$ with respect to the Riemannian connection on (M, g). The subspaces $\mathfrak{H}_{(p,u)}$, $\mathfrak{V}_{(p,u)}$ are called the horizontal and vertical subspaces respectively. Consequently the Lie algebra of smooth vector fields $\mathfrak{X}(TM)$ on the tangent bundle TM admits the decomposition $\mathfrak{X}(TM) = \mathfrak{H} \oplus \mathfrak{V}$, where $\mathfrak{H}$ is called the horizontal distribution and $\mathfrak{V}$ is called the vertical distribution on the tangent bundle TM. For each $X_p \in T_pM$, the horizontal lift of X_p to a point $z = (p, u) \in TM$ is the unique vector $X^h_z \in \mathfrak{H}_z$ such that $d\pi(X^h_z) = X_p \circ \pi$ and the vertical lift of X_p to a point $z = (p, u) \in TM$ is the unique vector $X^v_z \in \mathfrak{V}_z$ such that $X^v_z(df) = X_p(f)$ for all functions $f \in C^\infty(M)$, where df is the function defined by $(df)(p, u) = u(f)$. Also for a vector field $X \in \mathfrak{X}(M)$, the horizontal lift of X is a vector field $X^h \in \mathfrak{X}(TM)$ whose value at a point (p, u) is the horizontal lift of $X(p)$ to (p, u), the vertical lift X^v of X is defined similarly. For $X \in \mathfrak{X}(M)$ the horizontal and vertical lifts X^h, X^v of X are the uniquely determined vector fields on TM satisfying

$$d\pi(X^h_z) = X_{\pi(z)}, K(X^h_z) = 0_{\pi(z)},\ d\pi(X^v_z) = 0_{\pi(z)}, K(X^v_z) = X_{\pi(z)}$$

Also we have for a smooth function $f \in C^\infty(M)$ and vector fields $X, Y \in \mathfrak{X}(M)$, that, $(fX)^h = (f \circ \pi)X^h$, $(fX)^v = (f \circ \pi)X^v$, $(X + Y)^h = X^h + Y^h$ and $(X + Y)^v = X^v + Y^v$. If $\dim M = m$ and (U, ϕ) is a chart on M with local coordinates $x^1, x^2, \ldots, x^m$, then $(\pi^{-1}(U), \bar{\Phi})$ is a chart on TM with local coordinates $x^1, \ldots, x^m, y^1, \ldots, y^m$, where $x^i = x^i \circ \pi$ and $y^i = dx^i$,$i = 1, \ldots, m$. Throughout this paper we use Einstein summation, that is, the repeated indices are summed on their range. For horizontal and vertical lifts we have

Lemma 2.1 ([3]). *Let (M, g) be a Riemannian manifold and $X, Z \in \mathfrak{X}(M)$ which locally are represented by $X = \xi^i \frac{\partial}{\partial x^i}$ and $Z = \eta^i \frac{\partial}{\partial x^i}$. Then the vertical and horizontal lifts X^v and X^h of X at the point $Z \in TM$ are given by*

$$(X^v)_Z = \xi^i \frac{\partial}{\partial y^i}, \quad (X^h)_Z = \xi^i \frac{\partial}{\partial x^i} - \xi^j \eta^k \Gamma^i_{jk} \frac{\partial}{\partial y^i}$$

where the coefficients Γ^i_{jk} are the Christoffel symbols of the connection ∇ on (M, g).

A Riemannian metric $\bar{g}$ on the tangent bundle TM is said to be natural metric with respect to g on M if $\bar{g}_{(p,u)}(X^h, Y^h) = g_p(X, Y)$ and $\bar{g}_{(p,u)}(X^h, Y^v) = 0$, for all vector fields $X, Y \in \mathfrak{X}(M)$ and $(p, u) \in TM$, that is the projection map $\pi\colon TM \to M$ is the Riemannian submersion [6]

3. Tangent bundle of the hypersurface

Let M be an immersed hypersurface of the Euclidean space $(R^{n+1}, \langle, \rangle)$, where $\langle, \rangle$ is the Euclidean metric, with the immersion $f\colon M \to R^{n+1}$. Then we have the smooth maps

$$F = df\colon TM \to R^{2n+2}, \widetilde{\pi}\colon R^{2n+2} \to R^{n+1}$$

defined by $F(p, X_p) = (f(p), df_p(X_p))$ and $\widetilde{\pi}\,(x, y) = x$ for $x, y \in R^{n+1}$, where $df_p\colon T_pM \to R$ is the differential of the map f at $p \in M$. Clearly $f \circ \pi = \widetilde{\pi} \circ F$ holds, where $\pi\colon TM \to M$ is the projection of the tangent bundle. We have for the submersion $\widetilde{\pi}\colon (R^{2n+2}, \langle, \rangle) \to (R^{n+1}, \langle, \rangle)$, as $\widetilde{\pi}$ is linear $d\widetilde{\pi}_p = \widetilde{\pi}$, $p \in R^{2n+2}$, which implies that the vertical space $\bar{\mathfrak{V}}_p = \ker d\widetilde{\pi}_p = (0, R^{n+1})$ and since $\bar{\mathfrak{H}}_p \perp \bar{\mathfrak{V}}_p$ we get $\bar{\mathfrak{H}}_p = R^{2n+2}/\bar{\mathfrak{V}}_p = (R^{n+1}, 0)$. Also we see that $d\widetilde{\pi}$ preserves lengths of horizontal vectors, that is, $\langle X, Y \rangle = \langle d\widetilde{\pi}(X), d\widetilde{\pi}(Y) \rangle$ for $X, Y \in \bar{\mathfrak{H}}$ where $d\widetilde{\pi} = \begin{bmatrix} I_{(n+1)\times(n+1)} & 0_{(n+1)\times(n+1)} \end{bmatrix}$ consequently it follows that $\widetilde{\pi}\colon (R^{2n+2}, \langle, \rangle) \to (R^{n+1}, \langle, \rangle)$ is a Riemannian submersion (cf. [6]).

If $x^1, \ldots, x^n$ are the local coordinates on M then the corresponding coordinates on TM are $x^1, x^2, \ldots, x^n, y^1, y^2, \ldots, y^n$ where $x^i = x^i \circ \pi, y^i = dx^i, i = 1, \ldots, n$. Similarly if $u^1, \ldots, u^{n+1}$ are the local coordinates on R^{n+1} then we get a corresponding coordinates $u^1, \ldots, u^{n+1}, v^1, \ldots, v^{n+1}$ on R^{2n+2} where we know that

$$\begin{aligned} \left(\frac{\partial}{\partial u^i}\right)^v &= \frac{\partial}{\partial v^i} \\ \left(\frac{\partial}{\partial u^i}\right)^h &= \frac{\partial}{\partial u^i}, \quad i = 1, \ldots, n+1. \end{aligned}$$

Let us denote by $D, \bar{D}$ the Euclidean connections on R^{n+1}, R^{2n+2} respectively, then recall that the connection coefficients (Christoffel symbols) Γ^k_{ij} of the Euclidean connections are zero.

For the Riemannian submersion $\widetilde{\pi}\colon R^{2n+2} \to R^{n+1}$ we have the following:

Theorem 3.1. *$\widetilde{\pi}\colon R^{2n+2} \to R^{n+1}$ is the Riemannian submersion with totally geodesic fibers R^{n+1}, that is, $T = 0$. The tensor field A on R^{2n+2} also vanishes.*

Proof. Recall that for $E, F \in \mathfrak{X}(R^{2n+2})$ we have [6]

$$T_E F = \mathfrak{H}(\bar{D}_{\mathfrak{V}E}\mathfrak{V}F) + \mathfrak{V}(\bar{D}_{\mathfrak{V}E}\mathfrak{H}F)$$
$$A_E F = \mathfrak{V}(\bar{D}_{\mathfrak{H}E}\mathfrak{H}F) + \mathfrak{H}(\bar{D}_{\mathfrak{H}E}\mathfrak{V}F).$$

Let $E = X + U, F = Y + V$ where $X, Y \in \bar{\mathfrak{H}}, U, V \in \bar{\mathfrak{V}},$, that is $X = a^i \frac{\partial}{\partial u^i}$, $Y = b^i \frac{\partial}{\partial u^i}$, $U = c^i \frac{\partial}{\partial v^i}$ and $V = d^i \frac{\partial}{\partial v^i}$. Then we have

$$\begin{aligned} T_E F &= \mathfrak{H}(\bar{D}_U V) + \mathfrak{V}(\bar{D}_U Y) \\ &= \mathfrak{H}(U(d^i)\frac{\partial}{\partial v^i}) + \mathfrak{V}(U(b^i)\frac{\partial}{\partial u^i}) = 0, \\ A_E F &= \mathfrak{V}(\bar{D}_X Y) + \mathfrak{H}(\bar{D}_X V) \\ &= \mathfrak{V}(X(b^i)\frac{\partial}{\partial u^i}) + \mathfrak{H}(X(d^i)\frac{\partial}{\partial v^i}) = 0. \end{aligned}$$

□

The following theorem is a consequence of the fact that an immersion of M in N induces an immersion of TM in TN, yet we sketch the proof for the sake of our need for an explicit expression for the differential of the induced immersion of TM in TN.

Theorem 3.2. *The map $F\colon TM \to R^{2n+2}$ is an immersion.*

Proof. Let $p \in M$ and $P = (p, X_p) \in TM$, then we have for local coordinates $x^1, \ldots, x^n$ around p, $X_p = y^i(P)(\frac{\partial}{\partial x^i})_p$ and $F(P) = df(p, X_p) = (f(p), df_p(X_p))$. The matrix for $df_p\colon T_pM \to T_{f(p)}R^{n+1}$ is the $(n+1) \times n$ matrix.

$$df_p = \begin{bmatrix} \frac{\partial f^1}{\partial x^1}(p) & \cdots & \cdots & \frac{\partial f^1}{\partial xn}(p) \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial f^{n+1}}{\partial x^1}(p) & \cdots & \cdots & \frac{\partial f^{n+1}}{\partial xn}(p) \end{bmatrix}$$

where $f^\alpha = u^\alpha \circ f$, $\alpha = 1, \ldots, n+1$. This gives

$$df_p(X_p) = \begin{bmatrix} \frac{\partial f^1}{\partial x^i}(p)y^i(P) \\ \vdots \\ \frac{\partial f^{n+1}}{\partial x^i}(p)y^i(P) \end{bmatrix}$$

consequently

$$F(P) = (f^1(p), f^2(p), \ldots, f^{n+1}(p), \frac{\partial f^1}{\partial x^i}(p)y^i(P), \ldots, \frac{\partial f^{n+1}}{\partial x^i}(p)y^i(P))$$

that is

$$F = (f^1 \circ \pi, f^2 \circ \pi, \ldots, f^{n+1} \circ \pi, (\frac{\partial f^1}{\partial x^i} \circ \pi)y^i, \ldots, (\frac{\partial f^{n+1}}{\partial x^i} \circ \pi)y^i).$$

Thus the matrix for $dF_P\colon T_P(TM) \to T_{F(P)}(R^{2n+2})$ is the $(2n+2)\times 2n$ matrix.

$$dF_P = \begin{bmatrix} \frac{\partial F^1}{\partial x^1}(P) & \cdots & \frac{\partial F^1}{\partial x^n}(P) & \frac{\partial F^1}{\partial y^1}(P) & \cdots & \frac{\partial F^1}{\partial y^n}(P) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\partial F^{n+1}}{\partial x^1}(P) & \cdots & \frac{\partial F^{n+1}}{\partial x^n}(P) & \frac{\partial F^{n+1}}{\partial y^1}(P) & \cdots & \frac{\partial F^{n+1}}{\partial y^n}(P) \\ \frac{\partial F^{n+2}}{\partial x^1}(P) & \cdots & \cdots & \cdots & \cdots & \frac{\partial F^{n+2}}{\partial y^n}(P) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\partial F^{2n+2}}{\partial x^1}(P) & \cdots & \cdots & \cdots & \cdots & \frac{\partial F^{2n+2}}{\partial y^n}(P) \end{bmatrix}$$

Note that for $\alpha = 1, \ldots, n+1$ and $j = 1, \ldots, n$ we have:

$$\frac{\partial F^\alpha}{\partial x^j}(P) = \frac{\partial (f^\alpha \circ \pi)}{\partial x^j}(p) = \frac{\partial f^\alpha}{\partial x^j}(p),$$

$$\frac{\partial F^{n+1+\alpha}}{\partial y^j}(P) = \frac{\partial((\frac{\partial f^\alpha}{\partial x^i} \circ \pi)y^i)}{\partial y^j}(P) = \frac{\partial f^\alpha}{\partial x^j}(p),$$

$$\frac{\partial F^\alpha}{\partial y^j}(P) = \frac{\partial f^\alpha}{\partial y^j}(p) = 0,$$

$$\frac{\partial F^{n+1+\alpha}}{\partial x^j}(P) = \frac{\partial((\frac{\partial f^\alpha}{\partial x^i} \circ \pi)y^i)}{\partial x^j}(P) = \frac{\partial^2 f^\alpha}{\partial x^j \partial x^i}(p) y^i(P)$$

thus we arrive at

$$dF_P = \begin{bmatrix} df_{p_{(n+1)\times n}} & 0_{(n+1)\times n} \\ (\frac{\partial^2 f^i}{\partial x^j \partial x^k}(p) y^k(P))_{(n+1)\times n} & df_{p_{(n+1)\times n}} \end{bmatrix}.$$

Hence dF_P has rank $2n$ that is $F\colon TM \to R^{2n+2}$ is an immersion. □

Thus the tangent bundle TM of the hypersurface M of the Euclidean space R^{n+1} is a submanifold of R^{2n+2}. We denote the induced Riemannian metrics on M and TM respectively by g and $\overline{g}$ respectively. Also we denote by $\nabla, \bar{\nabla}$ the Riemannian connections on M, TM respectively. We denote by N the unit normal vector field of the orientable hypersurface M. For the hypersurface M of the Euclidean space R^{n+1} we have the following Gauss and Weingarten formulae

$$D_X Y = \nabla_X Y + \langle S(X), Y \rangle N \tag{1}$$

$$D_X N = -S(X) \tag{2}$$

where $X, Y \in \mathfrak{X}(M)$ and S denotes the Weingarten map $S\colon \mathfrak{X}(M) \to \mathfrak{X}(M)$. Similarly for the submanifold TM of the Euclidean space R^{2n+2} we have the Gauss and Weingarten formulae:

$$\overline{D}_X Y = \bar{\nabla}_X Y + h(X, Y) \tag{3}$$

$$\overline{D}_X \hat{N} = -\overline{S}_{\hat{N}}(X) + \nabla^{\perp}_X \hat{N} \tag{4}$$

where $X, Y \in \mathfrak{X}(TM)$ and $\overline{S}_{\hat{N}}$ denotes the Weingarten map in the direction of the normal $\hat{N}$ which is $\overline{S}_{\hat{N}}\colon \mathfrak{X}(TM) \to \mathfrak{X}(TM)$, and is related to the second fundamental form h by

$$\left\langle h(X,Y),\hat{N}\right\rangle = \left\langle \bar{S}_{\hat{N}}(X),Y\right\rangle.$$

Also we observe that for $X \in \mathfrak{X}(M)$ the vertical lift X^v of X to TM, as $X^v \in \ker d\pi$ we have $d\pi(X^v) = 0$ that is $df(d\pi(X^v)) = 0$ or equivalently we get $d(f\circ\pi)(X^v) = 0$, that is $d(\widetilde{\pi}\circ F)(X^v) = 0$ which gives $dF(X^v) \in \ker d\widetilde{\pi} = \overline{\mathfrak{V}}$. Moreover we have the following lemmas:

Lemma 3.1. *For* $P = (p, X_p) \in TM$

$$dF_P(X_P^v) = (df_p(X_p))^v.$$

Proof. For $X = \xi^i \frac{\partial}{\partial x^i}$ we know that $X_P^v = \xi^i \frac{\partial}{\partial y^i}$. Thus we have

$$dF_P(X_P^v) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \frac{\partial f^1}{\partial x^i}(p)\xi^i \\ \vdots \\ \frac{\partial f^{n+1}}{\partial x^i}(p)\xi^i \end{bmatrix}$$

and on the other hand

$$df_p(X_p) = \begin{bmatrix} \frac{\partial f^1}{\partial x^i}(p)\xi^i \\ \vdots \\ \frac{\partial f^{n+1}}{\partial x^i}(p)\xi^i \end{bmatrix}.$$

Thus we get $(df_p(X_p))^v = dF_P(X_P^v)$. □

Remark. On a Riemannian manifold (M,g) for a smooth function $f \in C^\infty(M)$, the Hessian of the function f is defined by $H_f(X,Y) = X(Y(f)) - \nabla_X Y(f)$, $X,Y \in \mathfrak{X}(M)$, where ∇ is the Riemannian connection on M. If $X = \xi^i \frac{\partial}{\partial x^i}$ and $Y = \eta^j \frac{\partial}{\partial x^j}$ then we have

$$\begin{aligned} H_f(X,Y) &= X(\eta^j \frac{\partial f}{\partial x^j}) - \xi^i(\nabla_{\frac{\partial}{\partial x^i}} \eta^j \frac{\partial}{\partial x^j})(f) \\ &= X(\eta^j)\frac{\partial f}{\partial x^j} + \eta^j X(\frac{\partial f}{\partial x^j}) - \xi^i\eta^j\Gamma_{ij}^k \frac{\partial f}{\partial x^k} - \xi^i \frac{\partial \eta^j}{\partial x^i}\frac{\partial f}{\partial x^j} \\ &= \xi^i\eta^j \frac{\partial^2 f}{\partial x^i \partial x^j} - \xi^i\eta^j\Gamma_{ij}^k \frac{\partial f}{\partial x^k} \end{aligned}$$

where Γ_{ij}^k are the Christoffel symbols for the Riemannian connection. Thus at a point p if $X_p = \lambda^i(\frac{\partial}{\partial x^i})_p$ and $Y_p = \mu^j(\frac{\partial}{\partial x^j})_p$ we have

$$H_f(X_p,Y_p) = \lambda^i\mu^j \frac{\partial^2 f}{\partial x^i \partial x^j}(p) - \lambda^i\mu^j\Gamma_{ij}^k(p)\frac{\partial f}{\partial x^k}(p).$$

Lemma 3.2. *Let N be the unit normal vector field to the hypersurface M and $P=(p,X_p)\in TM$. Then the horizontal lift Y_P^h of $Y_p\in T_pM$ satisfies*

$$dF_P(Y_P^h)=(df_p(Y_p))^h+V_P$$

where $V_P\in\mathfrak{V}_P$ is given by $V_P=\langle S_p(X_p),Y_p\rangle\, N_P^v$.

Proof. Since

$$dF_P=\begin{bmatrix} & df_p & & 0\\ \frac{\partial^2 f^1}{\partial x^1\partial x^k}(p)y^k(P) & \cdots & \frac{\partial^2 f^1}{\partial x^n\partial x^k}(p)y^k(P) & \\ \vdots & \vdots & \vdots & df_p\\ \frac{\partial^2 f^{n+1}}{\partial x^1\partial x^k}(p)y^k(P) & \cdots & \frac{\partial^2 f^{n+1}}{\partial x^n\partial x^k}(p)y^k(P) & \end{bmatrix}$$

for $X_p=\xi^i\left(\frac{\partial}{\partial x^i}\right)_p$ and $Y_p=\eta^j\left(\frac{\partial}{\partial x^j}\right)_p$ as $Y_P^h=\eta^i\left(\frac{\partial}{\partial xi}\right)_P-\xi^k\eta^j\Gamma_{jk}^i(p)\left(\frac{\partial}{\partial y^i}\right)_P$ we have

$$\begin{aligned} dF_P(Y_P^h) &= \begin{bmatrix} df_p(Y_p)\\ \frac{\partial^2 f^1}{\partial x^\alpha\partial x^k}(p)y^k(P)\eta^\alpha-\xi^k\eta^j\Gamma_{jk}^\alpha(p)\frac{\partial f^1}{\partial x^\alpha}(p)\\ \vdots\\ \frac{\partial^2 f^{n+1}}{\partial x^\alpha\partial x^k}(p)y^k(P)\eta^\alpha-\xi^k\eta^j\Gamma_{jk}^\alpha(p)\frac{\partial f^{n+1}}{\partial x^\alpha}(p)\end{bmatrix}\\ &= \begin{bmatrix} df_p(Y_p)\\ 0\end{bmatrix}+\begin{bmatrix}0\\ H_{f^1}(Y_p,X_p)\\ \vdots\\ H_{f^{n+1}}(Y_p,X_p)\end{bmatrix}. \end{aligned}$$

(Note that $X_p=\xi^i(\frac{\partial}{\partial x^i})_p=y^i(P)(\frac{\partial}{\partial x^i})_p$, that is, $\xi^i=y^i(P)$). Consequently we get

$$dF_P(Y_P^h)=(df_p(Y_p))^h+V_P$$

where $V_P\in\mathfrak{V}_P$ and $V_P=H_{f^\alpha}(Y_p,X_p)\frac{\partial}{\partial v^\alpha}$. We know that to compute the horizontal lift Y_P^h at $P=(p,X_p)$ we need to assume that $\nabla_Y X=0$ (that is, X is parallel along integral curves of Y) (cf. [3], p.8). Thus we have from Gauss equation

$$D_{df(Y)}df(X)=\nabla_Y X+\langle S(X),Y\rangle\, N=\langle S(X),Y\rangle\, N.$$

Now for $df(X)=\lambda^\alpha\frac{\partial}{\partial u^\alpha}$, $\lambda^\alpha=df(X)(u^\alpha)=X(u^\alpha\circ f)=X(f^\alpha)$ and that D being Euclidean connection:

$$\begin{aligned} D_{df(Y)}df(X) &= YX(f^\alpha)\frac{\partial}{\partial u^\alpha}\\ &= (YX(f^\alpha)-(\nabla_Y X)(f^\alpha))\frac{\partial}{\partial u^\alpha}\\ &= H_{f^\alpha}(Y,X)\frac{\partial}{\partial u^\alpha}. \end{aligned}$$

Thus

$$H_{f^\alpha}(Y,X)\frac{\partial}{\partial u^\alpha} = \langle S(X),Y\rangle N = \langle S(X),Y\rangle (h^\alpha \frac{\partial}{\partial u^\alpha})$$

implies $H_{f^\alpha}(Y,X) = \langle S(X)Y\rangle h^\alpha$ that is

$$V_P = \left(\langle S(X),Y\rangle h^\alpha \frac{\partial}{\partial v^\alpha}\right)(P) = \langle S_p(X_p), Y_p\rangle N_P^v$$

□

Lemma 3.3. *Let $\overline{N} = (N,0) \in \mathfrak{X}(R^{2n+2})$, where N is the unit normal vector field of the hypersurface M in R^{n+1}. Then*

(1) $\overline{N} = N^h$.

(2) *$\overline{N}$ is a normal vector field to TM as a submanifold of R^{2n+2}.*

Proof. 1. We denote by $\overline{\mathfrak{H}}$ and $\overline{\mathfrak{V}}$ the horizontal and vertical distributions of the tangent bundle TR^{n+1}. Then clearly $\overline{N} \in \overline{\mathfrak{H}}$, which implies $K(\overline{N}) = 0$, where K is the connection map of the connection D, and since the matrix of $d\widetilde{\pi}$ is $d\widetilde{\pi} = [I \;\; 0]$, we get $d\widetilde{\pi}(\overline{N}) = N \circ \widetilde{\pi}$. This proves $N^h = \overline{N}$. Note that we can prove this part from the known formula for the horizontal lift given in Lemma 2.1 as follows:

Since $N = h^\alpha \frac{\partial}{\partial u^\alpha}$ and Γ^k_{ji} for the connection D vanish, $N^h = (h^\alpha \circ \widetilde{\pi})\frac{\partial}{\partial u^\alpha} = \overline{N}$.

2. It is enough to prove that for any $X, Y \in \mathfrak{X}(M)$

$$\left\langle dF(X^h), \overline{N}\right\rangle = 0 \text{ and } \left\langle dF(Y^v), \overline{N}\right\rangle = 0.$$

Now since $\widetilde{\pi}$ is a Riemannian submersion we have

$$\begin{aligned}\left\langle dF(X^h), \overline{N}\right\rangle &= \left\langle (df(X))^h, N^h\right\rangle \\ &= \left\langle d\widetilde{\pi}(df(X))^h), d\widetilde{\pi}(N^h)\right\rangle = \langle df(X), N\rangle \circ \widetilde{\pi} = 0\end{aligned}$$

as $df(X) \in \mathfrak{X}(R^{n+1})$ and N be the normal vector field to M in R^{n+1}. Also by Lemma 3.1 since $dF(Y^v) = (df(Y))^v$ we have $\left\langle dF(Y^v), \overline{N}\right\rangle = 0$. This proves that $\overline{N}$ is normal vector field to TM. □

Remark. The Euclidean space R^{2n+2} has natural complex structure J, and if we put $\tilde{N} = J\overline{N}$ then from the definition of J we have $\tilde{N} = JN^h = N^v$. Now for $X, Y \in \mathfrak{X}(M)$ we have $\left\langle dF(X^h), \tilde{N}\right\rangle = \left\langle (df(X))^h + V, N^v\right\rangle = \langle V, N^v\rangle$ and $\left\langle dF(Y^v), \tilde{N}\right\rangle = \langle (df(Y))^v, N^v\rangle$. Let $Y = \eta^j \frac{\partial}{\partial x^j}$, then we have

$$df(Y) = (\frac{\partial f^\alpha}{\partial x^i}\eta^i)\frac{\partial}{\partial u^\alpha}, \quad (df(Y))^v = ((\frac{\partial f^\alpha}{\partial x^i}\eta^i)\circ \widetilde{\pi})\frac{\partial}{\partial v^\alpha}$$

and $N^v = (h^\alpha \circ \widetilde{\pi})\frac{\partial}{\partial v^\alpha}$ which implies

$$\left\langle dF(Y^v), \tilde{N}\right\rangle = ((\frac{\partial f^\alpha}{\partial x^i}\eta^i)h^\alpha)\circ \widetilde{\pi} = \langle df(Y), N\rangle \circ \widetilde{\pi} = 0.$$

But since $\langle V, N^v\rangle \neq 0$ in general, so $\tilde{N}$ can not be a normal vector field to TM.

We choose N^* as a unit normal vector field to TM in R^{2n+2} which is orthogonal to $\overline{N}$ so that for $X, Y \in \mathfrak{X}(TM)$ we have

$$h(X,Y) = \left\langle h(X,Y), \overline{N} \right\rangle \overline{N} + \langle h(X,Y), N^* \rangle N^* = \left\langle \bar{S}_{\overline{N}} X, Y \right\rangle \overline{N} + \left\langle \bar{S}_{N^*} X, Y \right\rangle N^*.$$

Lemma 3.4. *The unit normal N^* to TM is a vertical vector field on the tangent bundle TR^{n+1}.*

Proof. Take $U \in \mathfrak{X}(R^{n+1}) \mid_M$. Then we can express it as $U = df(X) + \varphi N$, $\varphi \in C^\infty(M)$, $X \in \mathfrak{X}(M)$, consequently we have

$$U^h = (df(X))^h + (\varphi \circ \pi)\overline{N} = dF(X^h) - V_p + (\varphi \circ \pi)\overline{N} \tag{5}$$

Now since $dF(X^h) = (df(X))^h + V$, if $(df(X))^h = Y^h + b\overline{N}$ and $V_p = \gamma N^v$ where Y^h, $b\overline{N}$ are the tangential and normal components of $(df(X))^h$ respectively and $\gamma = g(S(X), Y)$. We have $dF(X^h) = Y^h + b\overline{N} + \gamma N^v$, where $b\overline{N} + \gamma N^v$ must be tangential to TM (as $dF(X^h)$ is tangent to TM). Thus $g(b\overline{N} + \gamma N^v, N^*) = 0$ which implies $\gamma g(N^v, N^*) = 0$. Also $g(b\overline{N} + \gamma N^v, \overline{N}) = 0$ proves $b = 0$, that is $\gamma N^v = V_p$ must be tangential. Taking inner product in equation (3.5) with N^*, we get $\left\langle U^h, N^* \right\rangle = 0$ for each$U \in \mathfrak{X}(R^{n+1}) \mid_M$which implies N^* must be vertical. ☐

Lemma 3.5. *For $X \in \mathfrak{X}(M)$ and $\overline{N} = (N, 0) \in \mathfrak{X}(R^{2n+2})$ we have*

$$\bar{D}_{X^h}\overline{N} = (D_X N)^h \text{ and } \bar{D}_{X^v}\overline{N} = 0.$$

Proof. Expressing locally $\overline{N} = (h^\alpha \circ \widetilde{\pi})\frac{\partial}{\partial u^\alpha}$, $h^\alpha \in C^\infty(R^{n+1})$ we compute

$$\begin{aligned} \bar{D}_{X^h}\overline{N} &= (dF(X^h))(h^\alpha \circ \widetilde{\pi})\frac{\partial}{\partial u^\alpha} = (df(X))^h(h^\alpha \circ \widetilde{\pi})\frac{\partial}{\partial u^\alpha} \\ &= ((df(X))^h(h^\alpha) \circ \widetilde{\pi})\frac{\partial}{\partial u^\alpha} \end{aligned}$$

On the other hand we have $D_X N = (df(X))(h^\alpha)\frac{\partial}{\partial u^\alpha}$ and

$$(D_X N)^h = ((df(X))^h(h^\alpha) \circ \widetilde{\pi})\frac{\partial}{\partial u^\alpha} = \bar{D}_{X^h}\overline{N}$$

For the second relation we have

$$\bar{D}_{X^v}\overline{N} = (dF(X^v))(h^\alpha \circ \widetilde{\pi})\frac{\partial}{\partial u^\alpha} = ((df(X))^v(h^\alpha \circ \widetilde{\pi})\frac{\partial}{\partial u^\alpha} = 0.$$

☐

Corollary 3.1. *For $X \in \mathfrak{X}(M)$ we have $(S(X))^h = -\bar{D}_{X^h}\overline{N}$.*

Proof. From equation (3.2) and Lemma 3.5 we have $(S(X))^h = -(D_X N)^h = -\bar{D}_{X^h}\overline{N}$. ☐

Example. Take $M = S^2$ and $f\colon S^2 \to R^3$ the inclusion $f(z^1, z^2, z^3) = (z^1, z^2, z^3)$ for $(z^1, z^2, z^3) \in S^2$. Let $p = (z^1, z^2, z^3) \in S^2$ be a point with $z^3 > 0$ and take a chart (U, ϕ) around p where $U = \{(z^1, z^2, z^3) \in S^2 : z^3 > 0\ \}$ and

$$\phi\colon U \to B_1(0) \subset R^2, \phi(z^1, z^2, z^3) = (z^1, z^2),$$
$$\phi^{-1}(u^1, u^2) = (u^1, u^2, \sqrt{1 - (u^1)^2 - (u^2)^2}).$$

Let x^1, x^2 be the local coordinates on U and u^1, u^2, u^3 be the Euclidean coordinates on R^3. Then

$$\begin{aligned} f^\alpha &= u^\alpha \circ f = z^\alpha, \alpha = 1,2,3 \\ \frac{\partial f^i}{\partial x^j}(p) &= \delta^i_j,\ \ i,j = 1,2. \\ \frac{\partial f^3}{\partial x^i}(p) &= \frac{\partial (f^3 \circ \phi^{-1})}{\partial u^i}(\phi(p)) = \frac{\partial(\sqrt{1-(u^1)^2-(u^2)^2})}{\partial u^i}(\phi(p)) \\ &= \frac{-u^i}{\sqrt{1-(u^1)^2-(u^2)^2}}(\phi(p)) \\ &= \frac{-z^i}{z^3}, i = 1,2 \end{aligned}$$

and

$$df_p = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \frac{-z^1}{z^3} & \frac{-z^2}{z^3} \end{bmatrix}.$$

Now let $P = (p, X_p)$ where $X = \xi^i \frac{\partial}{\partial x^i} \in \mathfrak{X}(S^2)$, then

$$F = df = (f^1 \circ \pi, f^2 \circ \pi, f^3 \circ \pi, y^1, y^2, -\frac{u^i}{u^3} y^i)$$

where x^1, x^2, y^1, y^2 are the local coordinates with respect to the chart on TS^2 corresponding to (U, ϕ) on S^2. We get

$$\begin{aligned} \frac{\partial F^{3+\alpha}}{\partial x^i}(P) &= \frac{\partial (y^\alpha)}{\partial x^i}(P) = 0, \quad \alpha, i = 1,2 \\ \frac{\partial F^6}{\partial x^j}(P) &= -\frac{\partial(\frac{u^i y^i}{u^3})}{\partial x^j}(P) = -(\frac{u^3 y^i \delta^i_j\ - u^i y^i (\frac{-u^j}{u^3})}{(u^3)^2})(P) \\ &= \frac{-(u^3)^2 y^j - \sum_i u^i y^i u^j}{(u^3)^3}(P) = \frac{-(z^3)^2 \xi^j - z^i \xi^i z^j}{(z^3)^3}\ j = 1,2. \end{aligned}$$

Note that $y^\alpha(P) = \xi^\alpha, \alpha = 1,2$, so we get

$$dF_P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{-z^1}{z^3} & \frac{-z^2}{z^3} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{-(z^3)^2\xi^1 \; -z^i\xi^i z^1}{(z^3)^3} & \frac{-(z^3)^2\xi^2 \; -z^i\xi^i z^2}{(z^3)^3} & \frac{-z^1}{z^3} & \frac{-z^2}{z^3} \end{bmatrix}.$$

Now for $Y = \eta^i \frac{\partial}{\partial x^i} \in \mathfrak{X}(S^2)$ we have $Y^h = \eta^i \frac{\partial}{\partial x^i} - \eta^j \xi^k \Gamma^i_{jk} \frac{\partial}{\partial y^i}$ consequently

$$dF_P(Y^h_P) = \begin{bmatrix} \eta^1 \\ \eta^2 \\ \frac{-z^1\eta^1 - z^2\eta^2}{z^3} \\ -\eta^j\xi^k\Gamma^1_{jk} \\ -\eta^j\xi^k\Gamma^2_{jk} \\ \{(\frac{-(z^3)^2\xi^\alpha \; -z^i\xi^i z^\alpha}{(z^3)^3})\eta^\alpha + \frac{z^\alpha}{z^3}(\eta^j\xi^k\Gamma^\alpha_{jk})\} \end{bmatrix}$$

that is

$$dF_P(Y^h_P) = \begin{bmatrix} (df(Y_p))^h \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -\eta^j\xi^k\Gamma^1_{jk} \\ -\eta^j\xi^k\Gamma^2_{jk} \\ \{(\frac{-(z^3)^2\xi^\alpha \; -z^i\xi^i z^\alpha}{(z^3)^3})\eta^\alpha + \frac{z^\alpha}{z^3}(\eta^j\xi^k\Gamma^\alpha_{jk})\} \end{bmatrix}.$$

Now we need to compute the connection coefficients of Γ^k_{ji} of the connection ∇ with respect to this chart on S^2. Since

$$\frac{\partial}{\partial x^i} = \frac{\partial}{\partial u^i} - \frac{u^i}{u^3}\frac{\partial}{\partial u^3}$$

we get for $i, j = 1, 2$

$$\begin{aligned} g_{ij} &= g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = \left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right\rangle \\ &= \left\langle \frac{\partial}{\partial u^i} - \frac{u^i}{u^3}\frac{\partial}{\partial u^3}, \frac{\partial}{\partial u^j} - \frac{u^j}{u^3}\frac{\partial}{\partial u^3} \right\rangle = \delta^i_j + \frac{u^i u^j}{(u^3)^2} \end{aligned}$$

and consequently

$$(g_{ij}) = \begin{bmatrix} 1 + \left(\frac{u^1}{u^3}\right)^2 & \frac{u^1u^2}{(u^3)^2} \\ \frac{u^1u^2}{(u^3)^2} & 1 + \left(\frac{u^2}{u^3}\right)^2 \end{bmatrix}$$

and

$$(g^{ij}) = \begin{bmatrix} (u^3)^2 + (u^2)^2 & -u^1u^2 \\ -u^1u^2 & (u^3)^2 + (u^1)^2 \end{bmatrix}.$$

Using

$$\Gamma_{ij}^k = \frac{1}{2} g^{\alpha k} \left\{ \frac{\partial g_{i\alpha}}{\partial u^j} - \frac{\partial g_{ij}}{\partial u^\alpha} + \frac{\partial g_{\alpha j}}{\partial u^i} \right\}$$

we arrive at

$$\begin{aligned}
\Gamma_{11}^1 &= \frac{u^1((u^3)^2 + (u^1)^2)}{(u^3)^2} \\
\Gamma_{11}^2 &= \frac{u^2((u^3)^2 + (u^1)^2)}{(u^3)^2} \\
\Gamma_{12}^1 &= \Gamma_{21}^1 = \frac{(u^1)^2 u^2}{(u^3)^2} \\
\Gamma_{12}^2 &= \Gamma_{21}^2 = \frac{(u^2)^2 u^1}{(u^3)^2} \\
\Gamma_{22}^1 &= \frac{u^1((u^3)^2 + (u^2)^2)}{(u^3)^2} \\
\Gamma_{22}^2 &= \frac{u^2((u^3)^2 + (u^2)^2)}{(u^3)^2}
\end{aligned}$$

which gives

$$dF_P(Y_P^h) = \begin{bmatrix} (df(Y_p))^h \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ -u^1 \langle X, Y \rangle \\ -u^2 \langle X, Y \rangle \\ -u^3 \langle X, Y \rangle \end{bmatrix}$$

where $N = u^\alpha \frac{\partial}{\partial u^\alpha} \in \mathfrak{X}(R^3)$ is the unit normal vector field to S^2 and

$$\langle X, Y \rangle = \eta^1 \xi^1 + \eta^2 \xi^2 + \frac{1}{(u^3)^2} (\eta^1 u^1 + \eta^2 u^2)(\xi^1 u^1 + \xi^2 u^2).$$

Remark. We observe that the metrics defined on TM using the Riemannian metric of M (such as Sasaki metric, Cheeger-Gromoll metric, Oproiu metric) are natural metrics in the sense that the submersion $\pi\colon TM \to M$ becomes a Riemannian submersion with respect to these metrics. However, the induced metric on the tangent bundle TM of a hypersurface M of the Euclidean space R^{n+1}, as a submanifold of R^{2n+2} is not a natural metric because of the presence of the term V_P (see Lemma 3.2).

4. A special case

In this section we study the hypersurfaces $f\colon M \to R^{n+1}$ satisfying

$$dF_P(X_P^h) = (df_p(X_p))^h,$$

that is the hypersurfaces for which the vector field $V = 0$. We call these hypersurfaces generic hypersurfaces of the Euclidean space R^{n+1}. A trivial example of a generic hypersurface of the Euclidean space, is the totally geodesic hypersurface R^n of R^{n+1} (this follows from Lemma 3.2). The natural embedding $f: S^1 \to R^2$, $f(x,y) = (x,y)$ of the unit circle gives another example of a generic hypersurface. The tangent space at each point $p \in S^1$ is spanned by the unit vector $\xi_p = \left(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right)_p$ and that $\nabla_\xi \xi = 0$, that is $\Gamma^1_{11} = 0$ consequently, it can be easily verified that $dF_P(\xi^h_P) = (df_p(\xi_p))^h$.

Lemma 4.1. *For a generic hypersurface M of the Euclidean space R^{n+1} the induced metric $\bar{g}$ on TM as a submanifold of R^{2n+2} is a natural metric with respect to g on M.*

Proof. For $X, Y \in \mathfrak{X}(M)$ we have:

$$\begin{aligned}
\bar{g}(X^h, Y^h) &= \left\langle dF(X^h), dF(Y^h)\right\rangle \circ F = \left\langle (df(X))^h, (df(Y))^h\right\rangle \circ F \\
&= \langle df(X), df(Y)\rangle \circ \widetilde{\pi} \circ F = \langle df(X), df(Y)\rangle \circ f \circ \pi \\
&= g(X,Y) \circ \pi. \\
\bar{g}(X^h, Y^v) &= \left\langle dF(X^h), dF(Y^v)\right\rangle \circ F = \left\langle (df(X))^h, (df(Y))^v\right\rangle \circ F = 0.
\end{aligned}$$

□

Remark. Note that for a generic hypersurface M of R^{n+1} the submersion $\pi: TM \to M$ is a Riemannian submersion.

Recall that for the unit normal vector field $N = h^\alpha \frac{\partial}{\partial u^\alpha} \in \mathfrak{X}(R^{n+1})$ to M we have a unit normal vector field $\overline{N} = N^h = (h^\alpha \circ \widetilde{\pi})\frac{\partial}{\partial u^\alpha} \in \mathfrak{X}(R^{2n+2})$ to TM. Now put $N^* = J\overline{N} = N^v = (h^\alpha \circ \widetilde{\pi})\frac{\partial}{\partial v^\alpha}$ thus we have the following:

Lemma 4.2. *For a generic hypersurface M of R^{n+1}, $N^* = J\overline{N}$ is the normal vector field to TM in R^{2n+2} which is orthogonal to $\overline{N}$.*

Proof. For $X, Y \in \mathfrak{X}(M)$ we have $\left\langle dF(X^h), N^*\right\rangle = \left\langle (df(X))^h, N^v\right\rangle = 0$ and we know from the first section that $\langle dF(Y^v), N^*\rangle = \langle (df(Y))^v, N^v\rangle = 0$. That is $N^* = J\overline{N}$ is a normal vector field to TM. □

Lemma 4.3. *For a generic hypersurface M of R^{n+1}, $X \in \mathfrak{X}(M)$*

$$\bar{D}_{X^h} N^* = (D_X N)^v \bar{D}_{X^v} N^* = 0.$$

Proof. We have

$$\bar{D}_{X^h} N^* = (dF(X^h))(h^\alpha \circ \widetilde{\pi})\frac{\partial}{\partial v^\alpha} = ((df(X)(h^\alpha)) \circ \widetilde{\pi})\frac{\partial}{\partial v^\alpha}$$

and $D_X N = (df(X)(h^\alpha)\frac{\partial}{\partial u^\alpha}$ which gives

$$(D_X N)^v = ((df(X)(h^\alpha)) \circ \widetilde{\pi})\frac{\partial}{\partial v^\alpha} = \bar{D}_{X^h} N^*.$$

For the other equation we have

$$\bar{D}_{X^v}N^* = (dF(X^v))(h^\alpha \circ \widetilde{\pi})\frac{\partial}{\partial v^\alpha} = (df(X))^v(h^\alpha \circ \widetilde{\pi})\frac{\partial}{\partial v^\alpha} = 0.$$

□

Corollary 4.1. *For a generic hypersurface M of R^{n+1}, $(S(X))^v = -\bar{D}_{X^h}N^*$, $X \in \mathfrak{X}(M)$*

Proof. Since $S(X) = -D_X N$ we get $(S(X))^v = -(D_X N)^v = -\bar{D}_{X^h}N^*$. □

Corollary 4.2. *For a generic hypersurface M of R^{n+1} with $X \in \mathfrak{X}(M)$:*
1) $\bar{S}_{\overline{N}}X^h = (S(X))^h$,
2) $\bar{S}_{N^*}X^h = (S(X))^v$,
3) $\bar{S}_{\overline{N}}X^v = 0$,
4) $\bar{S}_{N^*}X^v = 0$.

Proof. 1) From Corollary 3.1 we have

$$(S(X))^h = -\bar{D}_{X^h}\overline{N} = -\left[-\bar{S}_{\overline{N}}(X^h) + \nabla^\perp_{X^h}\overline{N}\right] == \bar{S}_{\overline{N}}(X^h) - \nabla^\perp_{X^h}\overline{N}.$$

Equating the tangential and normal components we get

$$\nabla^\perp_{X^h}\overline{N} = 0 \text{ and } \bar{S}_{\overline{N}}(X^h) = (S(X))^h.$$

2) Similarly, from Corollary 4.1 we have

$$(S(X))^v = -\bar{D}_{X^h}N^* = \bar{S}_{N^*}(X^h) - \nabla^\perp_{X^h}N^*.$$

Equating the tangential and normal components we get:

$$\nabla^\perp_{X^h}N^* = 0 \text{ and } \bar{S}_{N^*}(X^h) = (S(X))^v.$$

3) From Lemma 3.5 we have

$$0 = \bar{D}_{X^v}\overline{N} = -\bar{S}_{\overline{N}}(X^v) + \nabla^\perp_{X^v}\overline{N}.$$

Equating the tangential and normal components we get:

$$\nabla^\perp_{X^v}\overline{N} = 0 \text{ and } \bar{S}_{\overline{N}}(X^v) = 0.$$

4) From Lemma 4.3 we have

$$0 = \bar{D}_{X^v}N^* = -\bar{S}_{N^*}(X^v) + \nabla^\perp_{X^v}N^*.$$

Equating the tangential and normal components we get

$$\nabla^\perp_{X^v}N^* = 0 \text{ and } \bar{S}_{N^*}(X^v) = 0.$$

□

Corollary 4.3. *For a generic hypersurface M of R^{n+1} with $X, Y \in \mathfrak{X}(M)$:*
1) $h(X^v, Y^v) = 0$,
2) $h(X^v, Y^h) = 0$,
3) $h(X^h, Y^v) = 0$,
4) $h(X^h, Y^h) = (\langle S(X), Y\rangle \circ \widetilde{\pi})N^h$.

Proof. Since $h(X,Y) = \left\langle \bar{S}_{\overline{N}}X, Y\right\rangle \overline{N} + \left\langle \bar{S}_{N^*}X, Y\right\rangle N^*$, using corollary 4.2 we get

$$\begin{aligned}
1)\ h(X^v,Y^v) &= \left\langle \bar{S}_{\overline{N}}X^v, Y^v\right\rangle \overline{N} + \left\langle \bar{S}_{N^*}X^v, Y^v\right\rangle N^* = 0,\\
2)\ h(X^v,Y^h) &= \left\langle \bar{S}_{\overline{N}}X^v, Y^h\right\rangle \overline{N} + \left\langle \bar{S}_{N^*}X^v, Y^h\right\rangle N^* = 0,\\
3)\ h(X^h,Y^v) &= \left\langle \bar{S}_{\overline{N}}X^h, Y^v\right\rangle \overline{N} + \left\langle \bar{S}_{N^*}X^h, Y^v\right\rangle N^*,\\
&= \left\langle (S(X))^h, Y^v\right\rangle \overline{N} + \left\langle X^h, \bar{S}_{N^*}Y^v\right\rangle N^* = 0,\\
4)\ h(X^h,Y^h) &= \left\langle \bar{S}_{\overline{N}}X^h, Y^h\right\rangle \overline{N} + \left\langle \bar{S}_{N^*}X^h, Y^h\right\rangle N^*,\\
&= \left\langle (S(X))^h, Y^h\right\rangle \overline{N} = (\langle S(X), Y\rangle \circ \widetilde{\pi})N^h.
\end{aligned}$$

□

Theorem 4.1. *For a generic hypersurface M of R^{n+1}, the tensor field T of the Riemannian submersion $\pi\colon TM \to M$ vanishes.*

Proof. We have for $E, F \in \mathfrak{X}(TM)$ (cf. [6])

$$T_EF = \mathfrak{H}(\bar{\nabla}_{\mathfrak{V}E}\mathfrak{V}F) + \mathfrak{V}(\bar{\nabla}_{\mathfrak{V}E}\mathfrak{H}F).$$

Thus if $X, Y \in \mathfrak{X}(M)$, then

(6) $$T_{X^h}Y^h = T_{X^h}Y^v = 0 \text{ as } T_E = T_{\mathfrak{V}E}$$

(7) $$T_{X^v}Y^v = \mathfrak{H}(\bar{\nabla}_{X^v}Y^v) = \mathfrak{H}(\bar{D}_{X^v}Y^v - h(X^v,Y^v))$$

but as $\bar{D}_{X^v}Y^v = X^v(\eta^i \circ \pi)\frac{\partial}{\partial y^i} = 0$ where $Y = \eta^i\frac{\partial}{\partial x^i}$. Then the Corollary 4.3 gives $T_{X^v}Y^v = 0$.

(8) $$T_{X^v}Y^h = \mathfrak{V}(\bar{\nabla}_{X^v}Y^h).$$

For $Z \in \mathfrak{X}(M)$ we use (6) to compute

$$\bar{g}(\bar{\nabla}_{X^v}Y^h, Z^v) = -\bar{g}(Y^h, \bar{\nabla}_{X^v}Z^v) = -\bar{g}(Y^h, \mathfrak{H}(\bar{\nabla}_{X^v}Z^v)) = -\bar{g}(Y^h, T_{X^v}Z^v) = 0$$

which implies $\mathfrak{V}(\bar{\nabla}_{X^v}Y^h) = 0 \Rightarrow T_{X^v}Y^h = 0$. Thus $T = 0$. □

Theorem 4.2. *For a generic hypersurface M of R^{n+1}, the tensor field A of the Riemannian submersion $\pi\colon TM \to M$ vanishes.*

Proof. For $E, F \in \mathfrak{X}(TM)$ we have (cf. [6])

$$A_EF = \mathfrak{V}(\bar{\nabla}_{\mathfrak{H}E}\mathfrak{H}F) + \mathfrak{H}(\bar{\nabla}_{\mathfrak{H}E}\mathfrak{V}F).$$

Taking $X, Y \in \mathfrak{X}(M)$ we compute

(9) $$A_{X^v}Y^h = A_{X^v}Y^v = 0 \text{ as } A_E = A_{\mathfrak{H}E}$$

(10) $$A_{X^h}Y^v = \mathfrak{H}(\bar{\nabla}_{X^h}Y^v) = \mathfrak{H}(\bar{D}_{X^h}Y^v - h(X^h,Y^v)) = \mathfrak{H}(\bar{D}_{X^h}Y^v)$$

As for $Y = \eta^i\frac{\partial}{\partial x^i}$ we have

$$\bar{D}_{X^h}Y^v = X^h(\eta^i \circ \pi)\frac{\partial}{\partial y^i} = X(\eta^i) \circ \pi\frac{\partial}{\partial y^i}.$$

and $D_XY = X(\eta^i)\frac{\partial}{\partial x^i}$. Thus we get $\bar{D}_{X^h}Y^v = (D_XY)^v$ consequently $A_{X^h}Y^v = 0$.

$$A_{X^h}Y^h = \mathfrak{V}(\bar{\nabla}_{X^h}Y^h). \tag{11}$$

Taking $Z \in \mathfrak{X}(M)$ we have

$\bar{g}(\bar{\nabla}_{X^h}Y^h, Z^v) = -\bar{g}(Y^h, \bar{\nabla}_{X^h}Z^v) = -\bar{g}(Y^h, \mathfrak{H}(\bar{\nabla}_{X^h}Z^v)) = -\bar{g}(Y^h, A_{X^h}Z^v) = 0$

which implies $\mathfrak{V}(\bar{\nabla}_{X^h}Y^h) = 0$ that is $A_{X^h}Y^h = 0$. Thus we have $A = 0$. □

Theorem 4.3. *If α is the mean curvature of the a generic hypersurface M of R^{n+1} and H be the mean curvature vector field for the submanifold TM of R^{2n+2} then we have*

$$H = \frac{1}{2}(\alpha \circ \pi)\overline{N}$$

Proof. Choosing normal coordinates on a normal neighbourhood of M we choose a local orthonormal frame $X^1, X^2, \ldots, X^n$ with respect to these local coordinates. Then we get a local orthonormal frame

$$X^{1^h}, X^{2^h}, \ldots, X^{n^h}, X^{1^v}, X^{2^v}, \ldots, X^{n^v}$$

on TM. We know that $\alpha = \frac{1}{n}\sum_{i=1}^n g(S(X^i), X^i)$, where $S\colon \mathfrak{X}(M) \to \mathfrak{X}(M)$ is the Weingarten map. Using Corollary 4.2 we compute

$$\begin{aligned} H &= \frac{1}{2n}\sum_{i=1}^n\{h(X^{i^h}, X^{i^h}) + h(X^{i^v}, X^{i^v})\} \\ &= \frac{1}{2n}\sum_{i=1}^n\{\left\langle \bar{S}_{\overline{N}}X^{i^h}, X^{i^h}\right\rangle \overline{N} + \left\langle \bar{S}_{N^*}X^{i^h}, X^{i^h}\right\rangle N^* \\ &\quad + \langle \bar{S}_{\overline{N}}X^{i^v}, X^{i^v}\rangle \overline{N} + \langle \bar{S}_{N^*}X^{i^v}, X^{i^v}\rangle N^*\} \\ &= \frac{1}{2n}\sum_{i=1}^n \left\langle (S(X^i))^h, X^{i^h}\right\rangle \overline{N} = \frac{1}{2n}\sum_{i=1}^n(\langle S(X^i), X^i\rangle \circ \widetilde{\pi})\overline{N} \\ &= \frac{1}{2n}\sum_{i=1}^n(g(S(X^i), X^i) \circ \pi)\overline{N} = \frac{1}{2}(\alpha \circ \pi)\overline{N} \end{aligned}$$

□

Finally, we prove the following theorem:

Theorem 4.4. *The tangent bundle TM of a generic hypersurface M of R^{n+1} is trivial.*

Proof. Since the fundamental tensors A and T of the Riemannian submersion $\pi\colon TM \to M$ are zero, both horizontal and vertical distributions $\mathfrak{H}$ and $\mathfrak{V}$ are integrable. Also the leaves of the distributions $\mathfrak{H}$ and $\mathfrak{V}$ are totally geodesic submanifolds of TM (cf. [6]). Moreover the leaves of $\mathfrak{V}$ are totally geodesic submanifolds of R^{2n+2} by corollary 4.3 and consequently are R^n. Moreover the restriction of π to the leaves of $\mathfrak{H}$ is an isometry thus leaves of $\mathfrak{H}$ are isometric to M and consequently we get that $TM = M \times R^n$ that is TM is trivial. □

Corollary 4.4. *The tangent bundle TS^2 of $f\colon S^2 \to R^3$, where f is the inclusion does not satisfy $dF(X^h) = (df(X))^h$, $X \in \mathfrak{X}(S^2)$, or equivalently S^2 not a generic hypersurface of R^3.*

Proof. If S^2 is a generic hypersurface, then by above theorem we get TS^2 is trivial. Which would imply that the Euler characteristic $\chi(S^2) = 0$, which is a contradiction as $\chi(S^2) = 2$. The proof also can be obtained from the example in section-3 by deriving a contradiction with the assumption that the vector field $V = 0$. □

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