Acta Mathematica Academiae Paedagogicae Nyíregyháziensis
23 (2007), 55–63
www.emis.de/journals
ISSN 1786-0091

ON THE FIXED POINT OF A COLLINEATION OF THE REAL PROJECTIVE PLANE

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Abstract. Using the "extended Euclidean plane" model we prove the existence of the fixed point of a collineation of the real projective plane. At first we obtain the collineation as a product of a reflection in a line, a reflection in a point and a central-axial collineation. Then we prove the existence of the fixed point of the product of the second and the third mappings, and also that it is possible to choose the center of the second one so that this fixed point will lie on the axis of the first one. We examine the locus of the mentioned fixed point, too.

1. Introduction

The theorem on the existence of the fixed point of a collineation of the real projective plane is proved most frequently by using analytical methods. There are synthetic ways, too: e.g. in [1], [3], [6] the fixed points are points of intersection of certain conics; in [4], [5] there is a special proof which is mainly based on Dedekind's axiom of continuity.

In this paper we will construct another synthetic proof. Our method will not be a pure projective one because we will use the so-called *Euclidean plane extended by ideal elements model* and Euclidean metrical concepts. The principle of the proof is to obtain the given collineation as a product of three transformations whose certain properties can be chosen arbitrarily. We determine these properties that a fixed point under the product of the second and the third transformations will exist, and also that this point will be fixed under the first one, too. Hence, this point is fixed under the given collineation.

Due to the principle of duality it is enough to prove either the existence of a fixed point or that of an invariant line. Our aim is to prove this theorem:

Theorem. *Any collineation of the real projective plane has either a fixed point or an invariant line.*

2000 *Mathematics Subject Classification.* 51M04, 51N20, 51A05.
Key words and phrases. real projective plane-geometry, fixed point of a collineation.

Any central-axial collineation has fixed points: e.g. its center. The ideal line is invariant under any affine collineation. Thus in the sequel we have to examine only the non-affine, non-central-axial collineations. The proof is based on the following three lemmas:

Lemma 1. *Any non-affine, non-central-axial collineation of the real projective plane can be obtained as a product of an opposite isometry and either an elation or a homology with a positive characteristical cross ratio.*

Lemma 2. *Let us consider the product of a reflection in a point and a non-affine elation with distinct centers. This product has an ordinary fixed point on the line of the centers.*

Lemma 3. *Let us consider the product of a reflection in a point and a non-affine homology with a positive characteristical cross ratio with distinct centers. This product has an ordinary fixed point on the line of the centers.*

We remark here that in this paper we use directed line segments. The following properties of the central-axial collineations will be important in the proofs:

In the case of the non-affine homology with center C, axis t and characteristical cross ratio λ if $P \neq C, P \notin t$, then $(P'PCM) = \lambda$, where $M := (CP) \cap t$. In the case of ordinary points $\frac{CP'}{MP'} = \lambda \frac{CP}{MP}$; if M is an ideal point, then $CP' = \lambda CP$.

In the case of the non-affine elation with center C, axis t and line of direction e (the image line of the ideal line; $e \parallel t$) if $P(\notin t)$ is an ordinary point, then $\frac{CP'}{PP'} = \frac{LC}{CP}$, where $L := (CP) \cap e$.

2. The proofs of the lemmas

Let us denote the ideal line by i, its image under the given collineation by i'.

Proof of Lemma 1. Consider a non-affine, non-central-axial collineation $\mathbf{\Phi}$. Let P', Q', R' denote three non-collinear ordinary points so that $(P'Q') \parallel i'$ and $R' \notin i'$. The originals of P', Q', R' under $\mathbf{\Phi}$ are P, Q, R, respectively; they are non-collinear, ordinary points, too. $\mathbf{\Phi}$ is uniquely determined by i, P, Q, R and their images under $\mathbf{\Phi}$. Consider the opposite similarity $\mathbf{S}$ which transforms P to P', Q to Q'. Let $X := \mathbf{S}(R)$, and $\mu(> 0)$ is the ratio of $\mathbf{S}$. Now consider the central-axial collineation $\mathbf{\Phi_1}$ with axis $(P'Q')$ which transforms i to i' and X to R'; the center of $\mathbf{\Phi_1}$ is denoted by C. Let us express $\mathbf{S}$ as the product of an opposite isometry $\mathbf{M}$ and a central dilatation $\mathbf{N}$ with center C. The ratio of $\mathbf{N}$ can be either μ or $-\mu$, we will choose it later. The product of $\mathbf{N}$ and $\mathbf{\Phi_1}$ is a line-preserving transformation with center C which transforms i to i'. Thus this product is a non-affine central-axial collineation $\mathbf{\Phi_2}$ with center C. The original collineation $\mathbf{\Phi}$ is obtained as the product of the opposite isometry $\mathbf{M}$ and the central-axial collineation $\mathbf{\Phi_2}$. Now we show that

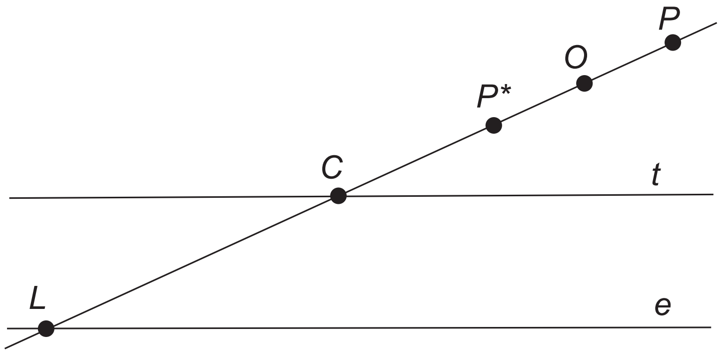

FIGURE 1

it is possible to choose the ratio of $\mathbf{S}$ so that $\mathbf{\Phi_2}$ will be either an elation or a homology with a positive characteristical cross ratio. If $\mathbf{\Phi_1}$ is an elation and $\mu = 1$, then let us choose 1 as the ratio of $\mathbf{S}$. If $\mathbf{\Phi_1}$ is an elation and $\mu \neq 1$, then $\mathbf{\Phi_2}$ is a homology whose characteristical cross ratio equals the ratio of $\mathbf{S}$. In this case let us choose μ as the ratio of $\mathbf{S}$. If $\mathbf{\Phi_1}$ is a homology, then $\mathbf{\Phi_2}$ is either an elation or a homology whose characteristical cross ratio equals the product of the ratio of $\mathbf{S}$ and the characteristical cross ratio of $\mathbf{\Phi_1}$, depending on this product whether it is 1 or not. In this case let us choose the ratio of $\mathbf{S}$ so that the product of this ratio and the characteristical cross ratio of $\mathbf{\Phi_1}$ will be positive. The proof is completed. □

Proof of Lemma 2. (Fig. 1.) We use the same notations in connection with the elation $\mathbf{\Phi_2}$ as at the end of Paragraph 1.; the center of the reflection in a point is denoted by O, $O \neq C$, and at first let $O \notin t$. We are looking for an ordinary point P on line (CO) for which $\mathbf{\Phi_2}(P^*) = P$, where P^* denotes the image of P under the reflection in point O. This equation holds — due to the formula in Paragraph 1. — iff $\frac{CP}{P^*P} = \frac{LC}{CP^*}$. Due to the reflection: $\frac{CP}{P^*P} = \frac{CO+OP}{2OP}$, $\frac{LC}{CP^*} = \frac{LC}{CO-OP}$. Let $x := \frac{OP}{CO}$ and $c := \frac{LC}{CO} (\neq 0)$; the second one is constant, independent of P. Applying these equations, if $x \neq 0, 1$, then we get the following condition for the fixed point:

$$\frac{1+x}{2x} = \frac{c}{1-x}, \quad x^2 + 2cx - 1 = 0.$$

Neither 0 nor 1 is a solution, because $c \neq 0$. The discriminant is $4c^2 + 4 > 0$, thus the equation has two distinct real solutions: $x_{1,2} = -c \pm \sqrt{c^2+1}$. Using $OP = xCO$ we get the ordinary fixed point P, if $O \notin t$.

If $O \in t$ then O is obviously fixed and $\frac{OO}{CO} = 0$. On the other hand, $c \to -\infty$ or $c \to \infty$ depending on whether O tends to t in the halfplane containing e or in the other one. $\lim\limits_{c\to\infty} \left(-c + \sqrt{c^2+1}\right) = \lim\limits_{c\to-\infty} \left(-c - \sqrt{c^2+1}\right) = 0$. Thus the formula mentioned above determines — as a limiting case — an ordinary fixed point in the case of $O \in t$, too. □

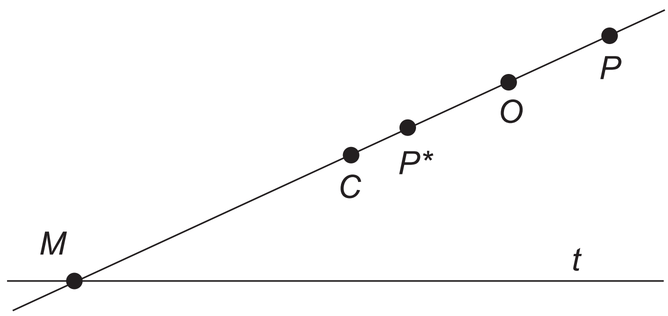

FIGURE 2

Proof of Lemma 3. (Fig. 2.) We use the same notations in connection with the homology $\mathbf{\Phi_2}$ as at the end of Paragraph 1.; the center of the reflection in a point is denoted by O, $O \neq C$, and at first let $(CO) \nparallel t$. We are looking for an ordinary point P on (CO), for which $\mathbf{\Phi_2}(P^*) = P$, where P^* denotes the image of P under the reflection in point O. This equation holds — due to the formula in Paragraph 1. — iff $\frac{CP}{MP} = \lambda \frac{CP^*}{MP^*}$, $\lambda \neq 1$, $\lambda > 0$. $\frac{CP}{MP} = \frac{CO+OP}{MO+OP}$, and due to the reflection: $\frac{CP^*}{MP^*} = \frac{CO-OP}{MO-OP}$. Let $x := \frac{OP}{CO}$ and $a := \frac{MO}{CO} (\neq 1)$; the second one is constant, independent of P. Applying these equations, if $x \neq \pm a$, then we get the following condition for the fixed point:

$$\frac{1+x}{a+x} = \lambda \frac{1-x}{a-x}, \quad x^2 + \frac{\lambda+1}{\lambda-1}(a-1)x - a = 0.$$

In the case of $x = \pm a$ either $a(a-1)\frac{2\lambda}{\lambda-1} = 0$ or $a(1-a)\frac{2}{\lambda-1} = 0$, so $\pm a$ are solutions iff $a = 0$. If $a = 0$, then $x_{1,2} = 0, \frac{\lambda+1}{\lambda-1}$. $x_2 \neq 0$, thus this solution determines a fixed point. $a = \frac{MO}{CO} = 0$ iff $M = O$, $O \in t$. In this case O is obviously a fixed point, so the $x_1 = 0$ solution determines a fixed point, too. In the sequel $a \neq 0$; let $r := \frac{\lambda+1}{\lambda-1}$ and $s := a - 1$. Now $x^2 + rsx - (s+1) = 0$, its discriminant is $r^2s^2 + 4s + 4$, $(r^2 > 0)$. The discriminant of this second formula is $16(1-r^2) = -\frac{64\lambda}{(\lambda-1)^2} < 0$. Thus the first discriminant is positive, so the equation determining the fixed point has two distinct solutions:

$$x_{1,2} = \frac{-rs \pm \sqrt{r^2s^2 + 4\,(s+1)}}{2}.$$

If in this formula $s = -1$ $(a = 0)$, then it gives the solutions of the "$a = 0$" case. Hence, this formula determines the fixed point not depending on a. Using $OP = xCO$ we get the ordinary fixed point P if $(CO) \nparallel t$.

If $(CO) \parallel t$ (M is an ideal point), then the condition for the fixed point P is $CP = \lambda CP^*$, $\lambda \neq 1$, $\lambda > 0$ (according to the formula in Paragraph 1.). $CO + OP = \lambda(CO - OP)$ so $\frac{OP}{CO} = \frac{\lambda-1}{\lambda+1} = \frac{1}{r}$. On the other hand, $s \to -\infty$ or $s \to \infty$ depending on whether O tends to the line containing C, being parallel to t in the halfplane containing t or in the other one. If $r > 0$, then $\lim\limits_{s\to\infty} \left(-rs + \sqrt{r^2s^2 + 4(s+1)}\right) = \lim\limits_{s\to-\infty} \left(-rs - \sqrt{r^2s^2 + 4(s+1)}\right) = \frac{2}{r}$. If

$r < 0$, then changing the signs of the square roots we get the same limits. Thus the formula mentioned above determines — as a limiting case — an ordinary fixed point in the case of $(CO) \parallel t$, too. $\square$

3. The proof of the Theorem

Let us consider a non-affine, non-central-axial collineation $\mathbf{\Phi}$, and — using Lemma 1. — let us obtain it as $\mathbf{\Phi} = \mathbf{\Phi_2 M}$, where $\mathbf{M}$ is an opposite isometry and $\mathbf{\Phi_2}$ is a non-affine central-axial collineation. $\mathbf{M}$ is either a reflection in a line or a glide reflection. If $\mathbf{M}$ is a reflection in a line, then the point of intersection of the axes is fixed under $\mathbf{\Phi}$, the proof is completed. Now $\mathbf{M}$ is a glide reflection, its invariant line is m, the center and axis of $\mathbf{\Phi_2}$ are C and t, respectively. If $C \in m$, then m is invariant under $\mathbf{\Phi}$, the proof is completed. If either $m \perp t$ or $m \parallel t$, then the ideal point of t is fixed under $\mathbf{\Phi}$, the proof is completed. In the sequel $C \notin m$, $m \not\perp t$, and m intersects t in an ordinary point T. Let us express $\mathbf{M}$ as the product of a reflection in a line and a reflection in a point: $\mathbf{M} = \mathbf{T_O T_d}$, $O \in m$, $d \perp m$, $O \notin d$. This factoring is not a unique one: e.g. we can choose O on m arbitrarily. $O \neq C$ because $O \in m$ and $C \notin m$; thus — according to Lemmas 2., 3. — there exists at least one ordinary fixed point of the product $\mathbf{\Phi_2 T_O}$ on (CO). We will prove that it is possible to choose O so that one of these fixed points will be on d. Thus this point will be fixed under $\mathbf{\Phi}$.

I. $\mathbf{\Phi_2}$ is an elation. (Fig. 3.) We use the same notations in connection with $\mathbf{\Phi_2}$ as above. Let K: $K \in m$, $(CK) \perp m$; $E := m \cap e$; O: an arbitrary point on m, but not T; $D := d \cap m$. The length and the direction of the segment OD are constants, these are determined by the glide reflection $\mathbf{M}$. $P(\in (CO))$ is one of the fixed points of $\mathbf{\Phi_2 T_O}$. Our aim is to make P incident to d; this holds, iff $\frac{OP}{CO} = \frac{OD}{KO}$. At the end of the proof of Lemma 2. we got the formula for $\frac{OP}{CO}$ using $c = \frac{LC}{CO}$ that is now equal to $\frac{ET}{TO}$. On the other hand

$$\frac{OD}{KO} = \frac{\frac{OD}{ET}}{\frac{KT}{ET} + \frac{TO}{ET}}.$$

Let $\alpha := \frac{OD}{ET}(\neq 0)$, $\beta := \frac{KT}{ET}(\neq 0)$; they are constants, independent of O. If $c \neq 0, -\frac{1}{\beta}$, then the condition for $P \in d$ is:

$$-c \pm \sqrt{c^2+1} = \frac{\alpha}{\beta + \frac{1}{c}}, \quad 2\alpha\beta c^3 + (\alpha^2 + 2\alpha - \beta^2)c^2 - 2\beta c - 1 = 0.$$

Neither 0 nor $-\frac{1}{\beta}$ is a solution, because α, $\beta \neq 0$. Moreover, the equation has degree 3, so it has at least one real solution. Applying $TO = \frac{1}{c}ET$ we get the appropriate O for $P \in d$. (We assumed $O \neq T$; we obviously got an O distinct from T.) Thus the case of elation is completed.

II. $\mathbf{\Phi_2}$ is a homology. (Fig 4.) We use the same notations in connection with $\mathbf{\Phi_2}$ as above. Points K, D and P are defined as in the previous case. Let $g : C \in g, g \parallel t$; $L := g \cap m$; O: an arbitrary point on m, but not L. The

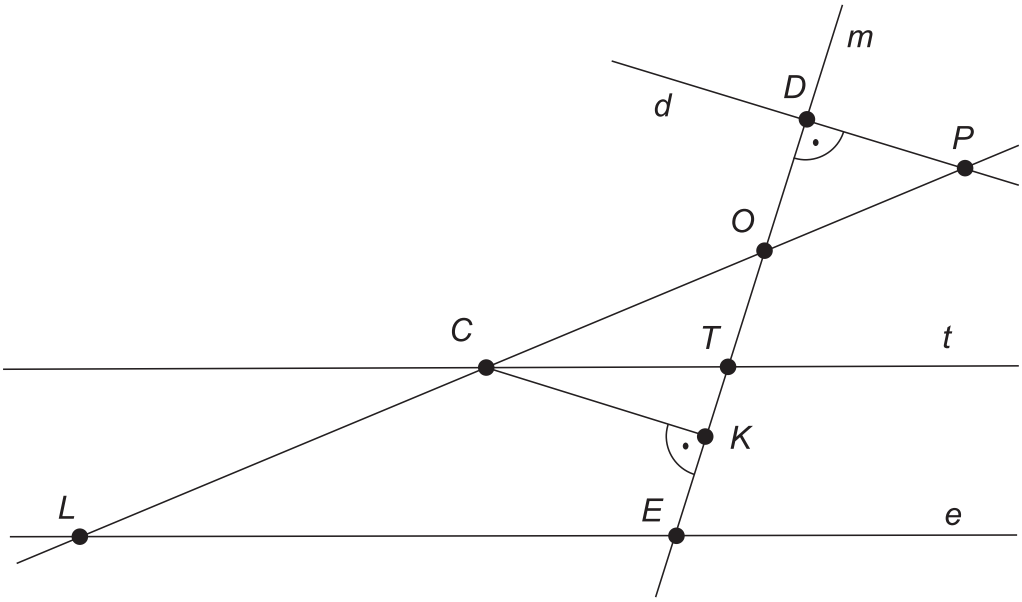

FIGURE 3

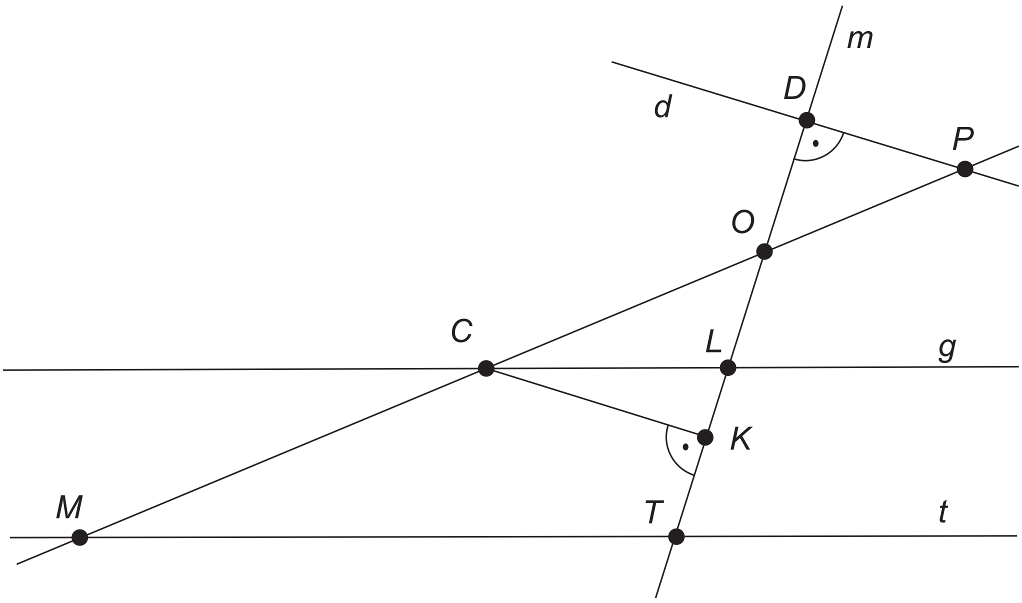

FIGURE 4

length and direction of OD are constants as in case I. The condition for $P \in d$ is again $\frac{OP}{CO} = \frac{OD}{KO}$. At the end of the proof of Lemma 3. we got the formula for $\frac{OP}{CO}$ using $r = \frac{\lambda+1}{\lambda-1}$ and $s = \frac{MO}{CO} - 1$. Now

$$\frac{MO}{CO} = \frac{TO}{LO} = \frac{\frac{TL}{OD}}{\frac{LK}{OD} + \frac{KO}{OD}} + 1.$$

Let $\alpha := \frac{TL}{OD} (\neq 0)$, $\beta := \frac{LK}{OD} (\neq 0)$; they are constants, independent of O. So $s = \frac{\alpha}{\beta + \frac{KO}{OD}}$, $\frac{OD}{KO} = \frac{s}{\alpha - \beta s}$. If $s \neq \frac{\alpha}{\beta}$, then the condition for $P \in d$ is:

$$\frac{-rs \pm \sqrt{r^2 s^2 + 4(s+1)}}{2} = \frac{s}{\alpha - \beta s},$$

$$\beta(\beta + r)s^3 + (\beta^2 - 2\alpha\beta - \alpha r - 1)s^2 + (\alpha^2 - 2\alpha\beta)s + \alpha^2 = 0.$$

Neither 0 nor $\frac{\alpha}{\beta}$ is a solution, because $\alpha \neq 0$. If this equation has degree either 1 or 3, it has at least one real solution. In the case of degree 2, $\beta + r = 0$ and the new main coefficient is not 0:

$$(r^2 + \alpha r - 1)s^2 + \alpha(\alpha + 2r)s + \alpha^2 = 0.$$

Its discriminant is $\alpha^2(\alpha + 2r)^2 - 4(r^2 + \alpha r - 1)\alpha^2 = \alpha^2(\alpha^2 + 4) > 0$, thus the equation has a real solution in this case, too. Finally, we have to prove that the equation has at least degree 1. If it was false, then $r(r + \alpha) = 1$, $\alpha = -2r$ and their consequence $-r^2 = 1$ would hold, which is impossible. Applying $LO = \frac{1}{s} TL$ we get the appropriate O for $P \in d$. (We assumed $O \neq L$; we obviously got an O distinct from L.) Thus the case of homology is completed, too, the Theorem is proved.

4. The locus of the fixed points

Finally, we examine the locus of the fixed points of $\mathbf{\Phi_2 T_O}$ while O "runs" on m. We use the analytical way.

I. $\mathbf{\Phi_2}$ is an elation. (Fig. 3.) We use the coordinate system as follows: $T(0;0)$, t: $y = 0$, e: $y = -1$. $C(k;0)$ $(k \neq 0)$, m: $x + by = 0$, $O(-bo;o)$ $(o \neq 0)$. Then $\overrightarrow{CO}(-bo - k; o)$, $L\left(\frac{ok+k+bo}{o}; -1\right)$, $\overrightarrow{LC}\left(\frac{-k-bo}{o}; 1\right)$. $\overrightarrow{LC} = c\overrightarrow{CO}$, so $c = \frac{1}{o}$. Using the formula that is at the end of the proof of Lemma 2. we get for the fixed point P:

$$\overrightarrow{OP} = \left(-\frac{1}{o} \pm \sqrt{\frac{1}{o^2} + 1}\right)\overrightarrow{CO} \text{ or } P = T$$

depending on whether $O \neq T$ $(o \neq 0)$ or $O = T$ $(o = 0)$. If $P(x;y)$, then:

$$x + bo = (-bo - k)\frac{-1 \pm \sqrt{1 + o^2}}{o} \text{ or } x = 0;$$

$$ox^2 + 2o^2xb - 2oxb - 2xk - 2b^2o^2 - 2bok - 2bo^2k - ok^2 = 0 \text{ or } x = 0.$$

$$y - o = o\frac{-1 \pm \sqrt{1 + o^2}}{o} \text{ or } y = 0;$$

$$y^2 - 2yo + 2y - 2o = 0 \text{ or } y = 0.$$

Using the computer algebra software "CoCoA" ([2]), we get the following equation for $(x;y)$:

$$by^2 + xy + 2x + (k + 2b)y = 0$$

The examination of the matrix of the equation shows that it is an equation of a hyperbola: the determinant of the matrix is $\det(A) = \frac{k}{2} \neq 0$, and its subdeterminant is $\det(A_{33}) = -\frac{1}{4} < 0$. A line passing through C has one common point with the hyperbola iff it is parallel to one of its asymptotes. (CT) has one point on the curve — $T(0;0)$ — thus the first asymptote is parallel to t; its equation is $y = -2$. The other (CO) lines ($O \in m$, $O \neq T$) have two points on the curve, so the second asymptote is parallel to m; its equation is $x + by = -k$. Consider the line s: $C \in s$, $s \parallel t$; s: $x + by = k$. The point of intersection of s and the hyperbola is $S(k + b; -1)$; this point is not a fixed point of $\mathbf{\Phi_2 T_O}$ for any $O \in m$.

II. $\mathbf{\Phi_2}$ is a homology. (Fig. 4.) We use the coordinate system as follows: $T(0;0)$, $t : y = 0$, $C(k;1)$. m: $x + by = 0$, $(k + b \neq 0)$, $O(-bo;o)$ $(o \neq 1)$, $L(-b;1)$. Then $\overrightarrow{CO}(-bo-k;o-1)$, $M\left(\frac{ok+ob}{o-1};0\right)$, $\overrightarrow{MC}\left(\frac{-k-bo}{o-1};1\right)$. $\overrightarrow{MC} = s\overrightarrow{CO}$, so $s = \frac{1}{o-1}$. Using the formula that is at the end of the proof of Lemma 3. we get for the fixed point P:

$$\overrightarrow{OP} = \frac{-\frac{r}{o-1} \pm \sqrt{\frac{r^2}{(o-1)^2} + \frac{4o}{o-1}}}{2}\overrightarrow{CO} \text{ or } \overrightarrow{OP} = \frac{1}{r}\overrightarrow{CO}$$

depending on whether $O \neq L$ $(o \neq 1)$ or $O = L$ $(o = 1)$. If $P(x;y)$, then:

$$x + bo = (-bo - k)\frac{-r \pm \sqrt{r^2 + 4o(o-1)}}{2(o-1)} \text{ or } x + b = (-b - k)\frac{1}{r};$$

$$ox^2 - x^2 + 2xbo^2 - 2xbo - xbor - xkr - b^2o^2r - bokr - b^2o^2 - 2bo^2k - k^2o = 0$$

$$\text{or } x = \frac{-rb - b - k}{r}.$$

$$y - o = (o-1)\frac{-r \pm \sqrt{r^2 + 4o(o-1)}}{2(o-1)} \text{ or } y = 1;$$

$$y^2 - 2oy + ry - or + o = 0 \text{ or } y = 1.$$

Using “CoCoA”, now we get the following equation for $(x;y)$:

$$by^2 + xy + (r-1)x + (k + br)y = 0.$$

The examination shows that it is an equation of a hyperbola, too: the determinant of the matrix is $\det(A) = \frac{(r-1)(b+k)}{4} = \frac{b+k}{2(\lambda-1)} \neq 0$, and its subdeterminant is $\det(A_{33}) = -\frac{1}{4} < 0$. As in the previous case, we get that the asymptotes are parallel to t and to m. Their equations are $y = 1 - r$ and $x + by = -k - b$. The line s and the point S are defined as in case I. Now s: $x + by = k + b$ and $S\left(\frac{2k+br+b}{2}; \frac{1-r}{2}\right)$.

Thus in both cases the locus of the fixed points of $\mathbf{\Phi_2 T_O}$ is a hyperbola except one point.

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Received October 4, 2005.

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