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SOME PROPERTIES OF OCTONION AND QUATERNION ALGEBRAS

CRISTINA FLAUT

ABSTRACT. In 1988, J.R. Faulkner has given a procedure to construct an octonion algebra on a finite dimensional unitary alternative algebra of degree three over a field K. Here we use a similar procedure to get a quaternion algebra. Then we obtain some conditions for these octonion and quaternion algebras to be split or division algebras. Then we consider the implications of the found conditions to the underlying algebra, when K contains a cubic root of unity.

1. PRELIMINARIES

A lot of concepts used in this paper as well as their properties can be found in details in R.D. Schafer's classical book *An Introduction to Nonassociative Algebras* [Sch66].

We recall only some definitions and results, which will be necessary in our paper. First, we define the notions used in that follows. K will denote, everywhere in the paper, a field with char $K \neq 2, 3$.

Definition 1.1. Let A be a nonassociative algebra over K.

i) The algebra A is a *flexible algebra* if

$$x\left(yx\right)=\left(xy\right)x, \forall x,y\in A.$$

ii) The algebra A is a *composition algebra* if there is a quadratic form $q\colon A\rightarrow K$ such that, for every $x,y\in A$, we have $q\left(xy\right)=q\left(x\right)q\left(y\right)$ and the associated bilinear form

$$f\colon A\times A\rightarrow K, f(x,y)=\frac{1}{2}[n(x+y)-n(x)-n(y)]$$

is nondegenerate. A unitary composition algebra is called a *Hurwitz algebra.*

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iii) The algebra A is a *power-associative algebra* if for every $x \in A$ the subalgebra generated by x is an associative algebra.

iv) The algebra A is an *alternative algebra* if

$$x^2 y = x(xy) \text{ and } yx^2 = (yx)x, \forall x, y \in A.$$

Each alternative algebra is a power-associative algebra.

v) If $A \neq 0$ and the equations

$$ax = b, \ ya = b, \forall a, b \in A, a \neq 0,$$

have unique solutions, then the algebra A is a *division algebra.*

v) A Hurwitz algebra A is called *a split Hurwitz algebra* if it satisfies one of the following equivalent conditions:

1) There are $x, y \in A$, $x \neq 0$, $y \neq 0$ such that $xy = 0$.
2) There is $x \in A$, $x \neq 0$ such that $q(x) = 0$.
3) There is $e \in A, e \neq 0, e \neq 1$, such that $e^2 = e$.

We note that each Hurwitz algebra is either split or it is a division algebra.

In [Fau88] J.R. Faulkner proved some relations in a unitary finite dimensional of degree three alternative algebra, having the generic minimum polynomial

$$P_x(\lambda) = \lambda^3 - T(x)\lambda^2 + S(x)\lambda - N(x) \cdot 1.$$

We recall only the relation $2S(x) = T(x)^2 - T(x^2)$, which we use in the next. The coefficient $T(x)$ is called the *trace* of x, while $N(x)$ the *norm* of x.

Definition 1.2. Let A be a composition algebra. Then its associated bilinear form f *is associative* (or *invariant*) if

$$f(xy, z) = f(x, yz), \forall x, y, z \in A.$$

If A is a composition algebra then its associated bilinear form f is associative if and only if, for the quadratic form q associated to f, we have the relation:

$$(xy)x = x(yx) = q(x)y, \forall x, y \in A. \tag{1.1}$$

Let ω be the cubic root of unity and ε be the root of the equation $x^2 + 3 = 0$. If μ is a root of the equation $3x^2 - 3x + 1 = 0$, then:

$$\mu^{-1} = 3(1-\mu), \ \omega = \mu^{-1}(\mu - 1) = 3\mu - 2, \tag{1.2}$$

$$\varepsilon = \mu^{-1}\omega = 3(2\mu - 1), \ \omega - \omega^2 = \varepsilon = 2\omega + 1.$$

Definition 1.3. Let A be a finite dimensional algebra over the field K, and $K \subset F$ be a field extension. The algebra A is a *separable algebra* if the algebra $A_F = F \otimes_K A$ is a direct sum of simple ideals, for every extension F of the field K. The algebra A is called a *central simple algebra* if the algebra A_F is a simple algebra, for every extension F of the field K.

If A is an associative central simple algebra, then each automorphism of A is inner [EP96].

Remark 1.4 ([McC69]). If A is a unitary finite dimensional alternative algebra of degree three over K, then the trace form T is nondegenerate if and only if A is a separable algebra.

Now, we suppose $\omega \in K$; then $\varepsilon, \mu \in K$. Let A be a unitary finite dimensional of degree three separable alternative algebra and

$$A_0 = \{x \in A \,/\, T(x) = 0\}$$

be its K-subspace of trace zero elements. The bilinear form S is nondegenerate over A if and only if S is nondegenerate over A_0 ([EM93]).

Proposition 1.5 ([Fau88])**.** *Let A be a finite dimensional unitary of degree three alternative algebra over the field K. Define the multiplication $*$ on A_0*

$$a * b = \omega ab - \omega^2 ba - \frac{2\omega + 1}{3} T(ab) \cdot 1, \forall a, b \in A_0. \tag{1.3}$$

Then S preserves composition, that is $S(a * b) = S(a) S(b)$. If A is a separable algebra over K, then the quadratic form S is nondegenerate and if $\dim A = 9$ *there exists an operation ∇ such that (A_0, ∇) becomes an octonion algebra.*

In Proposition 1.5, if $\dim A \in \{5, 9\}$, then we can find an operation ∇ such that (A_0, ∇) is a Hurwitz algebra (hence a quaternion algebra and octonion algebra).

Now we try to see if these algebras can be split or division algebras.

A. Elduque and H.C. Myung, in [EM93] proved that, if A is an alternative algebra over K with the generic minimum polynomial $P_x(\lambda) = \lambda^3 - T(x)\lambda^2 + S(x)\lambda - N(x) \cdot 1$ and the subspace A_0, and we define the multiplication $*$ by the relation (1.3), then the following identity holds:

$$(a * b) * a = a * (b * a) = S(a)\, b, \text{ for all } a, b \in A_0. \tag{1.4}$$

Moreover, S preserves composition and it is associative, so that $S(x * y, z) = S(x, y * z)$, for all $x, y, z \in (A_0, *)$. $(A_0, *)$ does not have a unit element and there is an element $a \in A_0$, such that $\{a, a*a\}$ is a linearly independent system. The above alternative algebra A is finite dimensional and separable if and only if S is nondegenerate.

In the same paper, they proved that the converse of this statement is true. Indeed, if $(B, *)$ is a nonunitary algebra over the field K,with its associated quadratic form S satisfying the condition (1.3) and if B has an element b_0 such that $\{b_0, b_0 * b_0\}$ are linearly independent, then we can build an alternative algebra A of degree three over K such that $(B, *)$ is isomorphic with the algebra $(A_0, *)$ defined above. Indeed, let $S(x, y)$ be the symmetric bilinear form associated to the quadratic form S. For $A = K \cdot 1 \oplus B$ if we define the following multiplication on A:

$$ab{=} - \frac{2S(a, b)}{3} \cdot 1 {+} \frac{1}{3}\left[\left(\omega^2\text{-}1\right) a * b\text{-} (\omega\text{-}1)\, b * a\right], \forall a, b \in B \tag{1.5}$$

and $1x = x1 = x, \forall x \in A$, then the algebra A is an alternative algebra of degree three.

2. Octonion algebras and quaternion algebras

By using the above procedure (i.e. the multiplication (1.5)), we obtained an alternative algebra $A = K \cdot 1 \oplus B$. Now we are looking for conditions on A to be associative. We get the following proposition.

Proposition 2.1. *The algebra $A = K \cdot 1 \oplus B$, constructed above, is associative if and only if:*

$$(a,c,b)^* + (b,a,c)^* = (a,b,c)^*, \forall a,b,c \in B, \tag{2.1}$$

with $(a,b,c)^* = (a*b)*c - a*(b*c)$.

Proof. Let $a,b,c \in B$, then we have:

$$\begin{aligned}
c\,(ab) &= c\left[\text{-}\frac{2S\,(a,b)}{3}\cdot 1 + \frac{1}{3}\left(\left(\omega^2\text{-}1\right) a*b - (\omega\text{-}1)\, b*a\right)\right] \\
&= -\frac{2S\,(a,b)}{3}c + \frac{\omega^2-1}{3}c\,(a*b) + \frac{\omega-1}{3}c\,(b*a) \\
&= -\frac{2S\,(a,b)}{3}c \\
&\quad + \frac{\omega^2-1}{3}\left[-\frac{2S\,(c,a*b)}{3}\cdot 1 + \frac{\omega^2-1}{3}c*(a*b) - \frac{\omega-1}{3}\,(a*b)*c\right] \\
&\quad - \frac{\omega-1}{3}\left[-\frac{2S\,(c,b*a)}{3} + \frac{\omega^2-1}{3}c*(b*a) - \frac{\omega-1}{3}\,(b*a)*c\right].
\end{aligned}$$

$$\begin{aligned}
(ca)\,b &= \left[\text{-}\frac{2S\,(c,a)}{3}\cdot 1 + \frac{1}{3}\left(\left(\omega^2-1\right) c*a - (\omega-1)\, a*c\right)\right] b \\
&= -\frac{2S\,(c,a)}{3}b + \frac{\omega^2-1}{3}\,(c*a)\,b - \frac{\omega\text{-}1}{3}\,(a*c)\,b \\
&= -\frac{2S\,(c,a)}{3}b \\
&\quad + \frac{\omega^2-1}{3}\left[-\frac{2S\,(c*a,b)}{3}\cdot 1 + \frac{\omega^2\text{-}1}{3}\,(c*a)*b - \frac{\omega-1}{3}b*(c*a)\right] \\
&\quad - \frac{\omega-1}{3}\left[-\frac{2S\,(a*c,b)}{3}\cdot 1 + \frac{\omega^2-1}{3}\,(a*c)*b - \frac{\omega-1}{3}b*(a*c)\right].
\end{aligned}$$

We use the relations:

$$(a*b)*c + (c*b)*a = 2S\,(a,c)\,b = 2S\,(c,a)\,b$$

obtained by linearization of the relation (1.4), and we get:

$$\left(\omega^2-1\right)^2 = -3\omega^2, \left(\omega^2-1\right)(\omega-1) = 3, (\omega-1)^2 = -3\omega.$$

Then:

$$\begin{aligned}
c\left(ab\right)-\left(ca\right)b &= -\frac{2S\left(a,b\right)}{3}c-\frac{2\left(\omega^{2}-1\right)}{9}S\left(c,a*b\right)+\frac{\left(\omega^{2}-1\right)^{2}}{9}c*\left(a*b\right)\\
&\quad-\frac{\left(\omega^{2}-1\right)\left(\omega-1\right)}{9}\left(a*b\right)*c+\frac{2\left(\omega-1\right)}{9}S\left(c,b*a\right)\\
&\quad-\frac{\left(\omega^{2}-1\right)\left(\omega-1\right)}{9}c*\left(b*a\right)+\frac{\left(\omega-1\right)^{2}}{3}\left(b*a\right)*c\\
&\quad+\frac{2S\left(c,a\right)}{3}b+\frac{2\left(\omega^{2}-1\right)}{9}S(c*a,b)-\frac{\left(\omega^{2}-1\right)^{2}}{9}(c*a)*b\\
&\quad+\frac{\left(\omega^{2}-1\right)\left(\omega-1\right)}{9}b*\left(c*a\right)-\frac{2\left(\omega-1\right)}{9}S\left(a*c,b\right)\\
&\quad+\frac{\left(\omega^{2}-1\right)\left(\omega-1\right)}{9}\left(a*c\right)*b-\frac{\left(\omega-1\right)^{2}}{9}b*\left(a*c\right)\\
&= -\frac{1}{3}\left(a*c\right)*b-\frac{1}{3}\left(b*c\right)*a-\frac{2\left(\omega^{2}\text{-}1\right)}{9}S\left(c,a*b\right)\\
&\quad-\frac{\omega^{2}}{3}c*\left(a*b\right)-\frac{1}{3}\left(a*b\right)*c+\frac{2\left(\omega-1\right)}{9}S\left(c,b*a\right)-\frac{1}{3}c*\left(b*a\right)\\
&\quad-\frac{\omega}{3}\left(b*a\right)*c+\frac{1}{3}\left(a*b\right)*c+\frac{1}{3}\left(c*b\right)*a+\frac{2\left(\omega^{2}-1\right)}{9}S\left(c*a,b\right)\\
&\quad+\frac{\omega^{2}}{3}\left(c*a\right)*b+\frac{1}{3}b*\left(c*a\right)-\frac{2\left(\omega-1\right)}{9}S\left(a*c,b\right)\\
&\quad+\frac{1}{3}\left(a*c\right)*b+\frac{\omega}{3}b*\left(a*c\right).
\end{aligned}$$

Since S is associative over B, we have:

$$S\left(c,a*b\right)=S(c*a,b) \text{ and } S\left(c,b*a\right)=S\left(b*a,c\right)=S\left(b,a*c\right).$$

It results that:

$$\begin{aligned}
c\left(ab\right)-\left(ca\right)b &= -\frac{1}{3}[(b*c)*a-b*(c*a)]-\frac{\omega}{3}[(b*a)*c-b*(a*c)]\\
&\quad+\frac{1}{3}[(c*b)*a-c*(b*a)]-\frac{\omega^{2}}{3}c*\left(a*b\right)+\frac{\omega^{2}}{3}\left(c*a\right)*b\\
&= -\frac{1}{3}\left(b,c,a\right)^{*}-\frac{\omega}{3}\left(b,a,c\right)^{*}+\frac{1}{3}\left(c,b,a\right)^{*}\\
&\quad+\frac{\omega^{2}}{3}\left(c,a,b\right)^{*}-\frac{\omega^{2}}{3}\left(c*a\right)*b+\frac{\omega^{2}}{3}\left(c*a\right)*b\\
&= -\frac{1}{3}\left(b,c,a\right)^{*}+\frac{1}{3}\left(c,b,a\right)^{*}+\frac{1}{3}\left(b,a,c\right)^{*},
\end{aligned}$$

therefore the required relation holds. □

If $(A,\cdot)$ is a flexible composition finite dimensional algebra and it satisfies the condition $(x\cdot y)\cdot x=x\cdot(y\cdot x)=f\left(x,\,x\right)y$, $\forall x,y\in A$ then A is a division algebra if and only if its associated bilinear form f has the property $f(x,x) \neq 0$, for each $x \neq 0$ [EM93]. By using the above conditions, we get that $(A_0, *)$ is a division algebra if and only if $S(x,x) \neq 0$ for all $x \neq 0$.

Definition 2.2. An associative algebra A is a *cyclic algebra* if there exists F, a maximal subfield of the algebra $A, F \neq K$, such that $K \subset F$ is a cyclic extension (i.e. a Galois extension with a cyclic associated Galois group).

Each associative finite dimensional central simple algebra, over an arbitrary field is a separable algebra and each finite dimensional of degree three division associative algebra is a cyclic algebra [Pie82].

Proposition 2.3. *Let $(A, *)$ be a composition algebra with an associative bilinear form f and $u \in A$ be a nonzero idempotent. If $\dim A \in \{4, 8\}$, then we find an operation ∇ such that (A, ∇) becomes a Hurwitz algebra with the norm q, and conversely. If $\dim A \neq 8$, then $x * y = \bar{x} \nabla \bar{y}$, where $\bar{x}$ is the conjugate of x in (A, ∇).*

Proof. Since f is associative, we have the relation (1.1). Therefore $u = (u * u) * u = q(u)\, u$ and then $q(u) = 1$. We obtain also $(u * x) * u = q(u)\, x = x, \forall x \in A$. By linearizing the relation (1.1), we get

$$(x * y) * z + (z * y) * x = 2f(x,z)\, y, \forall x, y, z \in A. \tag{2.2}$$

By relation (2.2) we have $(x * u) * u + u * x = 2f(u,x)\, u$, hence

$$((x * u) * u) * u = 2f(u,x)\, u - x$$

and we have

$$R_u^3(x) = 2f(u,x)\, u - x,$$

where $R_u \colon (A, *) \to (A, *)$ is the right multiplication. In [EP96] since $q(u) \neq 0$ we have $R_u = L_u^{-1}$ where L_u is the left multiplication. Defining

$$x \nabla y = (u * x) * (y * u),$$

(A, ∇) is a Hurwitz algebra with the norm q and u the unit element. Then $f(u,x)\, u - x$ is the conjugate of x so that

$$R_u^3(x) = 2f(u,x)\, u - x = \bar{x}.$$

Since

$$R_u^3(\bar{x}) = 2f(u,\bar{x})\, u - \bar{x} = 4f(u,x)\, u - 2f(u,x)\, u - 2f(u,x)\, u + x = x,$$

the map $\varphi \colon (A, \nabla) \to (A, \nabla)$, where $\varphi(x) = \bar{x} * u$ is an automorphism with the property $\varphi^3 = 1_A$. Now, defining

$$x \perp y = (\varphi(\bar{x}))\, \nabla \left(\varphi^{-1}(\bar{y})\right) = (x * u)\, \nabla\, (u * y),$$

we have $x \perp y = (u * (x * u)) * ((u * y) * u) = x * y$, therefore

$$x * y = (\varphi(\bar{x}))\, \nabla \left(\varphi^{-1}(\bar{y})\right). \tag{2.3}$$

If (A, ∇) is a quaternion algebra, then it is an associative central simple algebra and its automorphisms are inner. For an automorphism φ with

$$\varphi^3 = 1_A, \varphi \neq 1_A,$$

there exist an element $v \in (A, \nabla)$ such that $q(v) \neq 0$ and $\varphi(x) = v^{-1}\nabla x \nabla v$. Therefore we have $v^3 = a\nabla u, a \in K$. For $z = \frac{v^2}{q(v)}$, we have $z^3 = u$ and $q(z) = 1$, hence $\bar{z} = z^2$. Then we have

$$z\nabla x\nabla z^2 = v^{-1}\nabla x\nabla v = \varphi(x).$$

By the relation (2.3), we have

$$z * z = (\varphi(\bar{z}))\nabla\left(\varphi^{-1}(\bar{z})\right) = z\nabla\bar{z}\nabla z^2\nabla z^2\nabla\bar{z}\nabla z = z.$$

We compute

$$\begin{aligned} x * z &= z\nabla\bar{x}\nabla z^2\nabla z^2\nabla\bar{z}\nabla z = z\nabla\bar{x}\nabla z \\ &= \left(2f(z\nabla\bar{x}, u)\nabla u - \overline{z\nabla\bar{x}}\right)\nabla z \\ &= \left(2f(z\nabla\bar{x}, u) - x\nabla z^2\right)\nabla z \\ &= 2f(z, \bar{x})\nabla z - x = 2f(z, x)\nabla z - x. \end{aligned}$$

In the same way, $z * x = 2\ f(z, x)\nabla z - x$. It results that $x * z = z * x = \bar{x}$ and $\varphi : (A, \nabla) \to (A, \nabla), \varphi(x) = \bar{x} * z$, is an automorphism with the property $\varphi^3 = 1_A$. Then $x * y = (\varphi(\bar{x}))\nabla(\varphi^{-1}(\bar{y})) = (x * z)\nabla(z * y) = \bar{x}\nabla\bar{y}$. □

Proposition 2.4. *Let A be a finite dimensional of degree three alternative algebra with the generic minimum polynomial*

$$P_x(\lambda) = \lambda^3 - T(x)\lambda^2 + +S(x)\lambda - N(x)\cdot 1.$$

The algebra $(A_0, *)$ is a division algebra (with $\omega \in K$ or $\omega \notin K$) if and only if A_0 does not contain the nonzero elements $x \in A_0$ such that $x^2 \in A_0$.

Proof. We have $2S(x, x) = 2S(x) = T^2(x) - T(x^2)$. If there is an element $x_1 \in A_0, x_1 \neq 0$, such that $x_1^2 \in A_0$, we have $T(x_1) = T(x_1^2) = 0$. It results that

$$2S(x_1, x_1) = 2S(x_1) = T^2(x_1) - T\left(x_1^2\right) = 0,$$

therefore $x_1 = 0$, contradiction. □

Proposition 2.5. *Let A be a finite dimensional central simple of degree three alternative algebra over the field K. Define the multiplication $*$ on A_0:*

$$a * b = \omega ab - \omega^2 ba - \frac{2\omega + 1}{3}T(ab)\cdot 1.$$

Then S preserves composition, that is $S(a * b) = S(a)S(b)$. If

$$(a, c, b)^* + (b, a, c)^* = (a, b, c)^*, \forall a, b, c \in A_0$$

and $\dim A = 9$, then there is an operation ∇ such that (A_0, ∇) becomes an octonion algebra. This algebra is not a division algebra.

Proof. If $(a,c,b)^* + (b,a,c)^* = (a,b,c)^*$, then the algebra A is an associative central simple algebra and it is separable. (For a classification of central simple associative algebras, see [Pie82]). Therefore the quadratic form is nondegenerate on A and A_0. Then, there is an element $u \in A, u \neq 0$, such that $S(u) \neq 0$. We define $a\nabla b = (u * a) * (b * u)$ and the algebra (A_0, ∇) becomes an octonion algebra with the unit element $S^{-1}(u)\, u * u$. To end the proof we need the following lemma:

Lemma 2.6. *If* $\dim A_0 \in \{4,8\}$, *then the algebra* $(A_0,*)$ *is a division algebra if and only if* (A_0,∇) *is a division algebra.*

Proof of the Lemma. Indeed, we suppose that $(A_0,*)$ is a division algebra. The equations $a\nabla x = b$ and $y\nabla a = b$ can be written $(u*a)**(x*u) = b$ and $(u*y)*(a*u) = b$ and they have unique solutions.

Conversely, if (A_0,∇) is a division algebra, by using the relation (2.3), the equations $a*x=b$ and $y*a=b$, can be written $(\varphi(\bar{a}))\nabla(\varphi^{-1}(\bar{x})) = b$ and $(\varphi(\bar{y}))\nabla(\varphi^{-1}(\bar{a})) = b$ and they have unique solutions. $\square$

If A is not a division algebra, then $A \simeq \mathcal{M}_3(K)$. In this case, we find an element, for example $X = \begin{pmatrix} 0 & 0 & \gamma \\ 0 & \varepsilon & 0 \\ \gamma & 0 & \bar{\varepsilon} \end{pmatrix}$, where $\gamma^2 = 3$ and $\varepsilon^2 = -3$ with the property $X^2 = \begin{pmatrix} 3 & 0 & \gamma\bar{\varepsilon} \\ 0 & -3 & 0 \\ \gamma\bar{\varepsilon} & 0 & 0 \end{pmatrix}$, therefore $T(X^2) = 0$ and $(A_0,*)$ is not a division a algebra hence the octonion algebra (A_0,∇) is a split algebra.

If we take the element $Y = \begin{pmatrix} 0 & 0 & \sqrt{3} \\ 0 & i\sqrt{3} & 0 \\ \sqrt{3} & 0 & -i\sqrt{3} \end{pmatrix}$, we have

$$Y^2 = \begin{pmatrix} 3 & 0 & -3i \\ 0 & -3 & 0 \\ -3i & 0 & 0 \end{pmatrix}$$

and

$$Y^3 = \begin{pmatrix} -3i\sqrt{3} & 0 & 0 \\ 0 & -3i\sqrt{3} & 0 \\ 0 & 0 & -3i\sqrt{3} \end{pmatrix},$$

therefore $Y^3 - \alpha I_3 = 0_3$, with $\alpha = -3i\sqrt{3} \in K$ and we get the same result.

If A is a division algebra, then is a cyclic algebra, and we find in A an element $x \neq 0$ with the minimum polynomial $X^3 - \alpha, \alpha \in K, \alpha \neq 0$. It results that $T(x) = S(x) = T(x^2) = 0$, $x \in A_0$, then, by Lemma, $(A_0,*)$ is not a division algebra, hence (A_0,∇) is a split algebra. $\square$

Corollary 2.7. *If* $\omega \in K$, *then in the algebra* $\mathcal{M}_3(K)$, *there are only split octonion algebra.*

We know that, if the field K is algebraically closed, then the octonion algebra is split. From the Corollary 2.6., if $K = \mathbb{Q}(\omega)$ for example, in $\mathcal{M}_3(K)$ we have only octonion split algebras.

Remark 2.8. If A is a finite dimensional separable of degree three alternative algebra, A' is a subalgebra of A and $(A_0, *), (A'_0, *)$ are the algebras in Proposition 1.5., then A'_0 is a subalgebra of A', and conversely. Indeed, let $\alpha\colon A' \to A$ be an inclusion morphism, then

$$\alpha'\colon (A'_0, *) \to (A_0, *), \alpha'(x) = \alpha(x),$$

is an inclusion morphism. For the converse, we use the relation (1.5).

If A is a division central simple finite dimensional associative algebra of degree three over the field K, with $\dim A = 9$, and if A' is a subalgebra of A of dimension 5, then in $(A'_0, *)$ we have an operation ∇ such that (A'_0, ∇) becomes a quaternion algebra and, by Proposition 2.3., $x * y = \bar{x}\nabla\bar{y}$. Then the unity of (A'_0, ∇), e, is a nonzero idempotent in $(A'_0, *)$,

$$e * e = e = (\omega - \omega^2)e^2 - \frac{2\omega+1}{3}T\left(e^2\right) \cdot 1$$

therefore e generates in A a quadratic extension of the field K. This is not possible, since A has degree three. Then we do not have a quaternion algebra in A.

Corollary 2.9. *If $\omega \in K$, then in the algebra $\mathcal{M}_3(K)$ there are only split quaternion algebras.*

Proof. The algebra $\mathcal{B} = \{A \in \mathcal{M}_3(K) \;/\; A = \begin{pmatrix} a & 0 & 0 \\ 0 & b & c \\ 0 & d & e \end{pmatrix}, \; a+b+e = 0\}$ is a subalgebra of $\mathcal{M}_3(K)$. We define the algebra$(\mathcal{A}_0, *)$, where

$$\mathcal{A}_0 = \{A \in \mathcal{M}_3(K) \;/\; Tr(A) = 0\}.$$

The algebra $(\mathcal{B}, *)$ is a subalgebra of $(\mathcal{A}_0, *)$. As from algebra $(\mathcal{B}, *)$ we obtain the quaternion algebra $(\mathcal{B}, \nabla)$ and $a * b = \bar{a}\nabla\bar{b}$, then this quaternion algebra is a split algebra. Indeed, $(\mathcal{B}, *)$ is not a division algebra since, for example, $X = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \varepsilon & \gamma \\ 0 & \gamma & \bar{\varepsilon} \end{pmatrix} \in \mathcal{B}$ and $X^2 = 0$. $\square$

In a future paper we search for similar conditions for octonion and quaternion algebras, when the field K contains the cubic root of the unity, ω.

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Department of Mathematics and Informatics
Ovidius University,
Bd. Mamaia 124, 900527-Constantza,
Romania
E-mail address: cflaut@univ-ovidius.ro