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ε-ISOMETRIC APPROXIMATION PROBLEM

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Abstract. In this paper, some problems for isometric approximation is resolved.

1. Introduction

Let E and F be normed linear spaces. Hyers and Ulam [6] called the mapping $T\colon E \to F$ an absolute error ε-isometry if for any $\varepsilon \geq 0$,

$$\|x-y\| - \varepsilon \leq \|Tx - Ty\| \leq \|x-y\| + \varepsilon \tag{1}$$

for any $x, y \in E$. On the stability of isometry, Hyers and Ulam asked following questions:

1. For each *surjective* ε-isometry T, if there exists an isometric mapping $U\colon E \to F$, and a constant K such that

$$\|Tx - Ux\| \leq K(E,F)\varepsilon$$

for any $x \in E$ where the constant K depends only on E and F.

2. If the answer above is positive, what is the best K?

To start with studying these problems, without loss of generality, $T(0) = 0$ for T is ε-isometry, $T - T(0)$ is necessary ε-isometry. P.M. Grubern [4] in 1978, T.M. Rassias and P. Šemel [12] in 1993 gave that the positive answer.

The ε-isometry $T\colon E \to F$ is called Lipschitz ε-isometry if

$$(1-\varepsilon)\|x-y\| \leq \|Tx - Ty\| \leq (1+\varepsilon)\|x-y\| \tag{2}$$

for all $x, y \in E$.

Now, suppose that Lipschitz ε-isometry T is a linear operator, Benyamini [2], Alspach [1] and Dingguanggui [5] proved that there exists an isometric approximation of T. When T is nonlinear and surjective operator, K. Jarosz [7] obtained positive answer on $C_0(X) \to C_0(Y)$, where X, Y are locally compact Hausdorff spaces.

Withdrawing the condition of *surjective* and *linear*, how about Lipschitz ε-isometric approximation problem? G.M. Lövblom [9, 10] gave two local results for these problems, i.e. to restrict the problem on the unit ball $B_1(C(X)) \to B_1(C(Y))$ where X, Y are compact Hausdorff spaces, the answer is positive. Two counterexamples given show that as $E = F = l_1$ or $E = F = (L_1(0,1) \times R)_1$ the local problem is negative.

In this paper we restrict ourselves to the local question about absolute error ε-isometry (1) without the assumption of *surjective* and we have some changed for the definition of T as follows.

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$T\colon E \to F$ is an ε-isometry, meaning that

$$\|x-y\| - \varepsilon \le \|Tx - Ty\| \le \|x-y\| \tag{3}$$

for any $x, y \in E$.

Thanks to Lövblom's idea, we prove that the ε-isometric problem (3) on

$$B_1(C(X)) \to B_1(C(Y))$$

is positive, and on $B_1(E) \to B_1(F)$ where $E = F = l_1$ or $E = F = (L_1(0,1) \times R)$ the problem is negative.

2. ε-isometry on $B_1(C(X)) \to B_1(C(Y))$

Let X, Y be compact metric spaces with metrics d_1 and d_2 and let $B_R(C(X))$ denote the ball of $C(X)$ with center 0 and radius R.

Theorem 2.1. *Let $T\colon B_1(C(X)) \to B_1(C(Y))$ with $T(0) = 0$, and*

$$\|f-g\| - \varepsilon \le \|Tf - Tg\| \le \|f-g\| \tag{4}$$

for any $f, g \in B_1(C(X))$. Then there exists an isometry

$$U\colon B_{1-\delta_1(\varepsilon)}(C(X)) \to B_1(C(Y))$$

such that

$$\|Tf - Uf\| \le \varepsilon$$

on $B_{1-\delta_1(\varepsilon)}(C(X))$, where $\delta_1(\varepsilon) \to 0$ when $\varepsilon \to 0$.

The proof is based on the following Lemmas. Let a be fixed, $4\varepsilon < a \le 1$.

Definition 2.2 ([9])**.** Given $x_0 \in X$, we say that $f \in C(X)$ is a tentfunction at x_0 if for some $\delta > 0$

$$f(x) = \begin{cases} 1 - \frac{d_1(x_0,x)}{\delta}, & x \in B(x_0, \delta), \\ 0, & \text{otherwise.} \end{cases} \tag{5}$$

obviously, $f(x_0) = 1, \|f\| = 1$.

Lemma 2.3. *Let $\{f_n\} \subset B_1(C(X))$, $\{x_n\} \subset X$, $\{y_n\} \subset Y$ be sequences with $y_n \to y$ and f_n a tentfunction at x_n with* $\operatorname{supp}(f_n) = B(x_n, \delta_n)$ *where $\delta_n \to 0$ when $n \to \infty$.*

If for all n

$$2a - \varepsilon \le |T(af_n)(y_n) - T(-af_n)(y_n)|, \tag{6}$$

then $\lim\limits_{n\to\infty} x_n$ exists.

Proof. X is a compact metric space, so $\{x_n\}$ contains a convergent subsequence, say $\{x_{n'}\}$ with $\lim_{n'\to\infty} x_{n'} = x$. Assume that x_n is not convergent. Then for some $d > 0$, there exists, for every N, $n \ge N$ such that $d_1(x_n, x) \ge d$. Let $g \in C(X)$ with $0 \le g \le \frac{a}{2}$, $g = \frac{a}{2}$ on $B(x, \frac{d}{4})$ and with $\operatorname{supp}(g) \subset B(x, \frac{d}{2})$.

For each N it is possible to find $n, n' \ge N$ such that $\operatorname{supp}(f_{n'}) \subset B(x, \frac{d}{4})$ and $B(x_n, \delta_n) \bigcap B(x, \frac{d}{2}) = \emptyset$. Then we have

$$\|g - af_n\| = \frac{a}{2}, \|g + af_n\| = \frac{3a}{2}, \|g \pm af_n\| = a. \tag{7}$$

Because T is ε-isometry, therefore for any $y \in Y, f, g \in B_1(C(X))$

$$|T(g)(y) - T(f)(y)| \le \|g - f\|$$

Thus

$$-\|g-f\| + T(f)(y) \le T(g)(y) \le \|g-f\| + T(f)(y).$$

We get

$$T(af_{n'})(y_{n'}) - \frac{a}{2} \leq T(g)(y_{n'}) \leq T(af_{n'})(y_{n'}) + \frac{a}{2}. \tag{8}$$

$$T(-af_n)(y_n) - \frac{3a}{2} \leq T(g)(y_n) \leq T(-af_n)(y_n) + \frac{3a}{2}. \tag{9}$$

$$T(\pm af_n)(y_n) - a \leq T(g)(y_n) \leq T(\pm af_n)(y_n) + a. \tag{10}$$

By hypothesis of T with $T(0) = 0$, we have for all n.

$$\|T(\pm af_n)\| \leq a. \tag{11}$$

$$\begin{cases} T(af_n)(y_n) \geq T(-af_n)(y_n) + 2a - \varepsilon, \\ T(af_n)(y_n) \leq T(-af_n)(y_n) - 2a + \varepsilon. \end{cases} \tag{12}$$

From (8)–(12) we get

$$\begin{cases} a - \varepsilon \leq T(af_n)(y_n) \leq a, \\ -a \leq T(-af_n)(y_n) \leq -a + \varepsilon. \end{cases} \tag{13}$$

$$\begin{cases} -a \leq T(af_n)(y_n) \leq -a + \varepsilon, \\ a - \varepsilon \leq T(-af_n)(y_n) \leq a. \end{cases} \tag{14}$$

By (12) and (14) we obtain that

$$\pm\frac{a}{2} - \varepsilon \leq T(g)(y_{n'}) \leq \pm\frac{a}{2} + \varepsilon.$$

Thus we have

$$|T(g)(y_{n'})| \geq \frac{a}{2} - \varepsilon > \varepsilon \text{ and } |T(g)(y_n)| \leq \varepsilon.$$

Since $T(g) \in C(Y)$, $4\varepsilon < a \leq 1$ fixed and $d_2(y_{n'}, y_n) \to 0$ when $n, n' \to \infty$, this clearly gives a contradiction for n, n' large enough. Hence $\{x_n\}$ is convergent. □

Definition 2.4 ([9])**.** We say $y \in A_x$ if there exist sequences $\{f_n\}, \{x_n\}, \{y_n\}$ satisfying the conditions in Lemma 2.3 with $x = \lim x_n$ and $y = \lim y_n$.

Lemma 2.5. *The set* $\bigcup_{x \in X} A_x$ *is closed and mapping*

$$\varphi\colon \bigcup_{x \in X} A_x \to X, \;\; \varphi(y) = x, \;\; y \in A_x$$

is well-defined and continuous.

Proof. The proof of Lemma is same as G.M. Lövblom's [9] although the two definitions of isometry is different. □

Lemma 2.6. *Let* $y \in A_x$ *and let* $\{f_{kn}\}, \{x_{kn}\}$ *and* $\{y_{kn}\}$ *be any collection of sequences satisfying the conditions in Lemma 2.3. Then*

$$\lim_{n \to \infty} \operatorname{sign} T(af_n)(y_n) = \operatorname{sign} T(\frac{a}{2})(y).$$

Proof. For each y_n we have $\operatorname{sign} T(af_n)(y_n) = \operatorname{sign} T(\frac{a}{2})(y_n)$, and $|T(\frac{a}{2})(y)| > \varepsilon$. Indeed, by definition we have

$$|T(af_n)(y_n)| \geq 2a - \varepsilon - |T(-af_n)(y_n)| \geq a - \varepsilon \tag{15}$$

and by $\|\frac{a}{2} - af_n\| = \frac{a}{2}$ we get

$$T(af_n)(y_n) - \frac{a}{2} \leq T(\frac{a}{2})(y_n) \leq T(af_n)(y_n) + \frac{a}{2}. \tag{16}$$

Hence

$$T(\frac{a}{2})(y_n) \geq a - \varepsilon - \frac{a}{2} > \varepsilon, \text{ if } T(af_n)(y_n) \geq 0.$$

Similarly,

$$T(\frac{a}{2})(y_n) \leq -a + \varepsilon + \frac{a}{2} < -\varepsilon, \text{ if } T(af_n)(y_n) \leq 0.$$

Thus

$$\lim_{n\to\infty} \operatorname{sign} T(af_n)(y_n) = \operatorname{sign} T(\frac{a}{2})(y).$$

□

Lemma 2.7 ([9]). *Let $f_1, f_2 \in B_{1-\frac{a}{2}}(C(X)), x_0 \in X$ and*

$$\|f_1 - f_2\| = |f_1(x_0) - f_2(x_0)|$$

and $d > 0$ be such that $|f_i(x) - f_i(x_0))| \leq a,\ i = 1, 2,\ x \in B(x_0, d)$. For each n, let

$$p_n(x) = \begin{cases} 1 - \dfrac{nd_1(x_0, x)}{d}, & x \in B(x_0, \frac{d}{n}) \\ \min\limits_{1,2}\{1 - f_i(x_0) + f_i(x), 1 - a\}, & \textit{otherwise.} \end{cases}$$

$$q_n(x) = \begin{cases} -1 + \dfrac{nad_1(x_0, x)}{d}, & x \in B(x_0, \frac{d}{n}) \\ \max\limits_{1,2}\{-1 - f_i(x_0) + f_i(x), -1 + a\}, & \textit{otherwise.} \end{cases}$$

$$r_n(x) = \begin{cases} 1 - \dfrac{nd_1(x_0, x)}{d}, & x \in B(x_0, \frac{d}{n}) \\ 0, & \textit{otherwise.} \end{cases}$$

Then

$$\|f_i - p_n\| \to 1 - f_i(x_0)\ \ (n \to \infty),$$
$$\|f_i - q_n\| \to 1 + f_i(x_0)\ \ (n \to \infty),$$
$$\|p_n - ar_n\| = 1 - a,$$
$$\|q_n + ar_n\| = 1 - a.$$

Lemma 2.8. *Given $x_0 \in X$, let $f_1, f_2 \in B_{1-\frac{a}{2}}(C(X))$, and*

$$\|f_1 - f_2\| = |f_1(x_0) - f_2(x_0)|.$$

Then there exists a signal function $s\colon \bigcup\limits_{x\in X} A_x \to \{-1, 1\}$ and $y_0 \in \varphi^{-1}(x_0)$ such that $|T(f_i)(y_0) - s(y_0)f_i(x_0)| \leq \varepsilon$, $i = 1, 2$.

Proof. Let $K = \bigcup\limits_{x\in X} A_x$, $s(y) = \operatorname{sign} T(\frac{a}{2})(y)$ on K and let $x_0 \in X$, $f_1, f_2 \in B_{1-\frac{a}{2}}(C(X))$ such that $\|f_1 - f_2\| = |f_1(x_0) - f_2(x_0)|$ and p_n, q_n, r_n are the functions in Lemma 2.7. Clearly, $p_n, q_n \in B_1(C(X))$ and $\|p_n - q_n\| = 2$.

Because T is the $\varepsilon-$isometry, there exist $y_n \in Y$ for every n such that

$$2 - \varepsilon \leq |T(p_n)(y_n) - T(q_n)(y_n)| \leq 2. \tag{17}$$

The sequence $\{y_n\}$ contains a convergent subsequence, say $y_n \to y_0$. We shall now prove that

$$y_0 \in \varphi^{-1}(x_0) = A_{x_0}.$$

Since r_n is a tentfunction at x_0, $\frac{d}{n} \to 0$ and $y_n \to y_0$ we have $y_0 \in \varphi^{-1}(x_0) = A_{x_0}$ if we can prove that $-|T(ar_n)(y_n) - T(-ar_n)(y_n)| \le -2a + \varepsilon$.

Assume that $T(p_n)(y_n) \ge T(q_n)(y_n)$. By (17)we obtain

$$2 - \varepsilon \le T(p_n)(y_n) - T(q_n)(y_n),$$

therefore

$$\begin{aligned} -|T(ar_n)(y_n) - T(-ar_n)(y_n)| &\le T(-ar_n)(y_n) - T(ar_n)(y_n) \\ &\le T(-ar_n)(y_n) - T(q_n)(y_n) + T(p_n)(y_n) \\ &\quad - T(ar_n)(y_n) + T(q_n)(y_n) - T(p_n)(y_n) \\ &\le 1 - a + 1 - a + \varepsilon - 2 = -2a + \varepsilon. \end{aligned}$$

Thus $y_0 \in \varphi^{-1}(x_0) = A_{x_0}$.

The case $T(p_n)(y_n) \le T(q_n)(y_n)$ is proved similarly. We shall now prove that

$$|T(f_i)(y_0) - s(y_0)f_i(x_0))| \le \varepsilon,\ i = 1, 2.$$

$|T(p_n)(y_n)| \le 1$ and $|T(q_n)(y_n)| \le 1$ imply

$$\left\{ \begin{array}{l} 1 - \varepsilon \le T(p_n)(y_n) \le 1, \\ -1 \le T(q_n)(y_n) \le \varepsilon - 1. \end{array} \right. \tag{18}$$

or

$$\left\{ \begin{array}{l} -1 \le T(p_n)(y_n) \le \varepsilon - 1, \\ 1 - \varepsilon \le T(q_n)(y_n) \le 1. \end{array} \right. \tag{19}$$

One can easily check that $\operatorname{sign} T(p_n)(y_n) = \operatorname{sign} T(ar_n)(y_n)$. In fact, since

$$\|p_n - ar_n\| = 1 - a,$$

then

$$|T(p_n)(y_n) - T(ar_n)(y_n)| \le 1 - a.$$

From (18) and (19) we see

if $T(p_n)(y_n) \ge 1 - \varepsilon$,

$$T(ar_n)(y_n) \ge a - 2\varepsilon > 0. \tag{20}$$

if $T(p_n)(y_n) \le -1 + \varepsilon$,

$$T(ar_n)(y_n) \le -a + 2\varepsilon < 0. \tag{21}$$

By Lemma 2.6, $s(y_0) = \lim_{n\to\infty} \operatorname{sign} T(p_n)(y_n)$, so for n large enough we have

$$s(y_0) = \operatorname{sign} T(p_n)(y_n). \tag{22}$$

Hence for n large enough those inequalities can be rewritten in the form

$$1 \ge s(y_0)T(p_n)(y_n) \ge 1 - \varepsilon$$
$$-1 + \varepsilon \ge s(y_n)T(q_n)(y_n) \ge -1.$$

From Lemma 2.8 we obtain

$$\begin{aligned} -\varepsilon(n, f_i) + T(p_n)(y_n) - \varepsilon - (1 - f_i(x_0)) &\le T(f_i)(y_n) \\ &\le 1 - f_i(x_0) + T(p_n)(y_n) + \varepsilon(n, f_i); \\ -\varepsilon(n, f_i) + T(q_n)(y_n) - \varepsilon - (1 + f_i(x_0)) &\le T(f_i)(y_n) \\ &\le 1 + f_i(x_0) + T(q_n)(y_n) + \varepsilon(n, f_i), \end{aligned}$$

where $\varepsilon(n, f_i) \to 0$ when $n \to \infty$. Hence for n large enough we have

$$-\varepsilon(n, f_i) - \varepsilon + s(y_0)f_i(x_0)) \le T(f_i)(y_n) \le \varepsilon + s(y_0)f_i(x_0)) + \varepsilon(n, f_i).$$

Letting $n \to \infty$ we obtain

$$|T(f_i)(y_0) - s(y_0)f_i(x_0))| \le \varepsilon.$$

The proof is complete. □

Before the proof of the Theorem 2.1, we recall the famous *Michael Selected Theorem* [7]. Suppose that Ω is a paracompact and X is a Banach space, if F is a lower-semi-continuous multi-valued function on Ω, and $f(t)$ $(\forall t \in \Omega)$ is a closed convex set of X, then there exists a continuous function f satisfies $f(t) \in F(t)$ $(t \in \Omega)$.

The proof of Theorem 2.1. Let φ and s be as above. Since

$$s\colon K = \bigcup_{x \in X} A_x \to \{-1, 1\}$$

and K is closed we can find, by *Urysohn*'s Lemma, a continuous function

$$\bar{s}\colon Y \to [-1, 1]$$

with $\bar{s}|_K = s$.

Now, let $M_1(X) = B_1(C(X))^*$ be the unit ball of the Radon measure space on X endowed with the *weak*$*$-topology. Define a set valued map on Y, $\Psi\colon Y \to 2^{M_1}(X)$ by

$$\Psi(y) = \begin{cases} s(y)\delta_{\varphi(y)}, & y \in K, \\ \{\bar{s}(y)\mu, \ \mu \text{ is the probability measure of } M_1(X), & y \in Y \backslash K. \end{cases}$$

Clearly $\Psi(y)$ is a closed and convex subset of $M_1(X)$ for all $y \in Y$. Furthermore, we can check that the set is the $w*$- lower-semi-continuous.

Assume that $y_n \to y$ when $n \to \infty$ and $\nu \in \Psi(y)$. Thus

$$\nu = \begin{cases} s(y)\delta_{\varphi(y)}, & y \in K, \\ \bar{s}(y)\mu, \ \mu \text{ is some probability measure of } M_1(X)\ , & y \in Y \backslash K. \end{cases}$$

Let

$$\nu_n = \begin{cases} s(y_n)\delta_{\varphi(y_n)}, & y_n \in K, \\ \bar{s}(y_n)\mu', & y_n \in Y \setminus K. \end{cases}$$

Where

$$\mu' = \begin{cases} \delta_{\varphi(y)}, & y \in K, \\ \mu, & y \in Y \setminus K \end{cases}$$

is the probability measure of $M_1(X)$, hence $\nu_n \in \varphi_n(y_n)$.

We shall now prove that $\nu_n \stackrel{w^*}{\to} \nu$ when $n \to \infty$.

(1) If $y \in K$ and there is a subsequence $\{y_n\} \subset K$, φ is continuous implies $\delta_{\varphi(y_n)} \stackrel{w^*}{\to} \delta_{\varphi(y)}$ by $\nu_n \stackrel{w^*}{\to} \nu$ when $n \to \infty$.

(2) If $y \in K$ and there is a subsequence $\{y_n\} \subset Y \backslash K$, then $\nu_n = \bar{s}(y_n)\delta_{\varphi(y)} \stackrel{w^*}{\to} \nu$ when $n \to \infty$.

(3) If $y \notin K$, since $Y \backslash K$ is an open set, then it is necessary there exists N such that $y_n \in Y \backslash K$ for $n > N$, hence $\nu_n = \bar{s}(y_n)\mu \stackrel{w^*}{\to} \nu$ when $n \to \infty$.

We can find, by *Michael Selected Theorem*, a w^*- continuous function

$$\tilde{\Psi}\colon Y \to M_1(X),$$

satisfies $\tilde{\Psi}(y) \in \Psi(y)$. Furthermore we have that $\tilde{\Psi}(y) = s(y)\delta_{\varphi(y)}$ for all $y \in K$.

Now, for any $y \in Y, f \in B_{1-\frac{a}{2}}(C(X))$ define a map by

$$U(f)(y) = \sup\{\inf\{\tilde{\Psi}(y)(f),\ T(f)(y) + \varepsilon\}, T(f)(y) - \varepsilon\}.$$

Clearly $|T(f)(y) - \tilde{\Psi}((y)(f)| \le \varepsilon$ if and only if $U(f)(y) = \tilde{\Psi}(y)(f)$.

Since $\tilde{\Psi}(y)$ is w^*- continuous, we have $U(f)(y)$ is continuous on Y and hence $U(f) \in C(Y)$. We now prove that U is an isometry and to do this we first show that

$$|U(f_1)(y) - U(f_2)(y)| \le \|f_1 - f_2\|, \ \ \forall y \in Y. \tag{23}$$

(1) If $U(f_i)(y) = \tilde{\Psi}(y)(f_i)$, (23) is true.

(2) If $U(f_i)(y) = T(f_i(y) \mp \varepsilon$, let $U(f_1)(y) = T(f_1(y) - \varepsilon$, and $U(f_2)(y) = T(f_2(y) + \varepsilon$, then by definition of $U(f)(y)$

$$\begin{cases} U(f_1)(y) \geq \tilde{\Psi}(y)(f_1), \\ U(f_2)(y) \leq \Psi(y)(f_2). \end{cases}$$

Hence

$$U(f_2)(y) - U(f_1)(y) \leq \tilde{\Psi}(y)(f_2 - f_1) \leq \|f_1 - f_2\|,$$
$$U(f_1)(y) - U(f_2)(y) \leq \|f_1 - f_2\|.$$

(3) If $U(f_1)(y) = \tilde{\Psi}(y)(f_1)$ and $U(f_2(y)) = T(f_2(y)) + \varepsilon$, then

$$U(f_1)(y) \leq T(f_1(y) + \varepsilon,$$

thus

$$U(f_2)(y) - U(f_1)(y) \leq \tilde{\Psi}(y)(f_2 - f_1) \leq \|f_1 - f_2\|,$$
$$U(f_1)(y) - U(f_2)(y) \leq T(f_1)(y) + \varepsilon - T(f_2)(y) - \varepsilon \leq \|f_1 - f_2\|.$$

(4) The case $U(f_1)(y) = \tilde{\Psi}(y)(f_1)$, $U(f_2)(y) = T(f_2)(y) - \varepsilon$ is proved similarly.

Now we shall prove that

$$\|U(f_1) - U(f_2)\| \geq \|f_1 - f_2\|. \tag{24}$$

Given $x_0 \in X$ such that $\|f_1 - f_2\| = |f_1(x_0) - f_2(x_0)|$, then by Lemma 2.8, we can find a point $y_0 \in \varphi^{-1}(x_0) = A_{x_0} \subset K$ such that

$$|T(f_i)(y_0) - s(y_0)f_i(x_0)| \leq \varepsilon$$

and $s(y_0)f_i(x_0) = s(y_0)\delta_{\varphi(y_0)}f_i = \tilde{\Psi}(y_0)(f_i)$. Thus $U(f_i)(y_0) = \tilde{\Psi}(y_0)(f_i)$. Hence

$$\|U(f_1) - U(f_2)\| \geq |s(y_0)f_1(x_0) - s(y_0)f_2(x_0)| = \|f_1 - f_2\|.$$

Furthermore, for any $f \in B_{1-\frac{a}{2}}(C(X))$ we have

$$\|T(f) - U(f)\| \leq \varepsilon$$

(1) $U(f)(y) = \tilde{\Psi}(y)f$ is equivalent to

$$|T(f)(y) - U(f)(y)| \leq \varepsilon,$$

(2) If $U(f)(y) = T(f)(y) \pm \varepsilon$, clearly,

$$\|T(f) - U(f)\| \leq \varepsilon$$

and the proof is complete. □

3. The Counterexamples for $\varepsilon-$ Isometric Approximate Problem.

Theorem 3.1. *Let $M \geq 3$, and any $\varepsilon > 0$. Then there exists an ε-isometry*

$$T\colon B_1(l_1) \to B_1(l_1)$$

such that for any isometry U which defines on some subset of l_1 that contains $B_{\frac{6}{M}}(l_1)$, it is necessary to have $x \in B_{\frac{3}{M}}(l_1)$ with $\|Tx - Ux\| \geq \frac{2}{M^2}$.

Theorem 3.2. *Let $M \geq 3$, for any $\varepsilon > 0$, then there exists an ε-isometry*

$$T\colon B_1((L_1(0,1) \times R)_1) \to B_1((L_1(0,1) \times R)_1)$$

such that for any isometry U which defines on some subset of $(L_1(0,1) \times R)_1)$ that contains $B_{\frac{6}{M}}((L_1(0,1) \times R)_1)$, it is necessary to have $x \in B_{\frac{3}{M}}((L_1(0,1) \times R)_1)$ with $\|Tx - Ux\| \geq \frac{2}{M^2}$ (where $\|(f,r)\| = \|f\|_{L_1} + |r|)$ is the norm of $(L_1(0,1) \times R)_1$.

Lemma 3.3 ([10])**.** *Let* $n \in \mathbf{N}$, $\varepsilon = \frac{1}{n}$ *and* $a \in l_1$, $S_a = \{1, 2, \ldots, n\} \bigcap \mathrm{supp}(a)$. *Let*

$$T_1(a) = \begin{cases} a, & a_{n+1} < 0, \\ a + \frac{a_{n+1}}{M}(\varepsilon \sum e_i - e_{n+1}), & a_{n+1} > 0. \end{cases}$$

For any $a, b \in l_1$, *if* $\mathrm{card}(S_a), \mathrm{card}(S_b) \le \frac{M}{2}$ *and* $a_i, b_i \ge 0$, $(1 \le i \le n)$. *Then*

1) if $a_{n+1}, b_{n+1} \le 0$, *then*

$$\|T_1(a) - T_1(b)\| = \|a - b\|,$$

2) if $a_{n+1}, b_{n+1} \ge 0$, *then*

$$\|a - b\| \ge \|T_1(a) - T_1(b)\| \ge \|a - b\| - 2\varepsilon(\mathrm{card}(S_b))\frac{(a_{n+1} - b_{n+1})}{M},$$

3) if $a_{n+1} \ge 0 \ge b_{n+1}$, *then*

$$\|a - b\| \ge \|T_1(a) - T_1(b)\| \ge \|a - b\| - 2\varepsilon \sum_{S_b} \frac{a_{n+1}}{M}.$$

Furthermore

$$\|a - b\| \ge \|T_1(a) - T_1(b)\| \ge \|a - b\|(1 - \varepsilon).$$

Lemma 3.4 ([10])**.** *Let* $n \in \mathbf{N}$, $\varepsilon = \frac{1}{n}$, $a \in l_1$, *and* $S_a = \{1, 2, \ldots, n\} \bigcap \mathrm{supp}(a)$. *Let* $T_2(a) = \sum_{i=1}^{\infty} T_2(a_i e_i)$, *where*

$$T_2(a_i e_i) = \begin{cases} a_i e_{n+1+i}, & i \le n \text{ and } a_i < \frac{2}{M}, \\ (a_i - \frac{2}{M})e_i + (\frac{2}{M})e_{n+1+i}, & i \le n \text{ and } a_i \ge \frac{2}{M}, \\ a_{n+1} e_{n+1}, & i = n+1, \\ a_i e_{n+1}, & i > n+1. \end{cases} \tag{25}$$

Then T_2 *is an isometry and if* $a \in B_1(l_1)$, *then* $(T_2(a))_i \ge 0$,$1 \le i \le n$ *and* $\mathrm{card}(S_{T_2(a)}) \le \frac{M}{2}$.

Lemma 3.5 ([10])**.** *Let* T_1, T_2 *satisfy the conditions of the Lemma 3.3 and Lemma 3.4 and* $T = T_1 \circ T_2$. *Then for any isometry* U *which defines on some subset of* l_1 *that contains* $B_{\frac{6}{M}}(l_1)$, *it's necessary to have* $x \in B_{\frac{3}{M}}(l_1)$, *with* $\|Tx - Ux\| \ge \frac{2}{M^2}$.

Proof of Theorem 3.1. We should only show that T is an $\varepsilon-$isometry on $B_1(l_1)$. By Lemma 3.3 T_2 is an isometry, and if $a \in B_1(l_1)$, then $(T_2(a))_i \ge 0$, and $\mathrm{card}(S_{T_2}(a)) \le \frac{M}{2}$.
if $\mathrm{card}(S_a), \mathrm{card}(S_b) \le \frac{M}{2}$, and $a_i, b_i \ge 0$, then

$$\|a - b\| \ge \|T_1(a) - T_1(b)\| \ge \|a - b\| - \varepsilon. \tag{26}$$

Directly by Lemma 3.4 we get
(1) If $a_{n+1}, b_{n+1} \le 0$,

$$|T_1(a) - T_1(b)\| = \|a - b\|,$$

(2) If $a_{n+1} \ge b_{n+1} \ge 0$,

$$\|a - b\| \ge |T_1(a) - T_1(b)\| \ge \|a - b\| - 2\varepsilon(\mathrm{card}(s_b))\frac{(a_{n+1} - b_{n+1})}{M}$$

Since $\mathrm{card}(s_b) \le \frac{M}{2}$, clearly, $a_{n+1} - b_{n+1} \le a_{n+1} < 1$.
(3) If $a_{n+1} \ge 0 \ge b_{n+1}$,

$$\|a - b\| \ge \|T_1(a) - T_1(b)\| \ge \|a - b\| - 2\sum_{S_b} \varepsilon \frac{a_{n+1}}{M} \ge \|a - b\| - \varepsilon,\ (a_{n+1} \le 1).$$

Thus T is an ε-isometry. By Lemma 3.5, for any isometry U which defines on subset of l_1 that contains $B_{\frac{6}{M}}(l_1)$. It is necessary to have $x \in B_{\frac{3}{M}}(l_1)$ with $\|Tx - Ux\| \geq \frac{2}{M^2}$ □

Remark 3.6. The proof for Theorem 3.2 is gotten by revising Lövblom's [10] method.

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