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# WARPED PRODUCT SUBMANIFOLDS IN GENERALIZED COMPLEX SPACE FORMS

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Abstract. B.Y. Chen [5] established a sharp inequality for the warping function of a warped product submanifold in a Riemannian space form in terms of the squared mean curvature. Later, in [4], he studied warped product submanifolds in complex hyperbolic spaces.

In the present paper, we establish an inequality between the warping function $f$ (intrinsic structure) and the squared mean curvature $\|H\|^2$ and the holomorphic sectional curvature $c$ (extrinsic structures) for warped product submanifolds $M_1 \times_f M_2$ in any generalized complex space form $\widetilde{M}(c, \alpha)$.

## Introduction

The notion of *warped product* plays some important role in differential geometry as well as in physics [3]. For instance, the best relativistic model of the Schwarzschild space-time that describes the out space around a massive star or a black hole is given as a warped product.

One of the most fundamental problems in the theory of submanifolds is the immersibility (or non-immersibility) of a Riemannian manifold in a Euclidean space (or, more generally, in a space form). According to a well-known theorem on Nash, every Riemannian manifold can be isometrically immersed in some Euclidean spaces with sufficiently high codimension.

Nash's theorem implies, in particular, that every warped product $M_1 \times_f M_2$ can be immersed as a Riemannian submanifold in some Euclidean space. Moreover, many important submanifolds in real and complex space forms are expressed as a warped product submanifold.

Every Riemannian manifold of constant curvature $c$ can be locally expressed as a warped product whose warping function satisfies $\Delta f = cf$. For example, $S^n(1)$ is locally isometric to $(-\frac{\pi}{2}, \frac{\pi}{2}) \times_{\cos t} S^{n-1}(1)$, $\mathbf{E}^n$ is locally isometric to $(0, \infty) \times_x S^{n-1}(1)$ and $H^n(-1)$ is locally isometric to $\mathbf{R} \times_{e^x} \mathbf{E}^{n-1}$ (see [3]).

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## 1. Preliminaries

Let $\widetilde{M}$ be an almost Hermitian manifold with almost complex structure $J$ and Riemannian metric $g$. One denotes by $\widetilde{\nabla}$ the operator of covariant differentiation with respect to $g$ on $\widetilde{M}$.

**Definition 1.1.** If the almost complex structure $J$ satisfies

$$(\widetilde{\nabla}_X J)Y + (\widetilde{\nabla}_Y J)X = 0,$$

for any vector fields $X$ and $Y$ on $\widetilde{M}$, then the manifold $\widetilde{M}$ is called a *nearly-Kaehler manifold* [10].

*Remark* 1.2. The above condition is equivalent to

$$(\widetilde{\nabla}_X J)X = 0, \quad \forall X \in \Gamma T\widetilde{M}.$$

For an almost complex structure $J$ on the manifold $M$, the *Nijenhuis tensor field* is defined by

$$N_J(X,Y) = [JX, JY] - J[JX, Y] - J[X, JY] - [X, Y],$$

for any vector fields $X, Y$ tangent to $M$, where $[,]$ is the Lie bracket.

A necessary and sufficient condition for a nearly-Kaehler manifold to be Kaehler is the vanishing of the Nijenhuis tensor $N_J$. Any 4-dimensional nearly-Kaehler manifold is a Kaehler manifold.

*Example* 1.3. Let $S^6$ be the 6-dimensional unit sphere defined as follows. Let $\mathbf{E}^7$ be the set of all purely imaginary Cayley numbers. Then $\mathbf{E}^7$ is a 7-dimensional subspace of the Cayley algebra $C$. Let $\{1 = e_0, e_1, \dots, e_6\}$ be a basis of the Cayley algebra, 1 being the unit element of $C$. If $X = \sum_{i=0}^{6} x^i e_i$ and $Y = \sum_{i=0}^{6} y^i e_i$ are two elements of $\mathbf{E}^7$, one defines the *scalar product* in $\mathbf{E}^7$ by

$$< X, Y >= \sum_{i=0}^{6} x^i y^i,$$

and the *vector product* by

$$X \times Y = \sum_{i \neq j} x^i y^j e_i * e_j,$$

$*$ being the multiplication operation of $C$.

Consider the 6-dimensional unit sphere $S^6$ in $\mathbf{E}^7$:

$$S^6 = \{X \in \mathbf{E}^7 \mid < X, X >= 1\}.$$

The scalar product in $\mathbf{E}^7$ induces the natural metric tensor field $g$ on $S^6$. The tangent space $T_X S^6$ at $X \in S^6$ can naturally be identified with the subspace of $\mathbf{E}^7$ orthogonal to $X$. Define the endomorphism $J_X$ on $T_X S^6$ by

$$J_X Y = X \times Y, \text{ for } Y \in T_X S^6.$$

It is easy to see that

$$g(J_X Y, J_X Z) = g(Y, Z), \; Y, Z \in T_X S^6.$$

The correspondence $X \mapsto J_X$ defines a tensor field $J$ such that $J^2 = -I$. Consequently, $S^6$ admits an almost Hermitian structure $(J, g)$. This structure is a non-Kaehlerian nearly-Kaehlerian structure (its Betti numbers of even order are 0).

We will consider a class of almost Hermitian manifolds, called *RK-manifolds*, which contains nearly-Kaehler manifolds.

**Definition 1.4** ([9])**.** An *RK-manifold* $(\widetilde{M}, J, g)$ is an almost Hermitian manifold for which the curvature tensor $\widetilde{R}$ is invariant by $J$, i.e.

$$\widetilde{R}(JX, JY, JZ, JW) = \widetilde{R}(X, Y, Z, W),$$

for any $X, Y, Z, W \in \Gamma T\widetilde{M}$.

An almost Hermitian manifold $\widetilde{M}$ is of *pointwise constant type* if for any $p \in \widetilde{M}$ and $X \in T_p\widetilde{M}$ we have $\lambda(X, Y) = \lambda(X, Z)$, where

$$\lambda(X, Y) = \widetilde{R}(X, Y, JX, JY) - \widetilde{R}(X, Y, X, Y)$$

and $Y$ and $Z$ are unit tangent vectors on $\widetilde{M}$ at $p$, orthogonal to $X$ and $JX$, i.e. $g(X, X) = g(Y, Y) = 1$, $g(X, Y) = g(JX, Y) = g(X, Z) = g(JX, Z) = 0$.

The manifold $\widetilde{M}$ is said to be of *constant type* if for any unit $X, Y \in \Gamma T\widetilde{M}$ with $g(X, Y) = g(JX, Y) = 0$, $\lambda(X, Y)$ is a constant function.

Recall the following result [9].

**Theorem 1.5.** *Let $\widetilde{M}$ be an RK-manifold. Then $\widetilde{M}$ is of pointwise constant type if and only if there exists a function $\alpha$ on $\widetilde{M}$ such that*

$$\lambda(X, Y) = \alpha[g(X, X)g(Y, Y) - (g(X, Y))^2 - (g(X, JY))^2],$$

*for any $X, Y \in \Gamma T\widetilde{M}$.*

*Moreover, $\widetilde{M}$ is of constant type if and only if the above equality holds good for a constant $\alpha$.*

In this case, $\alpha$ is the *constant type* of $\widetilde{M}$.

**Definition 1.6.** A *generalized complex space form* is an RK-manifold of constant holomorphic sectional curvature and of constant type.

We will denote a generalized complex space form by $\widetilde{M}(c, \alpha)$, where $c$ is the constant holomorphic sectional curvature and $\alpha$ the constant type, respectively.

Each complex space form is a generalized complex space form. The converse statement is not true. The sphere $S^6$ endowed with the standard nearly-Kaehler structure is an example of generalized complex space form which is not a complex space form.

Let $\widetilde{M}(c, \alpha)$ be a generalized complex space form of constant holomorphic sectional curvature $c$ and of constant type $\alpha$. Then the curvature tensor $\widetilde{R}$ of $\widetilde{M}(c, \alpha)$ has the following expression [9]:

$$\begin{aligned} \widetilde{R}(X, Y)Z = {} & \frac{c + 3\alpha}{4}[g(Y, Z)X - g(X, Z)Y] \\ & + \frac{c - \alpha}{4}[g(X, JZ)JY - g(Y, JZ)JX + 2g(X, JY)JZ]. \end{aligned} \tag{1.1}$$

Let $M$ be an $n$-dimensional submanifold of a $2m$-dimensional generalized complex space form $\widetilde{M}(c, \alpha)$ of constant holomorphic sectional curvature $c$ and constant type $\alpha$. One denotes by $K(\pi)$ the sectional curvature of $M$ associated with a plane section $\pi \subset T_pM, p \in M$, and $\nabla$ the Riemannian connection of $M$, respectively.

Also, let $h$ be the second fundamental form and $R$ the Riemann curvature tensor of $M$. Then the equation of Gauss is given by

$$\tilde{R}(X,Y,Z,W) = R(X,Y,Z,W) + g(h(X,W),h(Y,Z)) - g(h(X,Z),h(Y,W)), \tag{1.2}$$

for any vectors $X, Y, Z, W$ tangent to $M$.

Let $p \in M$ and $\{e_1, \ldots, e_n, \ldots, e_{2m}\}$ an orthonormal basis of the tangent space $T_p\widetilde{M}(c,\alpha)$, such that $e_1, \ldots, e_n$ are tangent to $M$ at $p$. We denote by $H$ the mean curvature vector, that is

$$H(p) = \frac{1}{n}\sum_{i=1}^{n} h(e_i, e_i). \tag{1.3}$$

Also, we set

$$h_{ij}^r = g(h(e_i, e_j), e_r), \quad i, j \in \{1, \ldots, n\}, r \in \{n+1, \ldots, 2m\}. \tag{1.4}$$

and

$$\|h\|^2 = \sum_{i,j=1}^{n} g(h(e_i, e_j), h(e_i, e_j)). \tag{1.5}$$

For any tangent vector field $X$ to $M$, we put $JX = PX + FX$, where $PX$ and $FX$ are the tangential and normal components of $JX$, respectively. We denote by

$$\|P\|^2 = \sum_{i,j=1}^{n} g^2(Pe_i, e_j). \tag{1.6}$$

Let $M$ be a Riemannian $n$-manifold and $\{e_1, \ldots, e_n\}$ be an orthonormal frame field on $M$. For a differentiable function $f$ on $M$, the Laplacian $\Delta f$ of $f$ is defined by

$$\Delta f = \sum_{j=1}^{n} \{(\nabla_{e_j} e_j) f - e_j e_j f\}. \tag{1.7}$$

We recall the following result of Chen for later use.

**Lemma 1.7** ([1]). *Let $n \geq 2$ and $a_1, \ldots, a_n, b$ real numbers such that*

$$\left(\sum_{i=1}^{n} a_i\right)^2 = (n-1)\left(\sum_{i=1}^{n} a_i^2 + b\right)$$

*Then $2a_1a_2 \geq b$, with equality holding if and only if*

$$a_1 + a_2 = a_3 = \ldots = a_n.$$

## 2. Warped product submanifolds

Chen established a sharp relationship between the warping function $f$ of a warped product $M_1 \times_f M_2$ isometrically immersed in a real space form $\widetilde{M}(c)$ and the squared mean curvature $\|H\|^2$ (see [5]). In [7], we established a relationship between the warping function $f$ of a warped product $M_1 \times_f M_2$ isometrically immersed in a complex space form $\widetilde{M}(c)$ and the squared mean curvature $\|H\|^2$.

Let $(M_1, g_1)$ and $(M_2, g_2)$ be two Riemannian manifolds and $f$ a positive differentiable function on $M_1$. The *warped product* of $M_1$ and $M_2$ is the Riemannian manifold

$$M_1 \times_f M_2 = (M_1 \times M_2, g),$$

where $g = g_1 + f^2 g_2$ (see, for instance, [5]).

Let $x\colon M_1 \times_f M_2 \to \widetilde{M}(c, \alpha)$ be an isometric immersion of a warped product $M_1 \times_f M_2$ into a generalized complex space form $\widetilde{M}(c, \alpha)$. We denote by $h$ the second fundamental form of $x$ and $H_i = \frac{1}{n_i}\operatorname{trace} h_i$, where trace $h_i$ is the trace of $h$ restricted to $M_i$ and $n_i = \dim M_i$ $(i = 1, 2)$.

For a warped product $M_1 \times_f M_2$, we denote by $\mathcal{D}_1$ and $\mathcal{D}_2$ the distributions given by the vectors tangent to leaves and fibres, respectively. Thus, $\mathcal{D}_1$ is obtained from the tangent vectors of $M_1$ via the horizontal lift and $\mathcal{D}_2$ by tangent vectors of $M_2$ via the vertical lift.

Let $M_1 \times_f M_2$ be a warped product submanifold of a generalized complex space form $\widetilde{M}(c, \alpha)$ of constant holomorphic sectional curvature $c$ and constant type $\alpha$.

Since $M_1 \times_f M_2$ is a warped product, it is known that

$$\nabla_X Z = \nabla_Z X = \frac{1}{f}(Xf)Z, \tag{2.1}$$

for any vector fields $X, Z$ tangent to $M_1, M_2$, respectively.

If $X$ and $Z$ are unit vector fields, it follows that the sectional curvature $K(X \wedge Z)$ of the plane section spanned by $X$ and $Z$ is given by

$$K(X \wedge Z) = g(\nabla_Z \nabla_X X - \nabla_X \nabla_Z X, Z) = \frac{1}{f}\{(\nabla_X X)f - X^2 f\}. \tag{2.2}$$

We choose a local orthonormal frame

$$\{e_1, \ldots, e_n, e_{n+1}, \ldots, e_{2m}\},$$

such that $e_1, \ldots, e_{n_1}$ are tangent to $M_1$, $e_{n_1+1}, \ldots, e_n$ are tangent to $M_2$, $e_{n+1}$ is parallel to the mean curvature vector $H$.

Then, using (2.2), we get

$$\frac{\Delta f}{f} = \sum_{j=1}^{n_1} K(e_j \wedge e_s), \tag{2.3}$$

for each $s \in \{n_1 + 1, \ldots, n\}$.

From the equation of Gauss, we have

$$n^2\|H\|^2 = 2\tau + \|h\|^2 - n(n-1)\frac{c+3\alpha}{4} - 3\|P\|^2\frac{c-\alpha}{4}. \tag{2.4}$$

We set

$$\delta = 2\tau - n(n-1)\frac{c+3\alpha}{4} - 3\|P\|^2\frac{c-\alpha}{4} - \frac{n^2}{2}\|H\|^2. \tag{2.5}$$

Then, (2.4) can be written as

$$n^2\|H\|^2 = 2(\delta + \|h\|^2). \tag{2.6}$$

With respect to the above orthonormal frame, (2.6) takes the following form:

$$\left(\sum_{i=1}^{n} h_{ii}^{n+1}\right)^2 = 2\left\{\delta + \sum_{i=1}^{n}(h_{ii}^{n+1})^2 + \sum_{i\neq j}(h_{ij}^{n+1})^2 + \sum_{r=n+2}^{2m}\sum_{i,j=1}^{n}(h_{ij}^r)^2\right\}.$$

If we put $a_1 = h_{11}^{n+1}, a_2 = \sum_{i=2}^{n_1} h_{ii}^{n+1}$ and $a_3 = \sum_{t=n_1+1}^{n} h_{tt}^{n+1}$, the above equation becomes

$$\left(\sum_{i=1}^{3} a_i\right)^2 = 2\left\{\delta + \sum_{i=1}^{3} a_i^2 + \sum_{1\le i\ne j\le n} (h_{ij}^{n+1})^2 + \sum_{r=n+2}^{2m}\sum_{i,j=1}^{n} (h_{ij}^r)^2 \right.$$
$$\left. - - \sum_{2\le j\ne k\le n_1} h_{jj}^{n+1}h_{kk}^{n+1} - \sum_{n_1+1\le s\ne t\le n} h_{ss}^{n+1}h_{tt}^{n+1}\right\}.$$

Thus $a_1, a_2, a_3$ satisfy the Lemma of Chen (for $n = 3$), i.e.

$$\left(\sum_{i=1}^{3} a_i\right)^2 = 2\left(b + \sum_{i=1}^{3} a_i^2\right).$$

Then $2a_1a_2 \ge b$, with equality holding if and only if $a_1 + a_2 = a_3$. In the case under consideration, this means

$$\sum_{1\le j<k\le n_1} h_{jj}^{n+1}h_{kk}^{n+1} + \sum_{n_1+1\le s<t\le n} h_{ss}^{n+1}h_{tt}^{n+1} \ge \frac{\delta}{2} + \sum_{1\le\alpha<\beta\le n} (h_{\alpha\beta}^{n+1})^2 + \frac{1}{2}\sum_{r=n+2}^{2m}\sum_{\alpha,\beta=1}^{n} (h_{\alpha\beta}^r)^2. \tag{2.7}$$

Equality holds if and only if

$$\sum_{i=1}^{n_1} h_{ii}^{n+1} = \sum_{t=n_1+1}^{n} h_{tt}^{n+1}. \tag{2.8}$$

Using again the Gauss equation, we have

$$\begin{aligned} n_2\frac{\Delta f}{f} &= \tau - \sum_{1\le j<k\le n_1} K(e_j\wedge e_k) - \sum_{n_1+1\le s<t\le n} K(e_s\wedge e_t) = \\ &= \tau - \frac{n_1(n_1-1)(c+3\alpha)}{8} - \sum_{r=n+1}^{2m}\sum_{1\le j<k\le n_1} (h_{jj}^r h_{kk}^r - (h_{jk}^r)^2) \\ &\quad - 3\frac{c-\alpha}{4}\sum_{1\le j<k\le n_1} g^2(Je_j, e_k) - \frac{n_2(n_2-1)(c+3\alpha)}{8} \\ &\quad - \sum_{r=n+1}^{2m}\sum_{n_1+1\le s<t\le n} (h_{ss}^r h_{tt}^r - (h_{st}^r)^2) - 3\frac{c-\alpha}{4}\sum_{n_1+1\le s<t\le n} g^2(Je_s, e_t). \end{aligned} \tag{2.9}$$

Combining (2.7) and (2.9) and taking account of (2.3), we obtain

$$\begin{aligned} n_2\frac{\Delta f}{f} &\le \tau - \frac{n(n-1)(c+3\alpha)}{8} + n_1 n_2\frac{c+3\alpha}{4} - \frac{\delta}{2} \\ &\quad - 3\frac{c-\alpha}{4}\sum_{1\le j<k\le n_1} g^2(Je_j, e_k) - 3\frac{c-\alpha}{4}\sum_{n_1+1\le s<t\le n} g^2(Je_s, e_t) \\ &\quad - \sum_{1\le j\le n_1; n_1+1\le t\le n} (h_{jt}^{n+1})^2 - \frac{1}{2}\sum_{r=n+2}^{2m}\sum_{\alpha,\beta=1}^{n} (h_{\alpha\beta}^r)^2 \end{aligned} \tag{2.10}$$

$$+\sum_{r=n+2}^{2m}\sum_{1\le j<k\le n_1}((h^r_{jk})^2-h^r_{jj}h^r_{kk})+\sum_{r=n+2}^{2m}\sum_{n_1+1\le s<t\le n}((h^r_{st})^2-h^r_{ss}h^r_{tt})$$
$$=\tau-\frac{n(n-1)(c+3\alpha)}{8}+n_1n_2\frac{c+3\alpha}{4}-\frac{\delta}{2}-\sum_{r=n+1}^{2m}\sum_{1\le j\le n_1;n_1+1\le t\le n}(h^r_{jt})^2$$
$$-3\frac{c-\alpha}{4}\sum_{1\le j<k\le n_1}g^2(Je_j,e_k)-3\frac{c-\alpha}{4}\sum_{n_1+1\le s<t\le n}g^2(Je_s,e_t)$$
$$-\frac{1}{2}\sum_{r=n+2}^{2m}\left(\sum_{j=1}^{n_1}h^r_{jj}\right)^2-\frac{1}{2}\sum_{r=n+2}^{2m}\left(\sum_{t=n_1+1}^{n}h^r_{tt}\right)^2$$
$$\le\tau-\frac{n(n-1)(c+3\alpha)}{8}+n_1n_2\frac{c+3\alpha}{4}-\frac{\delta}{2}-3\frac{c-\alpha}{4}\sum_{1\le j<k\le n_1}g^2(Je_j,e_k)$$
$$-3\frac{c-\alpha}{4}\sum_{n_1+1\le s<t\le n}g^2(Je_s,e_t).$$

The equality sign of (2.10) holds if and only if

$$h^r_{jt}=0,\quad 1\le j\le n_1,n_1+1\le t\le n,n+1\le r\le 2m, \tag{2.10.1}$$

and

$$\sum_{i=1}^{n_1}h^r_{ii}=\sum_{t=n_1+1}^{n}h^r_{tt}=0,\quad n+2\le r\le 2m. \tag{2.10.2}$$

Obviously (2.10.1) is equivalent to the mixed totally geodesicness of the warped product $M_1\times_f M_2$ (i.e. $h(X,Z)=0$, for any $X$ in $\mathcal{D}_1$ and $Z$ in $\mathcal{D}_2$) and (2.8) and (2.10.2) imply $n_1H_1=n_2H_2$.

Using (2.5), we finally obtain

**Lemma 2.1.** *Let $x\colon M_1\times_f M_2\to\widetilde{M}(c,\alpha)$ be an isometric immersion of an $n$-dimensional warped product into a $2m$-dimensional generalized complex space form $\widetilde{M}(c,\alpha)$. Then:*

$$\frac{\Delta f}{f}\le\frac{n^2}{4n_2}\|H\|^2+n_1\frac{c+3\alpha}{4}+3\frac{c-\alpha}{4n_2}\sum_{1\le i\le n_1}\sum_{n_1+1\le s\le n}g^2(Je_i,e_s). \tag{2.11}$$

*where $n_i=\dim M_i, i=1,2,$ and $\Delta$ is the Laplacian operator of $M_1$.*

From the above Lemma, it follows

**Theorem 2.2.** *Let $x\colon M_1\times_f M_2\to\widetilde{M}(c,\alpha)$ be an isometric immersion of an $n$-dimensional warped product into a $2m$-dimensional generalized complex space form $\widetilde{M}(c,\alpha)$. Then:*

*i) If $c<\alpha$, then*

$$\frac{\Delta f}{f}\le\frac{n^2}{4n_2}\|H\|^2+n_1\frac{c+3\alpha}{4}. \tag{2.12}$$

*Moreover, the equality case of (2.12) holds identically if and only if $x$ is a mixed totally geodesic immersion, $n_1H_1=n_2H_2$, where $H_i, i=1,2,$ are the partial mean curvature vectors and $J\mathcal{D}_1\perp\mathcal{D}_2$.*

*ii) If $c = \alpha$, then*

$$\frac{\Delta f}{f} \le \frac{n^2}{4n_2}\|H\|^2 + n_1 \frac{c+3\alpha}{4}. \tag{2.13}$$

*Moreover, the equality case of (2.13) holds identically if and only if $x$ is a mixed totally geodesic immersion and $n_1 H_1 = n_2 H_2$, where $H_i$ $(i = 1, 2)$, are the partial mean curvature vectors.*

*iii) If $c > \alpha$, then*

$$\frac{\Delta f}{f} \le \frac{n^2}{4n_2}\|H\|^2 + n_1 \frac{c+3\alpha}{4} + 3\frac{c-\alpha}{8}\|P\|^2. \tag{2.14}$$

*Moreover, the equality case of (2.14) holds identically if and only if $x$ is a mixed totally geodesic immersion, $n_1 H_1 = n_2 H_2$, where $H_i, i = 1, 2$, are the partial mean curvature vectors and both $M_1$ and $M_2$ are totally real submanifolds.*

A submanifold $N$ in a Kaehler manifold $\widetilde{M}$ is called a *CR-submanifold* if there exists on $N$ a holomorphic distribution $\mathcal{D}$ whose orthogonal complementary distribution $\mathcal{D}^{\perp}$ is a totally real distribution, i.e., $J\mathcal{D}_x^{\perp} \subset T_p^{\perp}N$.

A CR-submanifold of a Kaehler manifold $\widetilde{M}$ is called a *CR-product* if it is a Riemannian product of a Kaehler submanifold and a totally real submanifold.

There do not exist warped product CR-submanifolds of the form $M_{\perp} \times_f M_{\top}$, with $M_{\perp}$ a totally real submanifold and $M_{\top}$ a complex submanifold, other then CR-products. A *CR-warped product* is a warped product CR-submanifold of the form $M_{\top} \times_f M_{\perp}$, by reversing the two factors [2].

Obviously, any CR-warped product submanifold, in particular any CR-product, satisfies $J\mathcal{D}_1 \perp \mathcal{D}_2$.

**Corollary 2.3.** *Let $M$ be an $n$-dimensional CR-warped product submanifold of a $2m$-dimensional generalized complex space form $\widetilde{M}(c, \alpha)$. Then:*

$$\frac{\Delta f}{f} \le \frac{n^2}{4n_2}\|H\|^2 + n_1 \frac{c+3\alpha}{4}. \tag{2.15}$$

*Moreover, the equality case of (2.15) holds identically if and only if $x$ is a mixed totally geodesic immersion, $n_1 H_1 = n_2 H_2$, where $H_i, i = 1, 2$, are the partial mean curvature vectors.*

We derive the following non-existence results.

**Corollary 2.4.** *Let $\widetilde{M}(c, \alpha)$ be a generalized complex space form, $M_1$ an $n_1$-dimensional Riemannian manifold and $f$ a differentiable function on $M_1$. If there is a point $p \in M_1$ such that $(\Delta f)(p) > n_1 \frac{c+3\alpha}{4} f(p)$, then there do not exist any minimal CR-warped product submanifold $M_1 \times_f M_2$ in $\widetilde{M}(c, \alpha)$.*

**Corollary 2.5.** *Let $\widetilde{M}(c, \alpha)$ be a generalized complex space form, with $c > \alpha$, $M_1$ an $n_1$-dimensional totally real submanifold of $\widetilde{M}(c, \alpha)$ and $f$ a differentiable function on $M_1$. If there is a point $p \in M_1$ such that $(\Delta f)(p) > n_1 \frac{c+3\alpha}{4} f(p)$, then there do not exist any totally real submanifold $M_2$ in $\widetilde{M}(c, \alpha)$ such that $M_1 \times_f M_2$ be a minimal warped product submanifold in $\widetilde{M}(c, \alpha)$.*

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