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# ON RIEMANNIAN TANGENT BUNDLES

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Abstract. We study the geometry of manifolds whose tangent bundle is endowed with a Riemannian metric. The Levi-Civita connection, Schouten-Van Kampen connection and Vrănceanu connection are the main tools for this study. We obtain characterizations of special classes of vertical foliations and compare the sectional curvatures of the horizontal distribution with respect to the above connections.

## Introduction

As it is well-known, the tangent bundle of a Riemannian manifold becomes a Riemannian manifold too. A method to construct a Riemannian metric on the tangent bundle of a Riemannian manifold was developed by Sasaki [5]. This metric has been called the Sasaki metric and has had a great role in the study of the geometry of the tangent bundle of a Riemannian manifold. More general, the tangent bundle of a Finsler manifold is endowed with the so called Sasaki-Finsler metric (see Bejancu–Farran [1], p. 48), which is completely determined by the fundamental function of the Finsler manifold.

The above two large classes of manifolds appear as particular cases of the manifolds we introduce and study in the present paper. Let $M$ be a manifold whose tangent bundle $TM$ is endowed with a Riemannian metric $G$. Then we call $(TM, G)$ a Riemannian tangent bundle of $M$. In the first section we consider the Schouten-Van Kampen and Vrănceanu connections induced by the Levi-Civita connection on $(TM, G)$ and obtain characterizations of both the vertical and horizontal distributions when these connections coincide. Next, in the second section we first deduce the structure equations which relate the curvature tensor fields of the Schouten–Van Kampen and Levi–Civita connections. Finally, in case $G$ is bundle-like for the vertical foliation we are able to compare the sectional curvatures of the horizontal distribution with respect to the above three connections.

## 1. Linear Connections on a Riemannian Tangent Bundle

Let $M$ be a real $n-$dimensional manifold and $TM$ the tangent bundle of $M$ with the canonical projection $\pi\colon TM \longrightarrow M$. Then a local chart $(\mathcal{U}, \varphi)$ on $M$ with local coordinates $(x^i)$ for $x \in M$, $i \in \{1, \ldots, n\}$, defines a local chart $(\pi^{-1}(\mathcal{U}), \Phi)$ on $TM$ with local coordinates $(x^i, y^i)$ for $y = y^i \frac{\partial}{\partial x^i} \in \pi^{-1}(\mathcal{U})$. The transformations of coordinates on $TM$ are given by

$$\tilde{x}^i = \tilde{x}^i(x^1, \ldots, x^n),\ \tilde{y}^i = J^i_j(x) y^j, \tag{1.1}$$

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where $J^i_j(x) = \frac{\partial \tilde{x}^i}{\partial x^j}$. As a consequence of (1.1) the local frame fields $\{\frac{\partial}{\partial x^i}\ ,\ \frac{\partial}{\partial y^i}\}$ and $\{\frac{\partial}{\partial \tilde{x}^i}\ ,\ \frac{\partial}{\partial \tilde{y}^i}\}$ are related by

$$\frac{\partial}{\partial x^i} = J^j_i(x)\frac{\partial}{\partial \tilde{x}^j} + J^j_{ik}(x)y^k\ \frac{\partial}{\partial \tilde{y}^j}, J^j_{ik}(x) = \frac{\partial^2 \tilde{x}^j}{\partial x^i \partial x^k} \tag{1.2}$$

and

$$\frac{\partial}{\partial y^i} = J^j_i(x)\frac{\partial}{\partial \tilde{y}^j}. \tag{1.3}$$

Throughout the paper all manifolds are paracompact, and mappings are smooth (differentiable of class $C^\infty$). We denote by $\mathcal{F}(M)$ the algebra of smooth functions on $M$ and by $\Gamma(TM)$ the $\mathcal{F}(M)$-module of smooth vector fields on $M$. Similar notations we use for any other manifold or vector bundle. Also, we use the Einstein convention, that is, repeated indices with one upper index and one lower index denotes summation over their range. If not stated otherwise, we use the indices: $i, j, k, \ldots \in \{1, \ldots, n\}$.

Next, we consider on $TM$ the *vertical distribution* $VTM$, which is the tangent distribution to the foliation $\mathcal{F}_V$ determined by the fibers of $\pi\colon TM \longrightarrow M$. Thus $VTM$ is locally spanned by $\{\frac{\partial}{\partial y^i}\}$, $i \in \{1, \ldots, n\}$. Also, we suppose that $TM$ admits a Riemannian metric $G$ and denote by ${}^v\!g$ the induced Riemannian metric by $G$ on $VTM$. Then the local components of ${}^v\!g$ are given by

$${}^v\!g_{ij}(x, y) = G\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right). \tag{1.4}$$

We call $(TM, G)$ a *Riemannian tangent bundle* of $M$. Note that $M$ needs not to be a Riemannian manifold. Examples of such manifolds are abundant. First, any Riemannian manifold has a Riemannian tangent bundle whose Riemannian metric is the well-known Sasaki metric (cf. Sasaki [5]). In a similar way, a Finsler manifold has a Riemannian tangent bundle whose Riemannian metric is the Sasaki-Finsler metric (cf. Bejancu–Farran [1], p. 48).

Now, we denote by $HTM$ the complementary orthogonal distribution to $VTM$ in $TTM$ with respect to $G$ and call it the *horizontal distribution* on $(TM, G)$. Thus we have the orthogonal decomposition.

$$TTM = VTM \oplus HTM. \tag{1.5}$$

Then on $\pi^{-1}(\mathcal{U})$ we express each $\frac{\partial}{\partial x^i}$ as follows

$$\frac{\partial}{\partial x^i} = A^j_i\frac{\partial}{\partial y^j} + \frac{\delta}{\delta x^i},$$

where $\frac{\delta}{\delta x^i} \in \Gamma(HTM)$. Thus $HTM$ is locally spanned by

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - A^j_i\frac{\partial}{\partial y^j},\ i \in \{1, \ldots, n\}. \tag{1.6}$$

By using (1.2), (1.3) and (1.6) we deduce that

$$\frac{\delta}{\delta x^i} = J^j_i(x)\frac{\delta}{\delta \tilde{x}^j}, \tag{1.7}$$

with respect to the transformations of coordinates (1.1). Moreover, from (1.6) it follows that $A^j_i$ are determined by the Riemannian metric $G$ as follows

$$A^j_i = G\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^k}\right)\ {}^v\!g^{kj}, \tag{1.8}$$

where ${}^v\!g^{kj}$ are the entries of the inverse matrix of the $n \times n$ matrix $[{}^v\!g_{kj}]$. By direct calculations using (1.6) we obtain the following.

**Proposition 1.1.** *Let $(TM, G)$ be a Riemannian tangent bundle of $M$. Then we have:*

$$(1.9)\qquad (a)\quad \left[\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right] = R^k{}_{ij}\frac{\partial}{\partial y^k}, \qquad (b)\quad \left[\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j}\right] = D_j{}^k{}_i\frac{\partial}{\partial y^k},$$

*where we put:*

$$(1.10)\qquad (a)\quad R^k{}_{ij} = \frac{\delta A^k_i}{\delta x^j} - \frac{\delta A^k_j}{\delta x^i}, \qquad (b)\quad D_j{}^k{}_i = \frac{\partial A^k_i}{\partial y^j}.$$

It is interesting to note that the local functions defined by (1.10) behave in a different way with respect to (1.1). More precisely, by using (1.3), (1.7) and (1.9) we deduce that

$$(1.11)\qquad \begin{aligned} &\text{(a)}\quad R^k{}_{ij}\,\frac{\partial \tilde{x}^h}{\partial x^k} = \tilde{R}^h{}_{st}\,\frac{\partial \tilde{x}^s}{\partial x^i}\,\frac{\partial \tilde{x}^t}{\partial x^j},\\ &\text{(b)}\quad D_i{}^k{}_j\,\frac{\partial \tilde{x}^h}{\partial x^k} = \tilde{D}_s{}^h{}_t\,\frac{\partial \tilde{x}^s}{\partial x^i}\,\frac{\partial \tilde{x}^t}{\partial x^j} + \frac{\partial^2 \tilde{x}^h}{\partial x^i \partial x^j}.\end{aligned}$$

Moreover, by using (1.9a) we can state the following.

**Proposition 1.2.** *The horizontal distribution on a Riemannian tangent bundle $(TM, G)$ is integrable if and only if we have*

$$(1.12)\qquad R^k{}_{ij} = 0,\ \forall\ i, j, k \in \{1, \dots, n\}.$$

Next, we consider the Levi–Civita connection $\tilde{\nabla}$ on $(TM, G)$ given by (cf. Kobayashi–Nomizu [3], p. 160)

$$(1.13)\qquad \begin{aligned} 2G(\tilde{\nabla}_X Y, Z) &= X(G(Y, Z)) + Y(G(Z, X)) - Z(G(X, Y))\\ &\quad + G([X, Y], Z) - G([Y, Z], X) + G([Z, X], Y),\end{aligned}$$

for all $X, Y, Z \in \Gamma(TTM)$. Also, we recall that $\tilde{\nabla}$ is torsion-free and metric connection, that is, we have:

$$(1.14)\qquad \tilde{\nabla}_X Y - \tilde{\nabla}_Y X - [X, Y] = 0,$$

and

$$(1.15)\qquad (\tilde{\nabla}_X G)(Y, Z) = X(G(Y, Z)) - G(\tilde{\nabla}_X Y, Z) - G(Y, \tilde{\nabla}_X Z) = 0,$$

for all $X, Y, Z \in \Gamma(TTM)$. In general, none of the distributions $VTM$ or $HTM$ is parallel with respect to $\tilde{\nabla}$. However, $\tilde{\nabla}$ can be used to construct such special linear connections on $TM$. Two of them we consider here (cf. Ianus [2]):

$$(1.16)\qquad \nabla_X Y = V\tilde{\nabla}_X VY + H\tilde{\nabla}_X HY,$$

and

$$(1.17)\qquad \nabla^*_X Y = V\tilde{\nabla}_{VX} VY + H\tilde{\nabla}_{HX} HY + V[HX, VY] + H[VX, HY],$$

for all $X, Y \in \Gamma(TTM)$, where $V$ and $H$ are the projection morphism of $TTM$ on $VTM$ and $HTM$ respectively. Taking into account that $\nabla$ and $\nabla^*$ have been first defined in [6] and [7] on non-holonomic manifolds (by using local coefficients), we call them the *Schouten–Van Kampen connection* and the *Vrănceanu connection* respectively. Also we denote by ${}^v\nabla$ and ${}^h\nabla$ the induced linear connections by $\nabla$ on $VTM$ and $HTM$ and call them the *vertical* and *horizontal Schouten–Van Kampen connections* respectively. Similarly, we define the *vertical* and *horizontal Vrănceanu connections* ${}^v\nabla^*$ and ${}^h\nabla^*$ induced by $\nabla^*$ on $VTM$ and $HTM$ respectively.

The above two special connections on $TM$ allow us to define geometric objects related with the decomposition (1.5). For instance, we can define the covariant derivative of the Riemannian metric $\overset{v}{g}$ on $VTM$ with respect to the vertical Vrănceanu connection as follows:

$$({}^{v}\nabla^{*}_{X}\ \overset{v}{g})(VY,VZ) = X(\overset{v}{g}(VY,VZ)) - \overset{v}{g}({}^{v}\nabla^{*}_{X}VY,VZ) - \overset{v}{g}(VY,\ {}^{v}\nabla^{*}_{X}VZ), \tag{1.18}$$

for all $X,Y,Z \in \Gamma(TTM)$. By using (1.17) and (1.15) into (1.18) we deduce that $\overset{v}{g}$ is *vertical parallel* with respect to ${}^{v}\nabla^{*}$, that is, we have

$$({}^{v}\nabla^{*}_{VX}\ \overset{v}{g})(VY,VZ) = 0,\ \forall X,Y,Z \in \Gamma(TTM). \tag{1.19}$$

However, $\overset{v}{g}$ is not parallel with respect to horizontal vector fields. More precisely, if we take $X = \frac{\delta}{\delta x^{i}},\ VY = \frac{\partial}{\partial y^{j}}, VZ = \frac{\partial}{\partial y^{k}}$ in (1.18) and use (1.4), (1.17) and (1.9b), we deduce that:

$$\overset{v}{g}_{jk|i} = ({}^{v}\nabla^{*}_{\frac{\delta}{\delta x^{i}}}\ \overset{v}{g})\left(\frac{\partial}{\partial y^{j}},\frac{\partial}{\partial y^{k}}\right) = \frac{\delta\ \overset{v}{g}_{jk}}{\delta x^{i}} - \ \overset{v}{g}_{hk}\ D_{j\ i}^{\ h} - \ \overset{v}{g}_{jh}\ D_{k\ i}^{\ h}. \tag{1.20}$$

Next, we denote by $\overset{h}{g}$ the induced Riemannian metric by $G$ on HTM and define

$$\begin{aligned}\left({}^{h}\nabla^{*}_{X}\ \overset{h}{g}\right)(HY,HZ) = X\left(\overset{h}{g}(HY,HZ)\right) - \ &\overset{h}{g}({}^{h}\nabla^{*}_{X}HY,HZ)\\ - \ &\overset{h}{g}(HY,\ {}^{h}\nabla^{*}_{X}HZ),\end{aligned} \tag{1.21}$$

for all $X,Y,Z \in \Gamma(TM)$. Then in a similar way as above we obtain

$$\begin{aligned}&\text{(a)}\quad ({}^{h}\nabla^{*}_{HX}\ \overset{h}{g})(HY,HZ) = 0,\\ &\text{(b)}\quad \overset{h}{g}_{jk\|i} = ({}^{h}\nabla^{*}_{\frac{\partial}{\partial y^{i}}}\ \overset{h}{g})\left(\frac{\delta}{\delta x^{j}},\frac{\delta}{\delta x^{k}}\right) = \frac{\partial\ \overset{h}{g}_{jk}}{\partial y^{i}},\end{aligned} \tag{1.22}$$

where we put

$$\overset{h}{g}_{jk} = G\left(\frac{\delta}{\delta x^{j}},\frac{\delta}{\delta x^{k}}\right). \tag{1.23}$$

Now, we recall that the Riemannian metric $G$ is *bundle-like* for the vertical foliation $\mathcal{F}_{V}$ if and only if (see Reinhart [4], p. 122.)

$$\frac{\partial\ \overset{h}{g}_{ik}}{\partial y^{j}} = 0,\ \forall\ i,\ j,\ k\ \in\{1,\ldots,n\}. \tag{1.24}$$

Thus by using (1.22) and (1.24) we can state the following.

**Theorem 1.1.** *Let $(TM,G)$ be a Riemannian tangent bundle of $M$. Then the induced Riemannian metric $\overset{h}{g}$ on $HTM$ is parallel with respect to the horizontal Vrănceanu connection if and only if $G$ is bundle-like for $\mathcal{F}_{V}$.*

Also, the above covariant derivatives enable us to find the local coefficients of the Levi-Civita connection on $(TM,G)$ as it is stated in the next theorem.

**Theorem 1.2.** *The Levi-Civita connection $\tilde{\nabla}$ on the Riemannian tangent bundle $(TM,G)$ is locally given by:*

$$\begin{aligned}&(a)\quad \tilde{\nabla}_{\frac{\delta}{\delta x^{j}}}\frac{\delta}{\delta x^{i}} = -\frac{1}{2}\ \overset{v}{g}^{kt}(\overset{h}{g}_{ij\|t} + \ \overset{v}{g}_{ts}R^{s}_{\ ij})\frac{\partial}{\partial y^{k}} + F_{i\ j}^{\ k}\frac{\delta}{\delta x^{k}},\\ &(b)\quad \tilde{\nabla}_{\frac{\delta}{\delta x^{j}}}\frac{\partial}{\partial y^{i}} = \left(\frac{1}{2}\ \overset{v}{g}^{kt}\ \overset{v}{g}_{ti|j} + D_{i\ j}^{\ k}\right)\frac{\partial}{\partial y^{k}}\\ &\qquad\qquad + \frac{1}{2}\ \overset{h}{g}^{kt}(\overset{h}{g}_{tj\|i} + \ \overset{v}{g}_{is}R^{s}_{\ tj})\ \frac{\delta}{\delta x^{k}}\\ &\qquad\qquad = \tilde{\nabla}_{\frac{\partial}{\partial y^{i}}}\ \frac{\delta}{\delta x^{j}}\ +\ D_{i\ j}^{\ k}\ \frac{\partial}{\partial y^{k}},\\ &(c)\quad \tilde{\nabla}_{\frac{\partial}{\partial y^{j}}}\ \frac{\partial}{\partial y^{i}} = C_{i\ j}^{\ k}\ \frac{\partial}{\partial y^{k}}\ -\ \frac{1}{2}\ \overset{v}{g}_{ij|t}\ \overset{h}{g}^{tk}\frac{\delta}{\delta x^{k}},\end{aligned} \tag{1.25}$$

*where we put:*

$$(1.26)\qquad \begin{aligned} &(a)\quad F_{i\ j}^{\ k} = \frac{1}{2}\ {}^{h}\!g^{kt}\left(\frac{\delta\ {}^{h}\!g_{ti}}{\delta x^j} + \frac{\delta\ {}^{h}\!g_{tj}}{\delta x^i} - \frac{\delta\ {}^{h}\!g_{ij}}{\delta x^t}\right),\\ &(b)\quad C_{i\ j}^{\ k} = \frac{1}{2}\ {}^{v}\!g^{kt}\left(\frac{\partial\ {}^{v}\!g_{ti}}{\partial y^j} + \frac{\partial\ {}^{v}\!g_{tj}}{\partial y^i} - \frac{\partial\ {}^{v}\!g_{ij}}{\partial y^t}\right).\end{aligned}$$

*Proof.* First, we take $X = \frac{\delta}{\delta x^j}, Y = \frac{\delta}{\delta x^i}$ and $Z = \frac{\partial}{\partial y^k}$ in (1.13) and by using (1.23), (1.9) and (1.4) we obtain

$$(1.27)\qquad 2G\left(\tilde{\nabla}_{\frac{\delta}{\delta x^j}}\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^t}\right) = -\ {}^{h}\!g_{ij\|t} - \ {}^{v}\!g_{ts}R^{s}{}_{ij}.$$

Similarly, we take $X = \frac{\delta}{\delta x^j}, Y = \frac{\delta}{\delta x^i}, Z = \frac{\delta}{\delta x^t}$ in (1.13) and by using (1.23) and (1.9a) we deduce that

$$(1.28)\qquad 2G\left(\tilde{\nabla}_{\frac{\delta}{\delta x^j}}\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^t}\right) = \frac{\delta\ {}^{h}\!g_{ti}}{\delta x^j} + \frac{\delta\ {}^{h}\!g_{tj}}{\delta x^i} - \frac{\delta\ {}^{h}\!g_{ij}}{\delta x^t}.$$

Then (1.25a) is a consequence of (1.27) and (1.28) via (1.26a). By similar calculations we obtain (1.25b) and (1.25c). □

The formulas from (1.25) can give some information about the vertical foliation as we see from the next theorem.

**Theorem 1.3.** *Let $(M, G)$ be a Riemannian tangent bundle of $M$. Then the induced Riemannian metric ${}^{v}\!g$ on $VTM$ is parallel with respect to the vertical Vrănceanu connection if and only if the vertical foliation is totally geodesic.*

*Proof.* By (1.19) and (1.20) we deduce that ${}^{v}\!g$ is parallel with respect to ${}^{v}\nabla^{*}$ if and only if

$$(1.29)\qquad {}^{v}\!g_{jk|i} = 0,\ \forall\, i, j, k \in \{1, \ldots, n\}.$$

Then from (1.25c) we see that (1.29) is equivalent to

$$(1.30)\qquad \tilde{\nabla}_{\frac{\partial}{\partial y^j}}\frac{\partial}{\partial y^i} = C_{i\ j}^{\ k}\frac{\partial}{\partial y^k}.$$

Finally, we note that the leaves of $\mathcal{F}_V$ are totally geodesic immersed in $(TM, G)$ if and only if (1.30) is satisfied. This completes the proof of the theorem. □

An interesting result is obtained by combining Theorems 1.1 and 1.3.

**Corollary 1.1.** *Let $(TM, G)$ be a Riemannian tangent bundle of $M$. Then $G$ is parallel with respect to Vrănceanu connection if and only if the vertical foliation $\mathcal{F}_v$ is totally geodesic and $G$ is bundle-like for $\mathcal{F}_v$.*

Next, we prove the following.

**Theorem 1.4.** *Let $(TM, G)$ be a Riemannian tangent bundle of $M$. Then $HTM$ is integrable and its leaves are totally geodesic immersed in $(TM, G)$ if and only if the horizontal Schouten–Van Kampen connection coincides with the horizontal Vrănceanu connection.*

*Proof.* Suppose that ${}^{h}\nabla = {}^{h}\nabla^{*}$, that is, for any $X, Y \in \Gamma(TTM)$ we have

$$\nabla_X HY = \nabla^{*}_X HY.$$

Then by using (1.16) and (1.17) we deduce that ${}^{h}\nabla = {}^{h}\nabla^{*}$ if and only if

$$H\tilde{\nabla}_{HY}VX = 0,$$

which is equivalent to

$$(1.31)\qquad G(\tilde{\nabla}_{HY}VX, HZ) = 0,\ \forall X, Y, Z \in \Gamma(TTM).$$

Finally, taking into account (1.15) we infer that (1.31) is equivalent to

$$\tilde{\nabla}_{HY}HZ \in \Gamma(HTM),\ \forall\, Y, Z \in \Gamma(TTM),$$

which in fact is the condition for $HTM$ to be autoparallel with respect to $\tilde{\nabla}$, that is, $HTM$ is integrable and its leaves are totally geodesic immersed in $(TM, G)$. $\square$

In a similar way it is proved the following theorem.

**Theorem 1.5.** *Let $(TM, G)$ be a Riemannian tangent bundle of $M$. Then the vertical foliation $\mathcal{F}_V$ is totally geodesic if and only if the vertical Schouten–Van Kampen connection coincides with the vertical Vrănceanu connection.*

Finally, by combining Theorems 1.4 and 1.5 we obtain the following.

**Corollary 1.2.** *A Riemannian tangent bundle $(TM, G)$ is a locally Riemannian product with respect to the decomposition (1.5) if and only if the Schouten–Van Kampen connection coincides with the Vrănceanu connection.*

## 2. Curvature of a Riemannian Tangent Bundle

Let $\tilde{\nabla}$ and $\nabla$ be the Levi–Civita connection and the Schouten–Van Kampen connection respectively on the Riemannian tangent bundle $(TM, G)$. Then taking into account (1.5) and (1.16) we put:

$$\tilde{\nabla}_X VY = \nabla_X VY + B(X, VY), \tag{2.1}$$

and

$$\tilde{\nabla}_X HY = B'(X, HY) + \nabla_X HY, \tag{2.2}$$

for any $X, Y \in \Gamma(TTM)$, where $B$ and $B'$ are given by

$$\text{(a) } B(X, VY) = H\tilde{\nabla}_X VY \text{ and (b) } B'(X, HY) = V\tilde{\nabla}_X HY. \tag{2.3}$$

By using (2.1) - (2.3) and (1.15) we deduce that

$$^{h}g(B(X, VY), HZ) \ +\ ^{v}g(B'(X, HZ), VY) = 0,\ \forall X, Y, Z \in \Gamma(TTM). \tag{2.4}$$

Taking into account that both distributions $VTM$ and $HTM$ are parallel with respect to the Schouten–Van Kampen connection we define the covariant derivates of $B$ and $B'$ as follows:

$$(\nabla_X B)(Y, VZ) = \nabla_X(B(Y, VZ)) - B(\nabla_X Y, VZ) - B(Y, \nabla_X VZ), \tag{2.5}$$

and

$$(\nabla_X B')(Y, HZ) = \nabla_X(B'(Y, HZ)) - B'(\nabla_X Y, HZ) - B'(Y, \nabla_X HZ), \tag{2.6}$$

for any $X, Y, Z \in \Gamma(TTM)$. Now, we denote by $\tilde{R}$ and $R$ the curvature tensor fields of $\tilde{\nabla}$ and $\nabla$ respectively, and state the following.

**Theorem 2.1.** *Let $(TM, G)$ be a Riemannian tangent bundle of $M$. Then we have the following equations:*

$$\begin{aligned} G(\tilde{R}(X, Y)VZ, VU) = &\ ^{v}g(R(X, Y)VZ, VU) \\ &+\ ^{h}g(B(X, VZ), B(Y, VU)) \\ &-\ ^{h}g(B(Y, VZ), B(X, VU)), \end{aligned} \tag{2.7}$$

$$\begin{aligned} G(\tilde{R}(X, Y)VZ, HU) = &\ ^{h}g((\nabla_X B)(Y, VZ) - (\nabla_Y B)(X, VZ), HU) \\ &+\ ^{h}g(B(T(X, Y), VZ), HU), \end{aligned} \tag{2.8}$$

$$G(\tilde{R}(X,Y)HZ,HU) = {}^{h}g(R(X,Y)HZ,HU) + {}^{v}g(B'(X,HZ),B'(Y,HU)) - {}^{v}g(B'(Y,HZ),B'(X,HU)), \tag{2.9}$$

$$G(\tilde{R}(X,Y)HZ,VU) = {}^{v}g((\nabla_X B')(Y,HZ) - (\nabla_Y B')(X,HZ),VU) + {}^{v}g(B'(T(X,Y),HZ),VU), \tag{2.10}$$

*for any* $X,Y,Z \in \Gamma(TTM)$, *where* $T$ *is the torsion tensor field of* $\nabla$.

*Proof.* By using (2.1) and (2.2) we obtain

$$\tilde{\nabla}_X\tilde{\nabla}_Y VZ = \nabla_X\nabla_Y VZ + B(X,\nabla_Y VZ) + B(X,B(Y,VZ)) + \nabla_X(B(Y,VZ)). \tag{2.11}$$

On the other hand, taking into account that

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y],$$

and by using (2.1) we infer that

$$\tilde{\nabla}_{[X,Y]}VZ = \nabla_{[X,Y]}VZ + B(\nabla_X Y,VZ) - B(\nabla_Y X,VZ) - B(T(X,Y),VZ), \tag{2.12}$$

Then by using (2.11), (2.12) and (2.5) we deduce that

$$\begin{aligned}\tilde{R}(X,Y)VZ &= [\tilde{\nabla}_X,\tilde{\nabla}_Y]VZ - \tilde{\nabla}_{[X,Y]}VZ\\ &= \{R(X,Y)VZ + B'(X,B(Y,VZ)) - B'(Y,B(X,VZ))\}\\ &\quad + \{(\nabla_X B)(Y,VZ) - (\nabla_Y B)(X,VZ) + B(T(X,Y),VZ)\}.\end{aligned} \tag{2.13}$$

Now, we take the $HTM$ - and $VTM$ - components in (2.13) and obtain (2.8) and

$$G(\tilde{R}(X,Y)VZ,VU) = {}^{v}g(R(X,Y)VZ,VU) + {}^{v}g(B'(X,B(Y,VZ)) - B'(Y,B(X,VZ)),VU). \tag{2.14}$$

Finally, by using (2.4) in (2.14) we obtain (2.7). By similar calculations we obtain (2.9) and (2.10). □

*Remark* 1. The formulas (2.8) and (2.10) are equivalent. This follows by direct calculations using (2.4) and properties of $\tilde{R}$.

Next, let $z \in TM$ and $W$ be a 2-dimensional subspace of $HTM_z$ which we call a horizontal plane. Take a basis $\{u,v\}$ of $W$ and define the number

$$K(u,v) = \frac{{}^{h}g(R(u,v)v,u)}{\Delta(u,v)}, \tag{2.15}$$

where we put

$$\Delta(u,v) = {}^{h}g(u,u)\,{}^{h}g(v,v) - ({}^{h}g(u,v))^2.$$

Taking into account that ${}^{h}g$ is parallel with respect to Schouten-Van Kampen connection, we deduce that $K(u,v)$ is independent of the basis $\{u,v\}$. Then we denote it by $K(W)$ and call it the *Schouten–Van Kampen sectional curvature* of HTM at $z \in TM$ with respect to the plane $W$. To define such an object for the Vrănceanu connection we need a study of its curvature tensor field $R^*$. This is because ${}^{h}g$, in general, is not parallel with respect to $\nabla^*$ (see Theorem 1.3). Now, we prove the following.

**Theorem 2.2.** *Let* $(TM,G)$ *be a Riemannian tangent bundle. Then* $G$ *is bundle-like for the vertical foliation* $\mathcal{F}_V$ *if and only if*

$$B'(HX,HY) + B'(HY,HX) = 0,\ \forall\ X,Y \in \Gamma(TTM). \tag{2.16}$$

*Proof.* By using (1.15) we deduce that (1.19) is equivalent to

$$G(VX, \tilde{\nabla}_{HY}HZ + \tilde{\nabla}_{HZ}HY) = 0. \tag{2.17}$$

Then by (2.2) it follows that (2.17) is equivalent to (2.16). □

**Lemma 2.1.** *Let $(TM, G)$ be a Riemannian tangent bundle of $M$, where $G$ is bundle-like for $\mathcal{F}_V$. Then the curvature tensor fields $R$ and $R^*$ are related by*

$$R(HX, HY)HZ = R^*(HX, HY)HZ - 2B(HZ, B'(HX, HY)). \tag{2.18}$$

*for any $X, Y, Z \in \Gamma(TTM)$.*

*Proof.* By using (1.16), (1.17) and (2.3a) we deduce that

$$\nabla_X HZ = \nabla^*_X HZ + B(HZ, VX), \forall\ X, Z \in \Gamma(TTM). \tag{2.19}$$

Then by direct calculations using (2.19) we obtain

$$R(HX, HY)HZ = R^*(HX, HY)HZ - B(HZ, V[HX, HY]). \tag{2.20}$$

Next, by using (1.14), (2.3b) and (2.16) we infer that

$$\begin{aligned} V[HX, HY] &= V\tilde{\nabla}_{HX}HY - V\tilde{\nabla}_{HY}HX \\ &= 2B'(HX, HY). \end{aligned} \tag{2.21}$$

Thus (2.18) follows from (2.20) by using (2.21). □

**Lemma 2.2.** *Let $(TM, G)$ as in Lemma 2.1. Then the curvature tensor field of the horizontal Vrănceanu connection satisfies the identity*

$${}^h g(R^*((HX, HY)HZ, HU) + {}^h g(R^*(HX, HY)HU, HZ) = 0, \tag{2.22}$$

*for any $X, Y, Z, U \in \Gamma(TTM)$.*

*Proof.* By using (2.18) and (2.4) we obtain

$$\begin{aligned} {}^h g(R(HX, HY)HZ, HU) = &\ {}^h g(R^*(HX, HY)HZ, HU) \\ &+ 2\ {}^v g(B'(HZ, HU), B'(HX, HY)). \end{aligned} \tag{2.23}$$

Then (2.22) follows from (2.23) by using (2.16) and taking into account that $R$ satisfies an identity as (2.22) (since ${}^h g$ is parallel with respect to $\nabla$). □

By using properties of $R^*$ (including (2.22)) we define the *Vrănceanu sectional curvature* $K^*(W)$ of the $HTM$ at $z \in TM$ with respect to the horizontal plane $W$ by (2.15), but with $R^*$ instead of $R$. Similarly, we have $\tilde{K}(W)$ given by (2.15), but with $G$ and $\tilde{R}$ instead if ${}^h g$ and $R$ respectively. In the next theorem we state an interesting relation between the above three sectional curvatures.

**Theorem 2.3.** *Let $(TM, G)$ be a Riemannian tangent bundle of $M$, where $G$ is bundle-like for $\mathcal{F}_V$. Then the Schouten-Van Kampen, Vrănceanu and Levi-Civita curvatures of the horizontal distribution are related by*

$$3K(W) = 2\tilde{K}(W) + K^*(W) \tag{2.24}$$

*for any horizontal plane $W$.*

*Proof.* Let $\{HX, HY\}$ be a basis of $W$. Then by using (2.9) and (2.16) we obtain

$$\begin{aligned} G(\tilde{R}(HX, HY)HY, HX) = &\ {}^h g(R(HX, HY)HY, HX) \\ &- {}^v g(B'(HX, HY), B'(HX, HY)), \end{aligned}$$

which implies

$$\tilde{K}(W) = K(W) - \frac{||B'(HX, HY)||^2}{\Delta(HX, HY)}. \tag{2.25}$$

On the other hand, by using (2.18), (2.4) and (2.16) we deduce that

$$\begin{aligned} {}^h g(R(HX,HY)HY,HX) = {}^h g(R^*(HX,HY)HY,HX) \\ - 2\, {}^v g(B'(HX,HY),B'(HX,HY)), \end{aligned}$$

which yields

$$K(W) = K^*(W) - 2\,\frac{||B'(HX,HY)||^2}{\Delta(HX,HY)}. \tag{2.26}$$

Thus (2.24) follows from (2.25) and (2.26). □

**Corollary 2.1.** *Let $(TM,G)$ as in Theorem 2.3. Then we have*

$$\tilde{K}(W) \leq K(W) \leq K^*(W). \tag{2.27}$$

*Moreover, one inequality becomes equality if and only if the other inequality is so, and this occurs if and only if the horizontal distribution is integrable and its leaves are totally geodesic immersed in $(TM,G)$.*

*Proof.* The inequalities in (2.27) follow from (2.25) and (2.26) since

$$\Delta(HX,HY) > 0.$$

If $\tilde{K}(W) = K(W)$, then from (2.25) we deduce that $B'(HX,HY) = 0$. Hence (2.26) yields $K(W) = K^*(W)$. Also, from (2.2) we deduce that $\tilde{\nabla}_{HX}HY \in \Gamma(HTM)$. Hence $HTM$ is integrable and its leaves are totally geodesic immersed in $(TM,G)$. The same reason is used if we start with the equality $K(W) = K^*(W)$. □

## References


[1] A. Bejancu and H. R. Farran. *Geometry of pseudo-Finsler submanifolds*. Kluwer Academic Publishers, Dordrecht, 2000.

[2] S. Ianuş. Some almost product structures on manifolds with linear connection. *Kōdai Math. Sem. Rep.*, 23:305–310, 1971.

[3] S. Kobayashi and K. Nomizu. *Foundations of differential geometry. Vol I.* Interscience Publishers, New York-London, 1963.

[4] B. L. Reinhart. *Differential geometry of foliations*, volume 99 of *Ergebnisse der Mathematik und ihrer Grenzgebiete*. Springer-Verlag, Berlin, 1983.

[5] S. Sasaki. On the differential geometry of tangent bundles of Riemannian manifolds. *Tôhoku Math. J. (2)*, 10:338–354, 1958.

[6] J. A. Schouten and E. R. van Kampen. Zur Einbettungs- und Krümmungstheorie nichtholonomer Gebilde. *Math. Ann.*, 103(1):752–783, 1930.

[7] G. Vrănceanu. Sur quelques points de la théorie des espaces non holonomes. *Bul. Fac. St. Cernăuti*, 5:177–205, 1931.


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