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FAMILY OF ANALYTIC FUNCTIONS OF COMPLEX ORDER

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Abstract. In this paper, we introduce the class $\mathcal{Q}_{\mathcal{T}}(\Phi, \Psi; \alpha, b)$ of analytic functions of complex order b and type $\alpha (0 \leq \alpha < 1)$. Coefficient inequalities, distortion theorems, closure theorems, radii of close-to-convexity, starlikeness, convexity and fractional calculus for functions belonging to the class $\mathcal{Q}_{\mathcal{T}}(\Phi, \Psi; \alpha, b)$ are obtained. Furthermore, we obtain the integral means inequality for the function $f(z)$ belongs to the class $\mathcal{Q}_{\mathcal{T}}(\Phi, \Psi; \alpha, b)$ with the extremal function of this class. Also, we consider q-δ-neighborhood for functions in this class.

1. Introduction and definitions

Let $\mathcal{A}$ denote the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk $\mathcal{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. A function $f(z) \in \mathcal{A}$ is said to be starlike of complex order b ($b \in \mathbb{C} \setminus \{0\}$) and type $\alpha (0 \leq \alpha < 1)$, that is $f(z) \in \mathcal{S}^*_\alpha(b)$, if and only if

$$\operatorname{Re}\left\{1 + \frac{1}{b}\left(\frac{zf'(z)}{f(z)} - 1\right)\right\} > \alpha \qquad (z \in \mathcal{U}; b \in \mathbb{C} \setminus \{0\}), \tag{1.2}$$

and is said to be convex of complex order b ($b \in \mathbb{C}\setminus\{0\}$) and type α ($0 \leq \alpha < 1$), denoted by $\mathcal{C}_\alpha(b)$ if and only if

$$\operatorname{Re}\left\{1 + \frac{1}{b}\frac{zf''(z)}{f'(z)}\right\} > \alpha \qquad (z \in \mathcal{U};\ b \in \mathbb{C} \setminus \{0\}). \tag{1.3}$$

Note that $\mathcal{S}^*_0(b) = \mathcal{S}^*(b)$ and $\mathcal{C}_0(b) = \mathcal{C}(b)$ the classes considered earlier by Nasr and Aouf [2] and Wiatrowski [7].

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Further, let $\mathcal{P}_\alpha(b)$ denote the class of functions $f(z) \in \mathcal{A}$ such that

$$\operatorname{Re}\left\{1 + \frac{1}{b}(f'(z) - 1)\right\} > \alpha \qquad (z \in \mathcal{U};\ b \in \mathbb{C} \setminus \{0\}). \tag{1.4}$$

Given two analytic functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $g(z) = z + \sum_{n=2}^{\infty} c_n z^n$ their convolution or Hadamard product $f(z) * h(z)$, is defined by

$$f(z) * g(z) = z + \sum_{n=2}^{\infty} a_n c_n z^n \qquad (z \in \mathcal{U}). \tag{1.5}$$

Let $\mathcal{T}$ denote the subclass of $\mathcal{A}$ whose members have the form:

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \qquad (a_n \geq 0) \tag{1.6}$$

we denote by $\mathcal{S}^*_\alpha[b]$, $\mathcal{C}_\alpha[b]$ and $\mathcal{P}_\alpha[b]$, respectively, the classes obtained by taking the intersections of $\mathcal{S}^*_\alpha(b)$, $\mathcal{C}_\alpha(b)$, and $\mathcal{P}_\alpha(b)$ with $\mathcal{T}$, that is,

$$\mathcal{S}^*_\alpha[b] = \mathcal{S}^*_\alpha(b) \cap \mathcal{T},\ \mathcal{C}_\alpha[b] = \mathcal{C}_\alpha(b) \cap \mathcal{T},\ \mathcal{P}_\alpha[b] = \mathcal{P}_\alpha(b) \cap \mathcal{T}. \tag{1.7}$$

We can obtain the above classes by using the following:

Definition 1.1. Given b $(b \in \mathbb{C} \setminus \{0\})$ and α $(0 \leq \alpha < 1)$. Let the functions

$$\Phi(z) = z + \sum_{n=2}^{\infty} \lambda_n z^n \text{ and } \quad \Psi(z) = z + \sum_{n=2}^{\infty} \mu_n z^n \tag{1.8}$$

be analytic in $\mathcal{U}$, such that $\lambda_n \geq 0, \mu_n \geq 0$ and $\lambda_n \geq \mu_n$ for $n \geq 2$, we say that $f(z) \in \mathcal{A}$ is in $\mathcal{Q}(\Phi, \Psi; \alpha, b)$ if $f(z) * \Psi(z) \neq 0$ and

$$\operatorname{Re}\left\{1 + \frac{1}{b}\left(\frac{f(z) * \Phi(z)}{f(z) * \Psi(z)} - 1\right)\right\} > \alpha \qquad (z \in \mathcal{U}) \tag{1.9}$$

Further, let

$$\mathcal{Q}_{\mathcal{T}}(\Phi, \Psi; \alpha, b) \ = \mathcal{Q}(\Phi, \Psi; \alpha, b) \ \cap \mathcal{T}. \tag{1.10}$$

We note that, by suitably choosing $\Phi(z)$,$\Psi(z)$ we obtain the above subclasses of $\mathcal{T}$ of complex order b and type

$$\alpha : \mathcal{Q}_{\mathcal{T}}\left(\frac{z}{(1-z)^2}, \frac{z}{1-z}; \alpha, b\right) = \mathcal{S}^*_\alpha[b]; \mathcal{Q}_{\mathcal{T}}\left(\frac{z+z^2}{(1-z)^3}, \frac{z}{(1-z)^2}; \alpha, b\right) = \mathcal{C}_\alpha[b];$$

and

$$\mathcal{Q}_{\mathcal{T}}\left(\frac{z}{(1-z)^2}, z; \alpha, b\right) = \mathcal{P}_\alpha[b].$$

In fact many new subclasses of $\mathcal{T}$ of complex order b and type α can be defined and studied by suitably choosing $\Phi(z)$,$\Psi(z)$. For example

$$\mathcal{Q}_{\mathcal{T}}\left(\frac{z}{1-z}, z; \alpha, b\right) = \left\{f(z) \in \mathcal{T}\colon \ \operatorname{Re}\left\{1 + \frac{1}{b}\left(\frac{f(z)}{z} - 1\right)\right\} > \alpha\right\},$$

and

$$\mathcal{Q}_{\mathcal{T}}\left(\frac{z+z^2}{(1-z)^3}, z; \alpha, b\right) = \left\{f(z) \in \mathcal{T} : \operatorname{Re}\left\{1 + \frac{1}{b}\left((zf'(z))' - 1\right)\right\} > \alpha\right\}$$

and so on.

In this paper, we shall obtain coefficient inequalities, distortion theorems, closure theorems, radii of close-to-convexity, starlikeness, convexity and fractional calculus for functions belonging to the class $\mathcal{Q}_{\mathcal{T}}(\Phi, \Psi; \alpha, b)$. Furthermore, we obtain the integral means inequality for the function $f(z)$ belongs to the class $\mathcal{Q}_{\mathcal{T}}(\Phi, \Psi; \alpha, b)$ with the extremal function of this class. Also, we consider q-δ-neighborhood for functions in this class.

2. Coefficient inequalities

Theorem 2.1. *Let the function $f(z)$ defined by (1.6) be in the class*

$$\mathcal{Q}_{\mathcal{T}}(\Phi, \Psi; \alpha, b).$$

Then

$$\sum_{n=2}^{\infty} \left[(\operatorname{Re}(b))\lambda_n + \left((1-\alpha)|b|^2 - \operatorname{Re}(b)\right)\mu_n\right]|a_n| \leq |b|^2(1-\alpha) \tag{2.1}$$

The result (2.1) is sharp.

Proof. Suppose that $f(z) \in \mathcal{Q}_{\mathcal{T}}(\Phi, \Psi; \alpha, b)$. Then

$$\operatorname{Re}\left\{1 + \frac{1}{b}\left(\frac{f(z) * \Phi(z)}{f(z) * \Psi(z)} - 1\right)\right\} > \alpha \qquad (z \in \mathcal{U}), \tag{2.2}$$

or equivalently

$$\operatorname{Re}\left\{\frac{1}{b}\left(\frac{-\sum_{n=2}^{\infty}(\lambda_n - \mu_n)a_n z^{n-1}}{1 - \sum_{n=2}^{\infty}\mu_n a_n z^{n-1}}\right)\right\} > \alpha - 1 \qquad (z \in \mathcal{U}) \tag{2.3}$$

Now choose values of z on the real axis and let $z \to 1^-$ through real values to find that

$$\left(\frac{-\sum_{n=2}^{\infty}(\lambda_n - \mu_n)|a_n|}{1 - \sum_{n=2}^{\infty}\mu_n |a_n|}\right)\operatorname{Re}\frac{1}{b} \geq \alpha - 1, \tag{2.4}$$

whence

$$\left(\frac{\sum_{n=2}^{\infty}(\lambda_n - \mu_n)|a_n|}{1 - \sum_{n=2}^{\infty}\mu_n |a_n|}\right)\frac{\operatorname{Re}(b)}{|b|^2} \leq 1 - \alpha, \tag{2.5}$$

and so

$$\sum_{n=2}^{\infty}(\lambda_n-\mu_n)\,|a_n| \le \frac{|b|^2}{\mathrm{Re}(b)}\,(1-\alpha)\left(1-\sum_{n=2}^{\infty}\mu_n\,|a_n|\right), \tag{2.6}$$

which is equivalent to (2.1).

The equality in (2.1) holds true for the functions $f(z)$ defined by

$$f(z)=z-\frac{|b|^2\,(1-\alpha)}{(\mathrm{Re}(b))\lambda_n+\big((1-\alpha)\,|b|^2-\mathrm{Re}(b)\big)\,\mu_n}z^n \qquad (n\ge 2). \tag{2.7}$$

□

Corollary 2.2. *Let the function $f(z)$ defined by (1.6) be in the class*

$$\mathcal{Q}_{\mathcal{T}}(\Phi,\Psi;\alpha,b).$$

Then

$$|a_n| \le \frac{|b|^2\,(1-\alpha)}{(\mathrm{Re}(b))\lambda_n+\big((1-\alpha)\,|b|^2-\mathrm{Re}(b)\big)\,\mu_n} \qquad (n\ge 2). \tag{2.8}$$

The result (2.8) is sharp for the function $f(z)$ given by (2.7).

Putting $\Phi(z)=z/(1-z)^2$ and $\Psi(z)=z/(1-z)$ in Theorem 2.1, we have

Corollary 2.3. *Let the function $f(z)$ defined by (1.6) be in the class $\mathcal{S}_{\alpha}^{*}[b]$. Then*

$$\sum_{n=2}^{\infty}\left[(\mathrm{Re}(b))n+\big((1-\alpha)\,|b|^2-\mathrm{Re}(b)\big)\right]|a_n| \le |b|^2\,(1-\alpha) \tag{2.9}$$

The result is sharp for

$$f(z)=z-\frac{|b|^2\,(1-\alpha)}{(\mathrm{Re}(b))n+\big((1-\alpha)\,|b|^2-\mathrm{Re}(b)\big)}z^n \qquad (n\ge 2).$$

Putting $\Phi(z)=(z+z^2)/(1-z)^3$ and $\Psi(z)=z/(1-z)^2$ in Theorem 2.1, we have

Corollary 2.4. *Let the function $f(z)$ defined by (1.6) be in the class $\mathcal{C}_{\alpha}^{*}[b]$. Then*

$$\sum_{n=2}^{\infty}\left[(\mathrm{Re}(b))n^2+\big((1-\alpha)\,|b|^2-\mathrm{Re}(b)\big)\,n\right]|a_n| \le |b|^2\,(1-\alpha) \tag{2.10}$$

The result is sharp for

$$f(z)=z-\frac{|b|^2\,(1-\alpha)}{(\mathrm{Re}(b))n^2+\big((1-\alpha)\,|b|^2-\mathrm{Re}(b)\big)\,n}z^n \qquad (n\ge 2).$$

Putting $\Phi(z)=z/(1-z)^2$ and $\Psi(z)=z$ in Theorem 2.1, we have

Corollary 2.5. *Let the function $f(z)$ defined by (1.6) be in the class $\mathcal{P}^*_\alpha[b]$. Then*

$$\sum_{n=2}^{\infty}[(\mathrm{Re}(b))n]\,|a_n| \leq |b|^2(1-\alpha) \tag{2.11}$$

The result is sharp for

$$f(z) = z - \frac{|b|^2(1-\alpha)}{(\mathrm{Re}(b))n}z^n \qquad (n \geq 2).$$

For the notational convenience we shall henceforth denote

$$\sigma_n(\alpha,b) = (\mathrm{Re}(b))\lambda_n + \left((1-\alpha)|b|^2 - \mathrm{Re}(b)\right)\mu_n \qquad (n \geq 2). \tag{2.12}$$

3. Growth and distortion theorems

Theorem 3.1. *Let the function $f(z)$ defined by (1.6) be in the class*

$$\mathcal{Q}_{\mathcal{T}}(\Phi,\Psi;\alpha,b).$$

If $\{\sigma_n(\alpha,b)\}_{n=2}^{\infty}$ is a non-decreasing sequence, then

$$|z| - \frac{|b|^2(1-\alpha)}{\sigma_2(\alpha,b)}|z|^2 \leq |f(z)| \leq |z| + \frac{|b|^2(1-\alpha)}{\sigma_2(\alpha,b)}|z|^2 \tag{3.1}$$

where $\sigma_2(\alpha,b) = (\mathrm{Re}(b))\lambda_2 + \left((1-\alpha)|b|^2 - \mathrm{Re}(b)\right)\mu_2$. The equality in (3.1) is attained for the function $f(z)$ given by

$$f(z) = z - \frac{|b|^2(1-\alpha)}{\sigma_2(\alpha,b)}z^2. \tag{3.2}$$

Proof. Note that

$$\sigma_2(\alpha,b)\sum_{n=2}^{\infty}|a_n| \leq \sum_{n=2}^{\infty}\sigma_n(\alpha,b)\,|a_n| \leq |b|^2(1-\alpha)$$

or, equivalently

$$\sum_{n=2}^{\infty}|a_n| \leq \frac{|b|^2(1-\alpha)}{\sigma_2(\alpha,b)}, \tag{3.3}$$

this last inequality following from Theorem 2.1. Thus we have

$$|f(z)| \geq |z| - \sum_{n=2}^{\infty}|a_n|\,|z|^n \geq |z| - |z|^2\sum_{n=2}^{\infty}|a_n| \geq |z| - \frac{|b|^2(1-\alpha)}{\sigma_2(\alpha,b)}|z|^2 \tag{3.4}$$

and

$$|f(z)| \leq |z| + \sum_{n=2}^{\infty}|a_n|\,|z|^n \leq |z| + |z|^2\sum_{n=2}^{\infty}|a_n| \leq |z| + \frac{|b|^2(1-\alpha)}{\sigma_2(\alpha,b)}|z|^2 \tag{3.5}$$

for $z \in \mathcal{U}$. From the inequalities (3.4) and (3.5) we obtain the inequality (3.1). □

Theorem 3.2. *The disk $|z| < 1$ is mapped onto a domain that contains the disk*

$$|w| < \frac{\sigma_2(\alpha, b) - |b|^2 (1-\alpha)}{\sigma_2(\alpha, b)}$$

by any $f(z) \in \mathcal{Q}_{\mathcal{T}}(\Phi, \Psi; \alpha, b)$. The theorem is sharp with the function $f(z)$ given by (3.2).

Theorem 3.3. *Let the function $f(z)$ defined by (1.6) be in the class*

$$\mathcal{Q}_{\mathcal{T}}(\Phi, \Psi; \alpha, b).$$

If $\{\sigma_n(\alpha, b)/n\}_{n=2}^{\infty}$ is a non-decreasing sequence, then

$$1 - \frac{2|b|^2 (1-\alpha)}{\sigma_2(\alpha, b)} |z| \le |f'(z)| \le 1 + \frac{2|b|^2 (1-\alpha)}{\sigma_2(\alpha, b)} |z|. \tag{3.6}$$

The equality in (3.6) is attained for the function $f(z)$ given by (3.2).

Proof. In view of Theorem 2.1,

$$\frac{\sigma_2(\alpha, b)}{2} \sum_{n=2}^{\infty} n |a_n| \le \sum_{n=2}^{\infty} \sigma_n(\alpha, b) |a_n| \le |b|^2 (1-\alpha). \tag{3.7}$$

that is,

$$\sum_{n=2}^{\infty} n |a_n| \le \frac{2|b|^2 (1-\alpha)}{\sigma_2(\alpha, b)}. \tag{3.8}$$

Form (3.8), we can easily prove that

$$|f'(z)| \ge 1 - \sum_{n=2}^{\infty} n |a_n| |z|^{n-1} \ge 1 - |z| \sum_{n=2}^{\infty} n |a_n| \ge 1 - \frac{2|b|^2 (1-\alpha)}{\sigma_2(\alpha, b)} |z| \tag{3.9}$$

and

$$|f'(z)| \le 1 + \sum_{n=2}^{\infty} n |a_n| |z|^{n-1} \le 1 + |z| \sum_{n=2}^{\infty} n |a_n| \le 1 + \frac{2|b|^2 (1-\alpha)}{\sigma_2(\alpha, b)} |z| \tag{3.10}$$

for $z \in \mathcal{U}$. Combining the inequalities (3.9) and (3.10) we obtain the inequality (3.6). □

4. Radii of close-to-convexity, starlikeness and convexity

Theorem 4.1. *Let the function $f(z)$ be defined by (1.6) be in the class*

$$\mathcal{Q}_{\mathcal{T}}(\Phi, \Psi; \alpha, b).$$

Then $f(z)$ is close-to-convex of complex order b in $|z| < r_1$, where

$$r_1 = r_1(\alpha, b) = \inf_n \left[\frac{\sigma_n(\alpha, b)}{|b|\, n\, (1-\alpha)} \right]^{1/(n-1)} \qquad (n \ge 2). \tag{4.1}$$

The result is sharp for the function $f(z)$ being given by (3.2).

Proof. We must show that $|f'(z) - 1| \leq |b|$ for $|z| < r_1$, where r_1 is given by (4.1). From (1.6) we have

$$|f'(z) - 1| < \sum_{n=2}^{\infty} n a_n \, |z|^{n-1} .$$

Thus $|f'(z) - 1| < |b|$ if

$$\sum_{n=2}^{\infty} \left(\frac{n}{|b|} \right) a_n \, |z|^{n-1} \leq 1. \tag{4.2}$$

But, by Theorem 2.1, (4.2) will be true if

$$\left(\frac{n}{|b|} \right) |z|^{n-1} \leq \frac{\sigma_n(\alpha, b)}{|b|^2 \,(1 - \alpha)},$$

that is, if

$$|z| \leq \left[\frac{\sigma_n(\alpha, b)}{|b| \, n \,(1 - \alpha)} \right]^{1/(n-1)} \qquad (n \geq 2). \tag{4.3}$$

Theorem 4.1 follows easily from (4.3). □

Theorem 4.2. *Let the function $f(z)$ be defined by (1.6) be in the class*

$$\mathcal{Q}_{\mathcal{T}}(\Phi, \Psi; \alpha, b).$$

Then $f(z)$ is starlike of complex order b in $|z| < r_2$, where

$$r_2 = r_2(\alpha, b) = \inf_n \left[\frac{\sigma_n(\alpha, b)}{|b| \,(n + |b| - 1)\,(1 - \alpha)} \right]^{1/(n-1)} \qquad (n \geq 2). \tag{4.4}$$

The result is sharp for the function $f(z)$ being given by (3.2).

Proof. It is sufficient to show that

$$\left| \frac{z f'(z)}{f(z)} - 1 \right| \leq |b|$$

for $|z| < r_2$, where r_2 is given by (4.4). From (1.6) we find that

$$\left| \frac{z f'(z)}{f(z)} - 1 \right| \leq \frac{\sum\limits_{n=2}^{\infty} (n-1) a_n \, |z|^{n-1}}{1 - \sum\limits_{n=2}^{\infty} a_n \, |z|^{n-1}}.$$

Thus $\left| \frac{z f'(z)}{f(z)} - 1 \right| \leq |b|$ if

$$\sum_{n=2}^{\infty} \left(\frac{n + |b| - 1}{|b|} \right) a_n \, |z|^{n-1} \leq 1 \tag{4.5}$$

But, by Theorem 2.1, (4.5) will be true if

$$\left(\frac{n+|b|-1}{|b|}\right)|z|^{n-1} \leq \frac{\sigma_n(\alpha,b)}{|b|^2(1-\alpha)}$$

that is, if

$$|z| \leq \left[\frac{\sigma_n(\alpha,b)}{|b|\,(n+|b|-1)\,(1-\alpha)}\right]^{1/(n-1)} \qquad (n \geq 2). \tag{4.6}$$

Theorem 4.2 follows easily from (4.6). □

Corollary 4.3. *Let the function $f(z)$ be defined by (1.6) be in the class*

$$\mathcal{Q}_{\mathcal{T}}(\Phi,\Psi;\alpha,b).$$

Then $f(z)$ is convex of complex order b in $|z| < r_3$, where

$$r_3 = r_3(\alpha,b) = \inf_n \left[\frac{\sigma_n(\alpha,b)}{|b|\,n(n+|b|-1)\,(1-\alpha)}\right]^{1/(n-1)} \qquad (n \geq 2). \tag{4.7}$$

The result is sharp for the function $f(z)$ being given by (3.2).

5. Fractional Calculus

In this section, we find it to be convenient to recall here the following of fractional calculus which were introduced by by Owa ([3], [4]).

Definition 5.1. The fractional integral of order δ is defined, for a function $f(z)$, by

$$D_z^{-\delta} f(z) = \frac{1}{\Gamma(\delta)} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{1-\delta}} d\zeta \qquad (\delta > 0), \tag{5.1}$$

where the function $f(z)$ is analytic in a simply-connected region of the z-plane containing the origin and the multiplicity of the function $(z-\zeta)^{\delta-1}$ is removed by requiring the function $\log(z-\zeta)$ to be real when $z-\zeta > 0$.

Definition 5.2. The fractional derivative of order δ is defined, for a function $f(z)$, by

$$D_z^{\delta} f(z) = \frac{1}{\Gamma(1-\delta)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{1-\delta}} d\zeta \qquad (0 \leq \delta < 1), \tag{5.2}$$

where the function $f(z)$ is constrained, and the multiplicity of the function $(z-\zeta)^{-\delta}$ is removed as in Definition 5.1.

Definition 5.3. Under the hypotheses of Definition 5.2, the fractional derivative of order $n+\delta$ is defined by

$$D_z^{n+\delta} f(z) = \frac{d^n}{dz^n} D_z^{\delta} f(z) \qquad (0 \leq \delta < 1;\, n \in N_0). \tag{5.3}$$

Remark 5.4. From Definition 5.1, we have $D_z^0 f(z) = f(z)$, which in view of Definition 5.3 yields $D_z^{n+0} f(z) = \frac{d^n}{dz^n} D_z^0 f(z) = f^{(n)}(z)$. Thus, $\lim_{\delta\to 0} D_z^{-\delta} f(z) = f(z)$ and $\lim_{\delta\to 0} D_z^{1-\delta} f(z) = f'(z)$.

Theorem 5.5. *Let the function $f(z)$ be defined by (1.6) be in the class*

$$\mathcal{Q}_{\mathcal{T}}(\Phi, \Psi; \alpha, b)).$$

If $\{\sigma_n(\alpha, b)\}_{n=2}^{\infty}$ is a non-decreasing sequence, then

$$\left|D_z^{-\delta} f(z)\right| \geq \frac{|z|^{1+\delta}}{\Gamma(2+\delta)} \left\{1 - \frac{2\,|b|^2\,(1-\alpha)}{\sigma_2(\alpha, b)(2+\delta)}\,|z|\right\} \tag{5.4}$$

and

$$\left|D_z^{-\delta} f(z)\right| \leq \frac{|z|^{1+\delta}}{\Gamma(2+\delta)} \left\{1 + \frac{2\,|b|^2\,(1-\alpha)}{\sigma_2(\alpha, b)(2+\delta)}\,|z|\right\} \tag{5.5}$$

for $\delta > 0$, and $z \in \mathcal{U}$. The result is sharp.

Proof. Let

$$\begin{aligned} F(z) &= \Gamma(2+\delta) z^{-\delta} D_z^{-\delta} f(z) \\ &= z - \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2+\delta)}{\Gamma(n+1+\delta)} a_n z^n = z - \sum_{n=2}^{\infty} \Delta(n) a_n z^n, \end{aligned} \tag{5.6}$$

where

$$\Delta(n) = \frac{\Gamma(n+1)\Gamma(2+\delta)}{\Gamma(n+1+\delta)} \qquad (n \geq 2). \tag{5.7}$$

It is easy to see that

$$0 < \Delta(n) \leq \Delta(2) = \frac{2}{2+\delta}. \tag{5.8}$$

Therefore, by using (3.3) and (5.8), we can see that

$$|F(z)| \geq |z| - \Delta(2)\,|z|^2 \sum_{n=2}^{\infty} a_n \geq |z| - \frac{2\,|b|^2\,(1-\alpha)}{\sigma_2(\alpha, b)(2+\delta)}\,|z|^2 \tag{5.9}$$

$$|F(z)| \leq |z| + \Delta(2)\,|z|^2 \sum_{n=2}^{\infty} a_n \leq |z| + \frac{2\,|b|^2\,(1-\alpha)}{\sigma_2(\alpha, b)(2+\delta)}\,|z|^2 \tag{5.10}$$

which prove the inequality of Theorem 5.5. Further, equalities are attained for the function $f(z)$ defined by

$$D_z^{-\delta} f(z) = \frac{z^{1+\delta}}{\Gamma(2+\delta)} \left\{1 + \frac{2\,|b|^2\,(1-\alpha)}{\sigma_2(\alpha, b)(2+\delta)} z\right\} \tag{5.11}$$

□

Theorem 5.6. *Let the function $f(z)$ be defined by (1.6) be in the class*

$$\mathcal{Q}_{\mathcal{T}}(\Phi, \Psi; \alpha, b).$$

If $\{\sigma_n(\alpha,b)/n\}_{n=2}^{\infty}$ is a non-decreasing sequence, then

$$\left|D_z^{\delta} f(z)\right| \geq \frac{|z|^{1-\delta}}{\Gamma(2-\delta)} \left\{1 - \frac{2\,|b|^2\,(1-\alpha)}{\sigma_2(\alpha,b)(2-\delta)}\,|z|\right\} \tag{5.12}$$

and

$$\left|D_z^{\delta} f(z)\right| \leq \frac{|z|^{1-\delta}}{\Gamma(2-\delta)} \left\{1 + \frac{2\,|b|^2\,(1-\alpha)}{\sigma_2(\alpha,b)(2-\delta)}\,|z|\right\} \tag{5.13}$$

for $0 \leq \delta < 1$, and $z \in \mathcal{U}$. The result is sharp.

Proof. Let

$$\begin{aligned} H(z) &= \Gamma(2-\delta) z^{\delta} D_z^{\delta} f(z) \\ &= z - \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\delta)}{\Gamma(n+1-\delta)} a_n z^n = z - \sum_{n=2}^{\infty} n\Omega(n) a_n z^n, \end{aligned} \tag{5.14}$$

where

$$\Omega(n) = \frac{\Gamma(n)\Gamma(2-\delta)}{\Gamma(n+1-\delta)} \qquad (n \geq 2). \tag{5.15}$$

Since

$$0 < \Omega(n) \leq \Omega(2) = \frac{1}{2-\delta}. \tag{5.16}$$

Therefore, by using (3.8) and (5.16), we can see that

$$|H(z)| \geq |z| - \Delta(2)\,|z|^2 \sum_{n=2}^{\infty} n a_n \geq |z| - \frac{2\,|b|^2\,(1-\alpha)}{\sigma_2(\alpha,b)(2-\delta)}\,|z|^2 \tag{5.17}$$

$$|H(z)| \leq |z| + \Delta(2)\,|z|^2 \sum_{n=2}^{\infty} a_n \leq |z| + \frac{2\,|b|^2\,(1-\alpha)}{\sigma_2(\alpha,b)(2-\delta)}\,|z|^2 \tag{5.18}$$

which give the inequalities of Theorem 5.6. Since equalities are attained for the function $f(z)$ defined by

$$D_z^{\delta} f(z) = \frac{z^{1-\delta}}{\Gamma(2-\delta)} \left\{1 + \frac{2\,|b|^2\,(1-\alpha)}{\sigma_2(\alpha,b)(2-\delta)} z\right\} \tag{5.19}$$

□

Remark 5.7. Letting $\delta = 0$ in Theorem 5.5, we have (3.1) of Theorem 3.1, and letting $\delta \longrightarrow 1$ in Theorem 5.6, we have (3.6) in Theorem 3.3.

6. Integral Means Inequalities

The following subordination result will be required in our present investigation.

Lemma 6.1 ([1]). *If f and g are analytic in $\mathcal{U}$ with $g \prec f$, then*

$$\int_0^{2\pi} \left|g(re^{i\theta})\right|^\delta d\theta \le \int_0^{2\pi} \left|f(re^{i\theta})\right|^\delta d\theta \tag{6.1}$$

where $\delta > 0$, $z = re^{i\theta}$ and $0 < r < 1$.

Applying Theorem 2.1 and Lemma 6.1, we prove the following

Theorem 6.2. *Let $\delta > 0$. If $f(z) \in \mathcal{Q}_{\mathcal{T}}(\Phi, \Psi; \alpha, b)$, and $\{\sigma_n(\alpha, b)\}_{n=2}^{\infty}$ is non-decreasing sequence, then, for $z = re^{i\theta}$, $0 < r < 1$, we have*

$$\int_0^{2\pi} \left|f(re^{i\theta})\right|^\delta d\theta \le \int_0^{2\pi} \left|f_2(re^{i\theta})\right|^\delta d\theta \tag{6.2}$$

where $f_2(z) = z - |b|^2 (1-\alpha) / \sigma_2(\alpha, b) z^2$.

Proof. Let

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \quad (a_n \ge 0,\ z \in \mathcal{U})$$

and

$$f_2(z) = z - |b|^2 (1-\alpha) / \sigma_2(\alpha, b) z^2,$$

then we must show that

$$\int_0^{2\pi} \left|1 - \sum_{n=2}^{\infty} a_n z^{n-1}\right|^\delta d\theta \le \int_0^{2\pi} \left|1 - \frac{|b|^2 (1-\alpha)}{\sigma_2(\alpha, b)} z\right|^\delta d\theta.$$

By Lemma 6.1, it suffices to show that

$$1 - \sum_{n=2}^{\infty} a_n z^{n-1} \prec 1 - \frac{|b|^2 (1-\alpha)}{\sigma_2(\alpha, b)} z.$$

Setting

$$1 - \sum_{n=2}^{\infty} a_n z^{n-1} = 1 - \frac{|b|^2 (1-\alpha)}{\sigma_2(\alpha, b)} w(z). \tag{6.3}$$

From (6.2) and (2.1), we obtain

$$\begin{aligned} |w(z)| &= \left|\sum_{n=2}^{\infty} \frac{\sigma_2(\alpha, b)}{|b|^2 (1-\alpha)} a_n z^{n-1}\right| \\ &\le |z| \sum_{n=2}^{\infty} \frac{\sigma_n(\alpha, b)}{|b|^2 (1-\alpha)} a_n \le |z|. \end{aligned}$$

This the completes the proof of the theorem. □

Remark 6.3. Taking different choices of $\Phi(z)$ and $\Psi(z)$ in Theorem 6.2, we can obtain integral means inequalities for functions belonging the classes $\mathcal{S}_{\alpha}^{*}[b]$, $\mathcal{C}_{\alpha}[b]$ and $\mathcal{P}_{\alpha}[b]$.

7. Neighborhoods of The Class $\mathcal{Q}_{\mathcal{T}}(\Phi, \Psi; \alpha, b)$.

For $f \in \mathcal{T}$ of the form (1.6) and $\delta \geq 0$, we define

$$M_{\delta}^{p}(f) = \{g \in \mathcal{T}\colon\ g(z) = z - \sum_{n=2}^{\infty} b_n z^n,\ \sum_{n=2}^{\infty} n^{p+1} |a_n - b_n| \leq \delta\}, \tag{7.1}$$

which was called the p-δ-neighborhood of f. So, for $e(z) = z$, we see that

$$M_{\delta}^{p}(e) = \{g \in \mathcal{T}\colon\ g(z) = z - \sum_{n=2}^{\infty} b_n z^n,\ \sum_{n=2}^{\infty} n^{p+1} |b_n| \leq \delta\}, \tag{7.2}$$

where p is a fixed positive integer. Note that $M_{\delta}^{0}(f) \equiv N_{\delta}(f)$ and $M_{\delta}^{1}(f) \equiv M_{\delta}(f)$. $N_{\delta}(f)$ called a δ-neighborhood of f by Ruscheweyh [5] and $M_{\delta}(f)$ was defined by Silverman [6].

In this section, we consider p-δ-neighborhood for function in the class

$$\mathcal{Q}_{\mathcal{T}}(\Phi, \Psi; \alpha, b).$$

Theorem 7.1. *If* $\{\sigma_n(\alpha, b)/n^{p+1}\}_{n=2}^{\infty}$ *is a non-decreasing sequence, then,*

$$\mathcal{Q}_{\mathcal{T}}(\Phi, \Psi; \alpha, b) \subset M_{\delta}^{p}(e),$$

where $\delta = 2^{p+1} |b|^2 (1-\alpha) / \sigma_2(\alpha, b)$.

Proof. It follows from (2.1) that if $f(z) \in \mathcal{Q}_{\mathcal{T}}(\Phi, \Psi; \alpha, b)$, then

$$\sum_{n=2}^{\infty} n^{p+1} a_n \leq \frac{2^{p+1} |b|^2 (1-\alpha)}{\sigma_2(\alpha, b)} \tag{7.3}$$

This gives that $\mathcal{Q}_{\mathcal{T}}(\Phi, \Psi; \alpha, b) \subset M_{\delta}^{p}(e)$. □

Putting $\Phi(z) = z/(1-z)^2$ and $\Psi(z) = z/(1-z)$ in Theorem 7.1, we have

Corollary 7.2. $\mathcal{S}_{\alpha}^{*}[b] \subset M_{\delta}^{p}(e)$, *where* $\delta = 2^{p+1} |b|^2 (1-\alpha) / [\mathrm{Re}(b) + (1-\alpha) |b|^2]$.

Putting $\Phi(z) = (z + z^2)/(1-z)^3$ and $\Psi(z) = z/(1-z)^2$ in Theorem 7.1, we have

Corollary 7.3. $\mathcal{C}_{\alpha}[b] \subset M_{\delta}^{p}(e)$, *where* $\delta = 2^{p} |b|^2 (1-\alpha) / [\mathrm{Re}(b) + (1-\alpha) |b|^2]$.

Putting $\Phi(z) = z/(1-z)^2$ and $\Psi(z) = z$ in Theorem 7.1, we have

Corollary 7.4. $\mathcal{P}_{\alpha}[b]. \subset M_{\delta}^{p}(e)$, *where* $\delta = 2^{p} |b|^2 (1-\alpha) / \mathrm{Re}(b)$.

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