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ON ORTHONORMAL PRODUCT SYSTEMS

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Dedicated to the 60th birthday of Professor W. R. Wade

Abstract. In this paper we collected the most significant results concerning product systems. In addition to that we formulate some open problems. Our purpose is to call the attention to this field which is connected to several other fields of mathematics and which was developed from dyadic analysis.

0. Introduction

Expansions by orthogonal and biorthogonal systems play an important role in mathematics and its applications. Several methods are available for constructing such systems. In a Hilbert space, for instance, the *Gram-Schmidt method* transforms a linearly independent system into an orthonormed one [A,1]. Orthogonal polynomials, the Franklin–system and its generalizations, orthogonal systems consisting of rational functions (discrete Laguerre, Kautz, Malmquist–Takenaka systems) are examples that can be derived this way [G;1,2,4].

On the other hand, orthonormed systems formed by *eigenfunctions of differential operators* are extensively used in mathematical physics. Another well known method is to take the *character system* of some groups. Then the tools of harmonic analysis, such as Haar measure, convolution etc., can be used. The trigonometric, the Walsh and more generally the Vilenkin systems can be viewed as character systems [A;2,5,6,7,8].

At the end of the sixties in the last century *G. Alexits* introduced the various concepts of multiplicative systems. The definition relies on the concept of product systems [A;1], [C;1-3]. The product system is formed by the collection of the finite products of the members of the original systems. The relation between the *Walsh* and the *Rademacher systems* served as a model for the definition. Using this terminology, the Walsh system is the product system of the Rademacher system in the Paley enumeration. By using appropriate conditions on the product system one can define the concepts of *multiplicative, strongly multiplicative* and *weakly multiplicative* systems. Namely, if the product system is orthogonal then the oroginal

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system is called *strongly multiplicative.* If the integrals of the members of the product system, except from the one with index 0, are all equal to 0 then the system is called *weakly multiplicative.* Many of the theorems originally proved for independent random variables were transferred to multiplicative systems. Among them are the theorems on the strong law of large numbers and on iterated logarithms [C;4-6,17]. Haar like systems, that share the good convergence properties of the Haar system, can be constructed by means of product systems [F;1-3].

In the middle of the 1970's the author generelizing the concept of product system introduced a new method for constructing orthogonal systems starting from some conditionally orthogonal functions [C;7-10]. Several classical systems, including the trigonometric system in a certain rearrangement, the Walsh system or the Vilenkin system, character systems of additive and multiplicative groups of local fields and UDMD–systems, can all be constructed using this method [A;6-7], [C;24]. Walsh–similar systems (WSS) recently introduced by Sendov [E;6,7] belongs to this class if we slightly modify the original concept. Moreover, some generalizations, introduced by *G. Gát* [C;26] can be obtained in this way. Orthonormed wavelet packets (see e.g. [C;24]) can also be originated by our method.

On of the key concept in our construction is the notion of product systems of conditionally orthogonal systems. Investigating such systems we can apply concepts and methods of probability theory, especially those from the area of martingales [C;8-23]. These systems have important theoretical properties that are useful in numerical computations, too. For instance Fourier–coefficients and partial sums can be computed applying fast algorythms similar to FFT [D;1-4]. Discrete orthonormal product systems of rational functions are introduced and applied in cotrol theory [G;1-6]. Moreover, applying some restrictions, the analogue of *M. Riesz's* on L^p norm convergence and *Carleson's theorem* on a.e. convergence of trigonometric Fourier–series also hold [A;9], [C;9,10,11]. Haar-like systems can be defined, starting from certain product systems [F;1,3].

Here we are not concerned with the summation processes related to product systems although quite a number of nice results have been proved in this area. For instance *G. Gát* [C;25] has solved a long standing problem posed by M. Taibleson in his book [A;8]. Namely, he proved that the statement analogous to Lebesgue's theorem on $(C,1)$ summability of trigonometric Fourier series is true for the character system of 2-adic field. For furher results on summation we refer the reader to the book of *F. Weisz* [A;10] and to the paper [C;26].

1. Conditional expectation

In this paper we investigate conditionally orthonormal systems and introduce the notion of product systems. To this end we fix a measure space $(\Omega, \mathcal{A}, \mu)$. The conditional expectation (CE) of the function f with respect to the sub-σ-algebra $\mathcal{B} \subseteq \mathcal{A}$ is denoted by $E^{\mathcal{B}} f$. The L^p-space of $\mathcal{B}$-measurable functions will be denoted by $L^p(\mathcal{B}) := L^p(\Omega, \mathcal{B}, \mu)$. Instead of $L^p(\Omega, \mathcal{A}, \mu)$ we write L^p and $\|\cdot\|_p$ $(0 < p \leq \infty)$ stands for the L^p-norm or L^p-quasinorm.

The conditional expectation can be characterised by the following two properties:

$$(1.1) \qquad i)\ \ E^{\mathcal{B}} f \in L^1(\mathcal{B}), \qquad ii)\ \ \int_B E^{\mathcal{B}} f \, d\mu = \int_B f \, d\mu$$

for every $f \in L^1$ and every $B \in \mathcal{B}$ [A;9]. It is well-known that $L^p \ni f \to E^{\mathcal{B}} f$ is a bounded linear projection onto $L^p(\mathcal{B})$ for every $1 \le p \le \infty$ and

$$\|E^{\mathcal{B}} f\|_p \le \|f\|_p \quad (1 \le p \le \infty). \tag{1.2}$$

The operator $E^{\mathcal{B}}$ is $\mathcal{B}$-homogeneous, i.e. if λ is $\mathcal{B}$-measurable and $f, \lambda f \in L^1$ then

$$E^{\mathcal{B}}(\lambda f) = \lambda E^{\mathcal{B}} f. \tag{1.3}$$

Furthermore, if $\mathcal{C} \subseteq \mathcal{B} \subseteq \mathcal{A}$ are sub-σ-algebras of $\mathcal{A}$ then

$$E^{\mathcal{B}}(E^{\mathcal{C}} f) = E^{\mathcal{C}}(E^{B} f) = E^{\mathcal{C}} f \tag{1.4}$$

for any $f \in L^1$. We note that if $\mathcal{B} := \{\Omega, \emptyset\}$ is the trivial σ-algebra then

$$E^{\mathcal{B}}(f) = \int_\Omega f \, d\mu, \tag{1.5}$$

i.e. CE is a generalization of the integral (see e.g. [A;9]).

The conditional expectation operator has a simple form if $\mathcal{B}$ is an atomic σ-algebra, i.e. if $\mathcal{B}$ is generated by the collection of pairwise disjoint subsets $B_j \in \mathcal{A}$ $(j = 1, 2, \cdots, m)$ of Ω :

$$\mathcal{B} : = \sigma\{B_j : j = 1, 2, \cdots, m\}, \ B_i \cap B_j = \emptyset$$
$$(1 \le i < j \le m), \ \mu(B_j) < \infty \ (1 \leqq j \leqq m).$$

The sets B_j $(j = 1, \cdots, m)$ are called the *atoms* of $\mathcal{B}$ and the $\mathcal{B}$-measurable functions are exactly the step functions, constant on the B_j's and wanishing autside $\cup_{j=1}^n B_j$. We denote this m-dimensional subspace of L^1 by $L(\mathcal{B})$. Obviously in this case $L^p(\mathcal{B}) = L(\mathcal{B})$ $(0 < p \le \infty)$ and the conditional expectation is of the form

$$E(f|\mathcal{B})(x) = \frac{1}{\mu(B_j)} \int_{B_j} f \, d\mu \quad (x \in B_j).$$

The fact that CE has properties similar to those of integral makes it possible to extend several concepts connected with the integral to CE. For example a finite or infinite system of functions $\varphi_n \in L^2$ $(n \in \mathcal{N})$ is called a $\mathcal{B}$-orthonormal system ($E^{\mathcal{B}}$-ONS) with respect the positive, $\mathcal{A}$ measurable weight function ρ, if

$$E^{\mathcal{B}}(\varphi_k \overline{\varphi}_\ell \rho) = \delta_{k\ell} \quad (k, \ell \in \mathcal{N}), \tag{1.6}$$

where $\delta_{k\ell}$ is the Kronecker-symbol. If $\mathcal{B} := \{\Omega, \emptyset\}$ is the trivial σ-algebra then we get the usual definition of ONS. Moreover (1.4) implies that each $E^{\mathcal{B}}$-ONS is an ONS in the usual sense.

Replacing the integral by CE in the definitions of Fourier-coefficients and Fourier partial sums we get the following: The function $E^{\mathcal{B}}(f\overline{\varphi}_n\rho)$ is called the n-th $\mathcal{B}$-Fourier coefficient of the function f with respect to the system $(\varphi_n, n \in \mathcal{N})$. The n-th partial sum of the $\mathcal{B}$-Fourier-series of f is defined by

$$S_0^{\mathcal{B}} f := 0, \ \ S_n^{\mathcal{B}} f := \sum_{k \in \mathcal{N}, k < n} E^{\mathcal{B}}(f\overline{\varphi}_n\rho)\varphi_n \quad (n \in \mathbb{N}^* := \{1, 2, \cdots\}). \tag{1.7}$$

In the case $\mathcal{B} := \{\Omega, \emptyset\}$ these notions coincide with the usual definitions of Fourier-coefficient and Fourier partial sums. The system Φ is called $\mathcal{B}$-complete with respect to the set of functions $\mathcal{F}$, if $f \in \mathcal{F}$ and $E^{\mathcal{B}}(f\overline{\varphi}_n) = 0$ $(n \in \mathcal{N})$ imply $f = 0$. The concept of $\mathcal{B}$-*biorthogonal systems* and $\mathcal{B}$-*biorthogonal expansions* can be defined in a similar way.

2. Product systems

To define product system we fix a collection of function systems

$$\Phi_k := \{\varphi_k^\ell : \ell = 0, 1, \cdots, m_k - 1\} \subset L^2, \quad \text{with} \quad \varphi_k^0 = 1 \ (k \in \mathbb{N}), \tag{2.1}$$

where $(m_k, k \in \mathbb{N})$ is a sequence of numbers for $(k \in \mathbb{N} := \{0, 1, \cdots\})$ satisfying $m_k \geq 2, m_k \in \mathbb{N}$. We shall use the *Cantor-expansion* of natural numbers with respect to the base

$$M_0 := 1, \ M_k := m_0 m_1 \cdots m_{k-1} \quad (k \in \mathbb{N}^*). \tag{2.2}$$

It is well-known that every number $n \in \mathbb{N}$ can be written uniquely in the form

$$n = \sum_{k=0}^{\infty} n_k M_k, \tag{2.3}$$

where $n_k \in \{0, 1, \cdots m_k - 1\}$ $(k \in \mathbb{N})$. Then for every n we define the (finite) product

$$\psi_n := \prod_{k=0}^{\infty} \varphi_k^{n_k} \qquad (n \in \mathbb{N}). \tag{2.4}$$

The system $\Psi = \{\psi_n : n \in \mathbb{N}\}$ is called the *product system* of the systems Φ_k $(k \in \mathbb{N})$. This type of product systems was introduced by the author in [C;9]. In the special case $m_k = 2$ $(k \in \mathbb{N})$ we have $M_k = 2^k$ $(k \in \mathbb{N})$ and (2.3) is the dyadic reprezentation of $n \in \mathbb{N}$. In this case we write φ_k instead of φ_k^1. The product system corresponding to this special case, i.e. the system

$$\psi_n := \prod_{n_k=1} \varphi_k \qquad (n = \sum_{k=0}^{\infty} n_k 2^k \in \mathbb{N}, \ n_k = 0, 1) \tag{2.5}$$

was introduced by Alexits in [A;1] and will be called *the binary product system of* $(\varphi_n, n \in \mathbb{N})$.

To get orthonormed product systems we fix a stochastic basis, i.e. an increasing sequence of sub-σ-algebras of $\mathcal{A}$:

$$\mathcal{A}_0 := \{\Omega, \emptyset\} \subset \mathcal{A}_1 \subset \cdots \subset \mathcal{A}_n \subset \cdots \subset \mathcal{A} \tag{2.6}$$

and a sequence Φ_k $(k \in \mathbb{N})$ of *adapted conditionally orthonormal systems* (AC-ONS) with respect to the sequence $(\rho_n, n \in \mathbb{N})$ of weight functions. This means that the functions in Φ_k are $\mathcal{A}_{k+1}$-measurable and Φ_k is $\mathcal{A}_k$- or E_k-orthonormed with respect to ρ_k:

$$\begin{aligned} &i) \quad \Phi_k \subset L^2_{\rho_k}(\mathcal{A}_{k+1}) \quad (k \in \mathbb{N}), \\ &ii) \quad E_k(\varphi_k^i \overline{\varphi_k^j} \rho_k) = \delta_{ij} \quad (0 \leq i, j < m_k, k \in \mathbb{N}), \end{aligned} \tag{2.7}$$

where E_k denotes the conditional expectation with respect to $\mathcal{A}_k$ and $L^2_{\rho_k}(\mathcal{A}_{k+1})$ is the space of $\mathcal{A}_{k+1}$-measurable functions, satisfying $\int_\Omega |f|^2 \rho_k \, d\mu < \infty$.

To get orthonormal product system we need some additional condition for the sequence of weigths. We will suppose that the sequence $(\rho_k, k \in \mathbb{N})$ consists of positive, adaptive and conditionally normalized functions, i.e.

$$\rho_k > 0, \quad \rho_k \in L^1(\Omega, \mathcal{A}_{k+1}, \mu), \quad E_k\rho_k = 1 \quad (k \in \mathbb{N}). \tag{2.8}$$

In this case the sequence

$$R_n := \prod_{k=0}^{n-1} \rho_k \quad (n \in \mathbb{N}^*) \tag{2.9}$$

forms a martingale with respect to $(\mathcal{A}_n, n \in \mathbb{N})$, i.e. R_n is $\mathcal{A}_n$-measurable and

$$E_n R_s = R_n \quad (n < s, n, s \in \mathbb{N}). \tag{2.10}$$

Indeed by (1.3) and (1.4) in the case $s > n$ we have

$$E_n(R_s) = E_n\big(E_{s-1}(R_{s-1}\rho_{s-1})\big) = E_n\big(R_{s-1}E_{s-1}\rho_{s-1}\big) = E_n R_{s-1}.$$

Hence we get

$$E_n(R_s) = E_n R_{n+1} = E_n\big(R_n\rho_n\big) = R_n E_n \rho_n = R_n$$

and (2.10) is proved. The martingale $(R_n, n \in \mathbb{N})$ is L^1-bounded and consequently conveges μ a.e. Namely from (2.10) we get

$$\int_\Omega R_s \, d\mu = E_0(E_1 R_s) = E_0(R_1) = E_0(\rho_0) = 1 \quad (s \in \mathbb{N}).$$

In this paper we will suppose that the partial products $(R_n, n \in \mathbb{N})$ form a *positive, regular martingele,* i.e. there exists a function $\rho > 0, \rho \in L^1$ with $\int_\Omega \rho \, d\mu = 1$ such that

$$R_n = E_n\rho, \quad \text{or equivalently} \quad \rho_n = \frac{E_{n+1}\rho}{E_n\rho} \quad (n \in \mathbb{N}). \tag{2.11}$$

In this case we call the sequence $(\rho_k, k \in \mathbb{N})$ of weight *regular.* It is easy to see that the regular $(\rho_k, k \in \mathbb{N})$ weights satisfy (2.8), in addition

$$E_n\Big(\prod_{k=n}^{\infty} \rho_k\Big) = 1 \quad (n \in \mathbb{N}). \tag{2.12}$$

Indeed, in this case $R_s \to \rho$ as $s \to \infty$ in L^1-norm and consequently

$$R_n = \lim_{s\to\infty} E_n R_s = E_n\rho = R_n E_n\Big(\prod_{k=n}^{\infty} \rho_k\Big).$$

Product systems of AC-ONS with $\rho_n = 1$ $(n \in \mathbb{N})$ has been introduced and investigated in [C;9].

It can be shown that if the sequence $(\rho_k, k \in \mathbb{N})$ is regular, then the product system Ψ is an ONS with respect to the weight function ρ.

Theorem 1. *Suppose that the $(\rho_n, n \in \mathbb{N})$ sequence of weigts is regular and satisfy (2.11). Let Ψ be the product system of AC-ONS satisfying (2.7). Then Ψ is an orthonormed system with respect to $\rho := \prod_{k=0}^{\infty} \rho_k$, i.e.*

$$\int_\Omega \psi_n \overline{\psi_m}\, \rho\, d\mu = \delta_{mn} \quad (m, n \in \mathbb{N}). \tag{2.13}$$

By (1.3) and (1.4) it is easy to see that the n-th Fourier-coefficient of $f \in L^2$ with respect to the system Ψ can be written in the form

(2.14)

$$\int_\Omega f \overline{\psi}_n\, \rho\, d\mu = E_0(\overline{\varphi_0^{n_0}} \rho_0 E_1(\overline{\varphi_1^{n_1}} \rho_1 \cdots E_{s-1}(\overline{\varphi_{s-1}^{n_{s-1}}} \rho_{s-1} E_s(f \overline{\theta_s})))$$
$$0 \leq n < M_s),$$

where

$$\theta_s := \prod_{k=s}^{\infty} \rho_k \quad (s \in \mathbb{N}).$$

The formula (2.14) is the basis of all FFT-algorithms (for details see [D;1-4]).

It can be proved (see [C;7-9]) that if the system Φ_n is $\mathcal{A}_n$-complete with respect to $L^2(\mathcal{A}_{n+1})$, and the stochastic basis $(\mathcal{A}_n, n \in \mathbb{N})$ generates $\mathcal{A}$ then the product system Ψ is complete with respect to L^2. Especially, if the $\mathcal{A}_n$'s are atomic σ-algebras generating $\mathcal{A}$ and the systems Φ_n are $\mathcal{A}_n$-complete with respect to $L(\mathcal{A}_{n+1})$ then terms of the product system are bounded functions and the system itself is complete with respect to L^p for $1 \leq p < \infty$.

The conditions on Φ_n can be relaxed. Namely, $\varphi_k^0 = 1$ $(k \in \mathbb{N})$ can be omitted. In that case (2.4) becomes an infinite products. We must guarantee that this product converges to a function in L^2. Let us assume for example that

$$iv) \quad \prod_{k=n_0}^{\infty} \phi_k^0 \quad \text{converges, and} \quad \sup_{s \geq n_0} \prod_{k=n_0}^{s} |\phi_k^0| \in L^2 \quad (n_0 \in \mathbb{N}) \tag{2.15}$$

holds for the system

$$\Phi'_k := (\phi_k^\ell : 0 \leqq \ell < m_k) \quad (k \in \mathbb{N}),$$

which is an AC-ONS for the weight function $\rho_k = 1$ $(k \in \mathbb{N})$. Thus one can define the product system

$$\psi'_n := \prod_{k=0}^{\infty} \phi_k^{n_k} \quad (n \in \mathbb{N}).$$

ψ'_0 is called the *starting function* of the product system.

This general concept of product systems has been introduced in [C;24], [E;4]. It can be shown that also in this case the product system $\Psi' = (\psi'_n, n \in \mathbb{N})$ is orthogonal with respect to the weight function $\rho = 1$.

We note that this concept is equivalent to the original one. Indeed, let

$$\rho_k := |\phi_k^0|^2 > 0 \quad (k \in \mathbb{N}).$$

if

$$\varphi_k^\ell := \frac{\phi_k^\ell}{\phi_k^0} \quad (0 \leqq \ell < m_k,\ k \in \mathbb{N})$$

then $\varphi_k^0 = 1$ $(k \in \mathbb{N})$. Moreover

$$E_k(\phi_k^i \overline{\phi_k^j}) = E_k(\varphi_k^i \overline{\varphi_k^j} \rho_k) = \delta_{ij} \qquad (0 \leqq i, j < m_k). \tag{2.16}$$

It follows from the connection between the system Ψ and Ψ' that

$$\int_\Omega \psi_r' \overline{\psi_s'}\, d\mu = \int_\Omega \psi_r \overline{\psi_s}\, \rho\, d\mu = \delta_{st} \quad (s, t \in \mathbb{N}). \tag{2.17}$$

On the basis of this observation it is enough to study product systems with finite many factors utilizing the transition from Ψ' to Ψ. Then we obtain an orthonormal system in Hilbert space $L^2(\Omega, \mathcal{A}, \mu)$ rather then in $L^2_\rho(\Omega, \mathcal{A}, \mu)$.

Even if the measure space $(\Omega, \mathcal{A}, \mu)$ is fixed we have infinitely many possibilities to construct product systems. Namely, we are free to choose the weight function ρ, the stochastic basis $(\mathcal{A}_n, n \in \mathbb{N})$, and the AC-ONS system Φ_n $(n \in \mathbb{N})$.

The $(\mathcal{A}_n, n \in \mathbb{N})$ stochastic basis is called atomic if the σ-algebra $\mathcal{A}_n$ $(n \in \mathbb{N})$ are generated by finite many atoms. Clearly, the atoms in $\mathcal{A}_n$ can be decomposed as finite unions of $\mathcal{A}_{n+1}$ type atoms. If every one of them is the union of exactly p pairwise different $\mathcal{A}_{n+1}$ atoms then the stochastic basis is called p-adic. In particular, if $p = 2$ then it is called *dyadic.* Such a stochastic basis is called *homogeneous* if the μ measure of every $\mathcal{A}_n$ atom is the same.

By the application of stopping times one can redefine the initial stochastic basis and so adjust it to the particular problem. Recall that the function $\tau : \Omega \to \mathbb{N} \cup \{\infty\}$ is called a *stopping time* with respect to the stochastic basis $(\mathcal{A}_n, n \in \mathbb{N})$ if for every $n \in \mathbb{N}$ we have

$$\{\tau = n\} = \{\omega \in \Omega : \tau(\omega) = n\} \in \mathcal{A}_n.$$

Then

$$\mathcal{A}_\tau := \{A \in \mathcal{A} : A \cap \{\tau = n\} \in \mathcal{A}_n \ (n \in \mathbb{N})\}$$

is a sub σ-algebra of $\mathcal{A}$. In the case of the constant stopping time $\tau(\omega) = n$ $(\omega \in \Omega)$ we obtain that $\mathcal{A}_\tau = \mathcal{A}_n$. It can be shown, that if $\tau_1 \leqq \tau_2$ holds for the stopping times τ_1, τ_2 then $\mathcal{A}_{\tau_1} \subseteq \mathcal{A}_{\tau_2}$. Therefore by taking a sequence $\tau_1 \leqq \tau_2 \leqq \cdots \leqq \tau_n \leqq \cdots$ of stopping times we can switch to the generated stochastic basis

$$\mathcal{A}_{\tau_1} \subseteq \mathcal{A}_{\tau_2} \subseteq \cdots \subseteq \mathcal{A}_{\tau_n} \subseteq \cdots .$$

This socalled stopping time technique is widely used in the theory of probability. It turned to be a successful method also in harmonic analysis [A;9]. We note that this is an effective method in image procession and image compression as well. Namely, using stopping times one can construct stochsatic basis and orthogonal system that are adapted to the image.

Sometimes instead of starting from a stochastic basis we take a system of functions $\eta_k \in L^2(\Omega, \mathcal{A}, \mu)$ $(k \in \mathbb{N})$ and take the monotonically increasing sequence $\mathcal{A}_k := \sigma\{\eta_\ell : 0 \leqq \ell < k\}$ of σ-algebras. Let us standardize η_k:

$$\varphi_k := \frac{\eta_k - E_k(\eta_k \rho_k)}{\sqrt{E_k |\eta_k - E_k(\eta_k \rho_k)|^2 \rho_k}} \quad (k \in \mathbb{N}). \tag{2.18}$$

Then for the systems $\Phi_k := \{1, \varphi_k\}$ $(k \in \mathbb{N})$ condition (2.7) is satisfied. Thus the binary product system is orthogonal with respect to the weight function ρ. In particular, if η_k $(k \in \mathbb{N})$ is a sequence of independent random variables with expectation 0 and variance 1 then

$$E_k(\eta_k) = \int_\Omega \eta_k \, d\mu = 0, \quad E_k(|\eta_k|^2) = \int_\Omega |\eta_k|^2 \, d\mu = 1.$$

Consequently, (2.7) holds for the system η_k $(k \in \mathbb{N})$ with $\rho_k = 1$ $(k \in \mathbb{N})$. Thus the binary product system is orthonormal.

Let $\eta_k^j \in L^2_{\rho_k}(\Omega, \mathcal{A}_{k+1}, \mu)$ $(0 \leqq j < m_k)$ be linearly independent with respect to $\mathcal{A}_{k+1}$. Using an ortogonalization process, similar to the Schmidt procedure, we can construct an AC-ONS system Φ_k $(k \in \mathbb{N})$ satisfying (2.7).

Using these constructions we can obtain the Vilenkin systems [A;2,5,6] and their generalization introduced by G. Gát [C;26]. In the next chapter we will investigate only binary product systems.

3. Examples

In this section we give examples for AC-ONR systems taking the dyadic stochastic basis and choosing the sequence $(\rho_k, k \in \mathbb{N})$ in a special way.

Let $\Omega := \mathbb{I} := [0, 1)$, $\mathcal{A}$ the collection of Lebesgue-measurable subset in $[0, 1)$ and μ the Lebesgue-measure. The atoms of $\mathcal{A}_n$ are the dyadic intervals

$$\mathcal{J}_n := \{[k2^{-n}, (k+1)2^{-n}) : \; 0 \leqq k < 2^n\}.$$

Several classic orthonormal system can be obtained as a binary product system of systems of the form $\Phi_k = \{1, \varphi_k\}$ $(k \in \mathbb{N})$ satisfying (2.7).

Denote r_n $(n \in \mathbb{N})$ the Rademacher system on $[0, 1)$, i.e.

$$r_n(x) = (-1)^{x_n} \quad \Big(x = \sum_{n=0}^{\infty} x_n 2^{-(n+1)} \in [0, 1), x_n = 0, 1\Big). \tag{3.1}$$

Obviously r_n is $\mathcal{A}_{n+1}$-measurable and $E_n(r_n) = 0$. Then binary product system of the sequence $(r_n, n \in \mathbb{N})$ is the *Walsh-Paley system.*

In the case $\varphi_0 := r_0, \varphi_n := r_n r_{n-1}$ $(n \in \mathbb{N}^*)$ we get another AC-ONS and the binary product system is *the system introduced by J. L. Walsh* in 1923. We note that this is a rearrangement of the Walsh-Paley system (see [A;6]). Other rearangement of the Walsh-system (see [A;7], [E;1]) can be obtained in this way too.

Many orthogonal systems in dyadic harmonic analysis are binary product systems of dyadic martingale differences. Let $\varphi_n : [0, 1) \to \mathbb{C}$ $(n \in \mathbb{N})$ be a sequence of functions satisfying

$$i) \;\; |\varphi_n| = 1, \quad ii) \;\; \varphi_n \in L(\mathcal{A}_{n+1}), \quad iii) \;\; E_n(\varphi_n) = 0 \;\; (n \in \mathbb{N}). \tag{3.2}$$

Thus $(\varphi_n, n \in \mathbb{N})$ is a sequence of dyadic martingale differences of modulus 1, which is called *unitary dyadic martingale difference* (UDMD) system [A;7], [C;22]. It is obvious, that the functions φ_n are of the form

$$\varphi_n = r_n \kappa_n, \quad \text{where} \;\; \kappa_n \in L(\mathcal{A}_n), \;\; \text{and} \;\; |\kappa_n| = 1 \;\; (n \in \mathbb{N}).$$

The characters of the additive and multiplicative groups of 2-adic and 2-series field can be obtained as product systems of UDMD-systems [A;7]. Especially, the characters of the additive group of 2-series field are the Walsh functions. In the case of the 2-adic field the additive characters are generated by the UDMD system where

$$\varphi_n(x) := \exp\Big(2\pi i(\frac{x_n}{2} + \frac{x_{n-1}}{2^2} + \cdots + \frac{x_0}{2^{n+1}})\Big) \ (n \in \mathbb{N}, x \in [0,1)) \tag{3.3}$$

and the x_n's are the binary coefficients defined in (3.1).

The product systems of these systems are closely connected to the discrete trigonometric system. For describing the characters of the multiplicative groups of 2-adic and 2-series field see [A;7].

To define *wighted Wals-functions* fix the function $\rho > 0, \rho \in L^1$ and consider the sequence of regular weight $(\rho_k, k \in \mathbb{N})$, defined in (2.11). Taking the standardized sequence of the Rademacher functions defined by (2.18), we get

$$r_k^\rho = \frac{r_k - E_k^\rho(r_k)}{\sqrt{1 - |E_k^\rho(r_k)|^2}}, \quad \text{where} \quad E_k^\rho(r_k) := \frac{E_k(r_k\rho)}{E_k(\rho)} \quad (k \in \mathbb{N}), \tag{3.4}$$

wich is called the *weighted Rademacher-system.* The binary product-system of $(r_n^\rho, n \in \mathbb{N})$ is called the *weighted Walsh system* and will be denoted by $(w_n^\rho, n \in \mathbb{N})$. For weight function ρ of the form

$$\rho := \prod_{k=0}^{\infty}(1 + a_k r_k) \quad \Big(a_k \in \mathbb{R}, |a_k| < 1 \ (k \in \mathbb{N}), \sum_{k=0}^{\infty}|a_k| < \infty\Big)$$

this class coincides with the set of *Wals-similar functions*, introduced by Sendov [E;6,7]. The general class was introduced end investigated in [C;24], [E;4]. By Theorem 1 we have that $\{w_k^\rho : k \in \mathbb{N}\}$ is an ONS in $L_\rho^2[0,1)$.

It is easy to see that the AC-ONS property is invariant with respect to Kronecker-product. Namely, suppose that for any $k \in \mathbb{N}$ the system

$$\Phi_k = \{\varphi_k^i : 0 \le i < m_k\} \subset L^2(\Omega, \mathcal{A}_{k+1}, \mu),$$
$$\hat{\Phi}_k = \{\hat{\varphi}_k^j : 0 \le j < \hat{m}_k\} \subset L^2(\hat{\Omega}, \hat{\mathcal{A}}_{k+1}, \hat{\mu})$$

are adaptive conditionally orthonormal systems and denote

$$(f \times g)(x, y) := f(x)g(y) \quad (x \in X, y \in \hat{X})$$

the Kronecker product of the functions f and g. Then the Kronecker product

$$\Phi_k \times \hat{\Phi}_k := \{\varphi_k^i \times \hat{\varphi}_k^j : 0 \le i < m_k, 0 \le j < \hat{m}_k\},$$
$$\Phi_k \times \hat{\Phi}_k \subset L^2(\Omega \times \hat{\Omega}, \mathcal{A}_{k+1} \times \hat{\mathcal{A}}_{k+1}, \mu \times \hat{\mu}) \quad (k \in \mathbb{N})$$

is an AC-ONS with respect to the stochastic basis $\mathcal{A}_k \times \hat{\mathcal{A}}_k$ $(k \in \mathbb{N})$. Denote Ψ and $\hat{\Psi}$ the corresponding product systems. Then $\Psi \times \hat{\Psi}$ is the product system of $(\Phi_k \times \hat{\Phi}_k, k \in \mathbb{N})$.

Especially, if $(X, \mathcal{A}, \mu) = (\hat{X}, \hat{\mathcal{A}}, \hat{\mu})$ is the Lebesgue-space on $[0,1)$, then $d(\mu \times \hat{\mu})(x,y) = dx\,dy$ is the two-dimensional Lebesgue-measure and $\mathcal{A} \times \hat{\mathcal{A}}$ is the collection of Lebesgue-measurable subset in $\mathbb{I}^2 := \mathbb{I} \times \mathbb{I}$. Let $\mathcal{A}_k = \hat{\mathcal{A}}_k$ the dyadic σ-algebra. Then $\mathcal{A}_k^2 := \mathcal{A}_k \times \hat{\mathcal{A}}_k$ is the σ-algebra, generated by the dyadic squares

$$\mathcal{J}_n \times \mathcal{J}_n := \{I \times J : I, J \in \mathcal{J}_n\} \ (n \in \mathbb{N}).$$

In this case the conditional expectation is of the form

$$(E_n f)(x,y) = \frac{1}{|I_n(x) \times I_n(y)|} \int_{I_n(x)\times I_n(y)} f(s,t)\,ds\,dt \quad (x, y \in [0,1), n \in \mathbb{N}),$$

where $I_n(x)$ denotes the interval in $\mathcal{J}_n$ containing x.

The systems $\{1, r_k \times 1, 1 \times r_k, r_k \times r_k\}$ $(k \in \mathbb{N})$ is AC-ONS with respect to the stochastic basis $(\mathcal{A}_k^2, k \in \mathbb{N})$ and $\mathcal{A}_k^2$-complete with respect to $L(\mathcal{A}_{k+1}^2)$.

Taking the transform of this system by any orthogonal matrix $B^k = [b_{ij}^k]$ of $\mathcal{A}_k^2$-measurable functions b_{ij}^k, the system

$$\varphi_k^i = b_{i0}^k + b_{i1}^k(r_k \times 1) + b_{i2}^k(1 \times r_k) + b_{i3}^k(r_k \times r_k) \quad (i = 0,1,2,3) \tag{3.5}$$

is $\mathcal{A}_{k+1}^2$-measurable and $\mathcal{A}_k^2$-orthonormal. The special case, if a_k^1, a_k^2, a_k^3 are real numbers and

$$A^k := M[a_k] := \frac{1}{\sqrt{1 + |a_k^1|^2 + |a_k^2|^2 + |a_k^3|^2}} \begin{pmatrix} 1 & a_k^1 & a_k^2 & a_k^3 \\ a_k^1 & -1 & a_k^3 & -a_k^2 \\ a_k^2 & -a_k^3 & -1 & a_k^1 \\ a_k^3 & a_k^2 & -a_k^1 & -1 \end{pmatrix} \tag{3.6}$$

is the orhogonal matrix generated by the vector $a_k = (a_k^1, a_k^2, a_k^3)$ has been investigated in [C;24]. Obviously if a_k^1, a_k^2, a_k^3 are $\mathcal{A}_k^2$-measurable functions then the system $\{\varphi_k^j : 0 \le j < 4\} \subset L(\mathcal{A}_{k+1}^2)$ generated by $M[a_k]$ is $\mathcal{A}_k$-orthonormal.

4. Dirichlet kernels, Paley's formula, partial sum operators

In this section we investigate norm-convergence with respect to the product system Ψ introduced in (1.11). For the M_n-th Dirichlet kernel of the product system we have the following product reprezentation:

$$D_{M_n}(s,t) := \sum_{k=0}^{M_n-1} \psi_k(s)\overline{\psi}_k(t) = \prod_{i=0}^{n-1} \Big(\sum_{j=0}^{m_i-1} \phi_i^j(s)\overline{\phi}_i^j(t) \Big). \tag{4.1}$$

This is the genaralization of the Paley's identity proved for the Walsh system. Denote $\mathcal{L}(\Psi)$ the linear hull of Ψ, i.e. the set of Ψ-polinomials.

Let the partial sums of the Fourier-series of $f \in L_\rho^2$ with respect to the system Ψ be denoted by

$$S_0 f := 0, \quad S_n f := \sum_{k=0}^{n-1} [f, \psi_k]\psi_k \quad (n \in \mathbb{N}^*), \tag{4.2}$$

where $[f,g] := \int_{\mathbb{I}} f\overline{g}\,\rho d\mu$ is the usual scalar product in L^2_ρ.

On the set of Ψ polynomials the operators S_{M_n} can be expressed by the conditional expectation:

$$S_{M_n} f = E_n f \quad (n \in \mathbb{N}, f \in \mathcal{L}(\Psi)).$$

The operator S_n can be expressed by the partial sums of the $\mathcal{A}_k$-Fourier-series with respect to the system Φ_k, i.e. by

$$S^0_k f := 0,\ S^j_k f := \sum_{i=0}^{j-1} E_k(f\overline{\varphi}^i_k \rho_k)\varphi^i_k \quad (j = 1, 2, \cdots, m_k, k \in \mathbb{N}). \tag{4.3}$$

Theorem 2. *Let Ψ be the product system of Φ_k $(k \in \mathbb{N})$. Then the partial sum operators with respect to the product system on $\mathcal{L}(\Psi)$ are of the form*

$$S_n f = \sum_{k=0}^{\infty} S^{n_k}_k (f\overline{\psi}^{k+1}_n)\psi^{k+1}_n \quad (f \in \mathcal{L}(\Psi), n \in \mathbb{N}), \tag{4.4}$$

where

$$\psi^j_n := \prod_{i=j}^{\infty} \phi^{n_i}_i.$$

If the system is unitary then the partial sums can be expressed by generalized martingale transforms. Namely from (4.4) we get

Corollary 1. *If the systems Φ_k $(k \in \mathbb{N})$ are unitary, i.e. if $|\phi^j_k| = 1$ $(0 \leqq j < m_k, k \in \mathbb{N})$ then S_n is defined on L^1_ρ and*

$$S_n f = \psi_n T_n(f\overline{\psi}_n) \quad (n \in \mathbb{N}, f \in L^1_\rho), \tag{4.5}$$

where

$$T_n g = \sum_{k=0}^{\infty} \phi^{n_k}_k S^{n_k}_k (g\overline{\phi}^{n_k}_k) \quad (g \in L^1_\rho) \tag{4.6}$$

is the generalized martingale transform operator.

5. Norm and a.e. convergence

The representation (4.5) and (4.6) can be used in convergence problems of Fourier seriese with respect to product systems.

The operator $T : L^p \to L^p$ is called of *strong type* $(\mathcal{A}_k, p)$ if there exists a number $K > 0$ such that for any $f \in L^p$

$$(E_k(|Tf|^p))^{1/p} \leqq K(E_k(|f|^p))^{1/p} \tag{5.1}$$

is satisfied. The infimum of the numbers K in (5.1) is called the $(\mathcal{A}_k, p)$-norm of the operator T and is denoted by $\|T\|_{(\mathcal{A}_k,p)}$. The usual L^p-norm is denoted by $\|\cdot\|_p$.

The uniform conditional $(\mathcal{A}_k, p)$-boundedness of the operators S^j_k implies the uniform L^p-norm boundedness of the partial sum operators S_n if the systems Φ_k are unitary [C;9]. This implies the L^p-norm convergence of the Fourier series with respect to the product sysstem.

Theorem 3. *Suppose that the systems* Φ_k $(k \in \mathbb{N})$ *are unitary and let* $\rho_k = 1$ $(k \in \mathbb{N})$. *If for some* $1 < p < \infty$

$$\sup_{0 \leqq j < m_k, k \in \mathbb{N}} \|S_k^j\|_{(\mathcal{A}_k, p)} < \infty,$$

then

$$\sup_n \|S_n\|_p < \infty.$$

This implies

Corollary 2. *The Vilenkin systems are basis in* L^p *if* $1 < p < \infty$.

In connection with these results we put

Problem 1. *Under what conditions is* Ψ *a basis in* L^p *in the general case.*

The a.e. convergence of the parial sums $S_n f$ depends on L^p boundedness of the maximal operators

$$S^* f := \sup_n |S_n f|, \quad S_k^* f := \sup_{0 \leqq j < m_k} |S_k^j f| \ (k \in \mathbb{N}).$$

For bounded generating sequence $(m_k, k \in \mathbb{N})$ the analogue of Carleson-Hunt theorem was proved in [C;10].

Theorem 4.. *Suppose that the systems* Φ_k $(k \in \mathbb{R})$ *are unitary and* $\sup_k m_k < \infty$. *Then* S^* *is bounded on* L^p *if* $1 < p < \infty$.

This implies

Corollary 3. *If* $1 < p < \infty$ *and* $f \in L^p$ *then* $S_n f$ $(n \in \mathbb{N})$ *converges a.e. In particular for bounded Vilenkin systems* $S_n f$ $(n \in \mathbb{N})$ *converges a.e.*

In connection with these result we have the following open

Problem 2. *Does the analogue of Theorem 2. hold for the maximal operators:*

$$\sup_k \|S_k^*\|_{(\mathcal{A}_k, p)} < \infty \quad \Rightarrow \quad \|S^*\|_p < \infty?$$

Problem 3. *Does* $S_n f$ $(n \in \mathbb{N})$ *converge a.e. for arbitrary Vilenkin system if* $f \in L^p$ $(1 < p < \infty)$*?*

6. Binary product systems

In this section we investigate the special case $m_k = 2, \ \rho_k = 1$ $(k \in \mathbb{N})$. The binary product system of the system $\Phi_k := \{1, \phi_k\}$ $(k \in \mathbb{N})$ is an ortonormal system if (2.7) i) ii) are satisfied. In particular if the functions $(\phi_k, k \in \mathbb{N})$ are independent and

$$\int_\Omega \phi_k \, d\mu = 0, \qquad \int_\Omega |\phi_k|^2 \, d\mu = 1 \quad (k \in \mathbb{N}), \tag{6.1}$$

then (2.7) i) ii) hold for the σ-field $\mathcal{A}_k := \sigma\{\phi_j : j < k\}$ $(k = 1, 2, \dots)$.

In the binary case condition (2.7) is equivalent to the fact that the sequnce $(\phi_k, k \in \mathbb{N})$ forms a *normalized martingale difference sequence* with respect to the stochastic basis $(\mathcal{A}_{k+1}, k \in \mathbb{N})$. Consequently the partial sums of the series

$$\sum_{k=0} a_k \phi_k = \sum_{k=0} a_k \psi_{2^k} \tag{6.2}$$

form an L^2 bounded martingale, if $\sum_{k=0}^{\infty} |a_k|^2 < \infty$. Thus the series (6.2) is μ a.e. convergent, i.e. $(\psi_{2^k}, k \in \mathbb{N})$ is a *convergence system.* This claim is a special case of the *Kolmogorov's three series theorem* which play an important part in the proof of the general theorem.

The series (6.2) is a strongly lacunary subseries of the orthogonal series

$$\sum_{n=0}^{\infty} c_n \psi_n. \tag{6.3}$$

¿From theorem 4 it follows

Corollary 4. *If $(\phi_n, n \in \mathbb{N})$ is a unitary martingale difference system, then its binary product system is a convergence system.*

In coonection with Corollary 4 we put

Problem 4. *Under what condition with respect to the system $(\phi_k, k \in \mathbb{N})$ does*

$$\sum_{n=0}^{\infty} |c_n|^2 < \infty$$

impy the μ a.e. convergence of (6.3).

Similar claims for *real, uniformly bounded weakly multiplicative systems* are proved in [C;4-6]. For the summary this type of results see [C;17].

7. Walsh functions with respect to weights

Let $\Omega := [0, 1)$, $\mathcal{A}$ the collection of Lebesgue-measurable sets in $[0, 1)$ and denote μ the measure generated by the positive weight function $\rho \in L^1[0, 1]$:

$$\mu(H) := \int_H \rho(t)\, dt \quad (H \in \mathcal{A}) \quad (\rho > 0, \ \int_0^1 \rho(t)\, dt = 1). \tag{7.1}$$

Starting from the dyadic stochastic basis $\mathcal{A}_k$ $(k \in \mathbb{N})$ we can constract binary product system, orthogonal with respect to ρ. Namely set

$$\phi_k := r_k^{\rho} := \frac{r_k - E_k^{\rho} r_k}{(E_k^{\rho} |r_k - E_k^{\rho} r_k|^2)^{1/2}} \tag{7.2}$$

for standartization of the Rademacher system $(r_k, k \in \mathbb{N})$. Here

$$(E_k^{\rho} f)(x) = \frac{\int_I f\, \rho dx}{\int_I \rho dx} \quad (x \in I) \tag{7.3}$$

is the condional expectation operator with respect to $\mathcal{A}_k$ and I denotes a dyadic interval with the lenght 2^{-k}. The system $(r_k^\rho, k \in \mathbb{N})$ is a dyadic martingale difference system in the space $([0,1), \mathcal{A}, \mu)$, which coincides with the Rademacher system, if $\rho = 1$.

The binary product system of this system is called the *weighted Walsh system* and will be denoted by $W^\rho = (w_n^\rho, n \in \mathbb{N})$. In the special case

$$\rho := \prod_{k=0}^{\infty}(1 + a_k r_k), \quad \sum_{k=0}^{\infty} |a_k| < \infty, \quad |a_k| < 1 \ (k \in \mathbb{N})$$

this system was introduced and investigated by *Sendov* and called *Walsh similar functions* [E;6,7].

It was shown in [C;24] (see also [E;4,5,6])

Theorem 5. *If*

$$\sum_{k=0}^{\infty} \|E_k^\rho(r_n)\|_\infty < \infty \tag{7.4}$$

then the systems W^ρ and W are eqivalent in L^p $(1 \leqq p \leqq \infty)$. Moreover the Fourier series with respect to W^ρ of any functions in L^p $(1 < p < \infty)$ converges in L^p norm and a.e.

The special weights

$$\rho(t) := t^q \quad (0 \leqq t \leqq 1, \ 0 < q \leqq 1) \tag{7.5}$$

was investigated in [E;5,8].

Theorem 6. *Let ρ be the weight function introduced in (7.5) and set*

$$p_0 := \frac{2\ln(2^{q+1})}{\ln((1+2^q)/2)} (> 2), \quad p_1 := \frac{p_0}{p_0 - 1}. \tag{7.6}$$

Then the system W^ρ is not uniformly bounded. Moreover if $1 \leqq p \leqq p_1$ or $p \geq p_0$ then W^ρ is not a Schauder basis in L^p_ρ.

The next question is open.

Problem 5. *Is weighted Walsh system W^ρ generated by the weight function (7.5) a Schauder basis in L^p_ρ, if $p_1 < p < p_0$ with p_0 and p_1 defined in (7.6) ?*

Theorem 6 was generalized by *P. Simon.* Moreover he has proved [E;5]

Theorem 7. *There exists a weight function ρ such that the system W^ρ form a Schauder basis in L^p_ρ exactly when $p = 2$.*

7. Rational UDMD systems

Using Blaschke functions discret rational ortonormal product systems can be construct. The Blaschke functions

$$B_a(z) := \frac{1-\bar{a}}{1-a}\,\frac{z-a}{1-\bar{a}z}$$
$$(a \in \mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}, z \in \mathbb{C})$$

are 1-1 maps on $\mathbb{D}$ and on $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ and $B_a(1) = 1$. The function

$$A_a(z) := B_a(z)B_{-a}(z) = B_{a^2}(z^2)$$

is a twofold map of $\mathbb{D}$ and of $\mathbb{T}$.

In order to define the UDMD systems in question we fix the sequence $\mathfrak{a} = (a_n \in \mathbb{D}, n \in \mathbb{N}^*)$ and introduce the 2^n-fold maps

$$\phi_0(z) := z, \quad \phi_n = A_{a_n} \circ \cdots \circ A_{a_1} \quad (n = 1, 2, \cdots)$$

and the sets

$$X^n := \{z \in \mathbb{C} : \phi_n(z) = 1\} = \{x_k^n : 0 \leqq k < 2^n\} \quad (n \in \mathbb{N}).$$

The points of X^n are easy to compute. Introduce on X^n the discrete measure μ defined by $\mu(\{x\}) := 2^{-n}$ $(xinX_n)$. Then the finite system $(\phi_k, 0 \leqq k < n)$ is a unitary dyadic martingale difference sequence (UDMD-system) on X^n. The product system $(\psi_m, 0 \leqq m < 2^n)$ is a discrete orthonormal system with respect to the scalar product

$$\langle f, g\rangle := 2^{-n} \sum_{x \in X^n} f(x)\overline{g(x)}$$

It is clair that in the case $a_1 = a_2 = \cdots = 0$ then we get the trigonometric system.

The values of discrete Fourier coefficients $\langle f, \psi_m\rangle$ $(m < 2^n)$ and the partial sums

$$(S_{2^n} f)(x) := \sum_{m=0}^{2^n - 1} \langle f, \psi_m\rangle \psi_m(x)$$

at $x \in X_n$ can be computed by using $O(n2^n)$ algebraic operations and the partial sums $S_{2^n} f$ interpolate the function f at X^n [G;3].

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