Acta Mathematica Academiae Paedagogicae Nyíregyháziensis
19 (2003), 89–100
www.emis.de/journals

FIXED POINTS AND GENERALIZED VECTOR EQUILIBRA IN GENERAL TOPOLOGICAL SPACES

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ABSTRACT. In this paper, we first generalize a fixed point theorem due to Tarafdar from topological vector spaces to general topological spaces without linear structure. By using our new fixed point theorem, some new existence theorems of solutions for generalized vector equilibrium problems and generalized implicit vector variational inequality problems are proved under noncompact settings of general topological spaces. Some special cases of our results are also discussed.

1. INTRODUCTION

Let X, Y and Z be topological spaces. Let $F\colon X \times X \to 2^Z$, $P\colon X \to 2^Z$ and $T\colon X \to 2^Y$ be set-valued mappings where 2^Z denotes the family of all nonempty subsets of Z and let $\phi\colon Y \times X \times X \to Z$ be a single-valued mapping. Then, we consider the *generalized vector equilibrium problem* (in short, GVEP) which is to find $\hat{x} \in X$ such that

$$(1) \qquad F(\hat{x}, y) \not\subseteq P(\hat{x}), \qquad \forall y \in X.$$

Let $f\colon X \times X \to Z$ is a single-valued mapping. If $F(x, y) = \{f(x, y)\}$, then the GVEP (1) reduces to the following *vector equilibrium problem* (in short, VEP) which is to find $\hat{x} \in X$ such that

$$(2) \qquad f(\hat{x}, y) \notin P(\hat{x}), \qquad \forall y \in X.$$

We also consider the following generalized implicit vector variational inequality problem (in short, GIVVIP) which is to find $\hat{x} \in X$ such that

$$(3) \qquad \text{for each } y \in X, \text{ there exists } \hat{s} \in T\hat{x} \text{ satisfying } \phi(\hat{s}, \hat{x}, y) \notin P(\hat{x}).$$

If $T\colon X \to Y$ is a single-valued mapping, then the GIVVIP (3) reduces to the following implicit vector variational inequality problem (in short, IVVIP) which is to find $\hat{x} \in X$ such that

$$(4) \qquad \phi(T(\hat{x}), \hat{x}, y) \notin P(\hat{x}), \qquad \forall y \in X.$$

Clearly, If we define a single-valued mapping $f\colon X \times X \to Z$ and a set-valued mapping $F\colon X \times X \to 2^Z$ by

$$f(x, y) = \phi(Tx, x, y), \qquad \forall (x, y) \in X \times X,$$

2000 *Mathematics Subject Classification.* 91B50.

Key words and phrases. Fixed point, generalized vector equilibria, contractible, topological space.

The first author was supported by the NNSF of China (19871059), the KOSEF of Korea and the NSF of Educational Department of Sichuan Province ([2000] 25). The second author was supported by the Korean Research Foundation, Project No 1998-15-D00021.

and

$$F(x,y) = \phi(Tx,x,y) = \bigcup_{s \in Tx} \phi(s,x,y), \qquad \forall (x,y) \in X \times X,$$

then each solution of the VEP (2) is a solution of the IVVIP (4) and each solution of the GVEP (1) is a solution of the GIVVIP (3).

If X is a nonempty convex subset of a topological vector space, Z is a topological vector space and for each $x \in X$, $P(x) = -\operatorname{int} C(x)$ or $P(x) = \operatorname{int} C(x)$ where $C(x)$ is a closed convex cone in Z with $\operatorname{int} C(x) \neq \emptyset$ and $C(x) \neq Z$, and $\operatorname{int} C(x)$ denotes the interior of $C(x)$, then the GVEP (1) and the VEP (2) was introduced and studied by Ansari et. al. [2], Ansari[1], Oettli and Schlöger [18, 19], Ansari and Yao [3], Song [21] and so on.

If X is a nonempty convex subset of a Hausdorff topological vector space E and Z is another Hausdorff topological vector space, $Y = L(E,Z)$ is the space of all continuous linear mappings from E into Z, then the GIVVIP (3) was introduced and studied by Lee and Kim [17].

The GVEP (1), VEP (2), GIVVIP (3) and IVVIP (4) include many classes of vector and generalized vector equilibrium problems, vector and generalized vector variational inequality problems as special cases (see, for example, [2]–[14] and references therein).

In the paper, we shall employ the fixed point technique to establish some new existence theorems of solutions for the GVEP (1) and the GIVVIP (3) in general noncompact topological spaces without linear structure. For this reason, we shall first obtain a generalization of the Tarafdar's fixed point theorem in [22] from topological vector spaces to general topological spaces without linear structure. By using our new fixed point theorem, some new existence theorems of solutions for the GVEP (1) and GIVVIP (3) without any monotonicity assumptions are proved in general noncompact topological spaces. Some special cases of our results are also discussed.

2. Fixed point theorems

In 1987, Tarafdar [22] proved a Fan – Browder type fixed point theorem equivalent Fan – Knaster – Kuratowski – Mazurkiewicz theorem in topological vector spaces. Since then some applications of this fixed point theorem in many different fields have given by many authors. In this section, we generalize this fixed point theorem from topological vector space to general noncompact topological space without linear structure.

A topological space X is said to be *contractible* if the identity mapping I_X on X is homotopic to a constant function. In particular, any convex set or star-shaped set in a topological vector space is contractible.

Let X be a topological space. A subset A of X is said to be compactly open (resp., compactly closed) in X if for each compact subset K of X, $A \bigcap K$ is open (resp., closed) in K. It is clear that each open subset of X must be compactly open in X, but the inverse is not true in general. Define the compact interior of A, denoted by $\operatorname{cint} A$, as

$$\operatorname{cint} A = \bigcup \{B \subseteq X : B \subseteq A, B \text{ is compactly open in } X\}.$$

It is easy to see that A is compactly open if and only if $A = \operatorname{cint} A$.

It is well known that a subset of a topological space X is called a *k-test set* if its intersection with each nonempty compact subset K of X is closed in K. A topological space X is called a *k-space* if each k-test set is closed in X (or equivalently, a subset B of X is open in X if and only if B is compactly open in X). For example, see Wilansky [24, p. 142] or Dugunji [13, p. 248] or Husain [15, p. 171–172]. However, there exist topological spaces which are not k-spaces. Indeed, the topological vector space $\mathbf{R}^{\mathbf{R}}$ is not a k-space, see Kelley [16, p. 240] or Wilansky [24, p. 143]. The product of two k-spaces need not be a k-space, see Husain [15, p. 174]. Hence the notions of the compactly open sets and the compact interior of a set both are true extension of the notions of the open sets and the interior of a set in a general topological space.

For a nonempty set X, we shall denote by $\mathcal{F}(X)$ the family of all nonempty finite subsets of X. The following result is a special case of Theorem 2.1 of Ding [11].

Theorem 2.1. *Let X be a topological space, K be a nonempty compact subset of X, and $G\colon X \to 2^X$ be a set-valued mapping such that*

(i) *for each nonempty compact subset D of X, $D = \bigcup_{y\in X}(\operatorname{cint} G^{-1}(y) \bigcap D)$,*
(ii) *for each $N \in \mathcal{F}(X)$, there exists a nonempty compact contractible subset L_N of X containing N such that for each compactly open subset U of X, the set $\bigcap_{x\in U}(G(x)\bigcap L_N)$ is empty or contractible, and*

$$L_N \setminus K \subseteq \bigcup_{y\in L_N} \operatorname{cint} G^{-1}(y).$$

Then there exists a point $\hat{x} \in X$ such that $\hat{x} \in G(\hat{x})$.

The following result is an equivalent variant of Theorem 2.1.

Theorem 2.2. *Let X be a topological space, K be a nonempty compact subset of X, and $H, G\colon X \to 2^X$ be two set-valued mappings such that*

(i) *for each $x \in X$, $H(x) \subseteq G(x)$,*
(ii) *for each compact subset D of X, $D = \bigcup_{y\in X}(\operatorname{cint} H^{-1}(y) \bigcap D)$,*
(iii) *for each $N \in \mathcal{F}(X)$, there exists a nonempty compact contractible subset L_N of X containing N such that for each compactly open subset U of X, the set $\bigcap_{x\in U}(G(x)\bigcap L_N)$ is empty or contractible, and*

$$L_N \setminus K \subseteq \bigcup_{y\in L_N} \operatorname{cint} H^{-1}(y).$$

Then there exists a point $\hat{x} \in X$ such that $\hat{x} \in G(\hat{x})$.

Proof. By (i), we have $H^{-1}(y) \subseteq G^{-1}(y)$ for each $y \in X$ and hence $\operatorname{cint} H^{-1}(y) \subseteq \operatorname{cint} G^{-1}(y)$ for each $y \in X$. The condition (ii) implies that for each nonempty compact subset D of X, $D = \bigcup_{y\in X}(\operatorname{cint} G^{-1}(y) \bigcap D)$. The condition (iii) implies $L_N \setminus K \subseteq \bigcup_{y\in L_N} \operatorname{cint} G^{-1}(y)$. All conditions of Theorem 2.1 is satisfied. By Theorem 2.1, the conclusion of Theorem 2.2 holds.

Remark 2.1. Clearly, if $H = G$ then Theorem 2.2 reduces to Theorem 2.1.

Corollary 2.1. *Let X be a compact contractible space and $H, G\colon X \to 2^X$ be two set-valued mappings such that*

(i) *$H(x) \subseteq G(x)$ for each $x \in X$,*
(ii) *$X = \bigcup_{y\in X} \operatorname{cint} H^{-1}(y)$,*
(iii) *for each open subset U of X, the set $\bigcap_{x\in U} G(x)$ is empty or contractible.*

Then there exists a point $\hat{x} \in X$ such that $\hat{x} \in G(\hat{x})$.

Proof. Note that X is a compact contractible space, by letting $L_N = K = X$ for each $N \in \mathcal{F}(X)$, it is easy to see that all conditions of Theorem 2.2 are satisfied. The conclusion of Corollary 2.1 holds from Theorem 2.2.

Remark 2.2. In Theorem 2.2 and Corollary 2.1, if the mappings H and G are replaced by H^{-1} and G^{-1}, respectively, then their conclusions still hold. Therefore Theorem 2.2 and Corollary 2.1 improve and generalize Theorem 4 of Park and Jeong [20] and Corollary 2.2 of Ding [8] to general noncompact topological space. For the related results, the readers may consult Tarafdar and Yuan [23] and Ding [5]–[10].

Now, by applying Theorem 2.2 and Corollary 2.1, we can obtain the following new generalization of Tarafdar's fixed point theorem in [22].

Theorem 2.3. *Let X be a topological space, $H, G\colon X \to 2^X$ be two set-valued mappings such that*

(i) *$H(x) \subseteq G(x)$ for each $x \in X$,*
(ii) *for each compact subset D of X, $D = \bigcup_{y\in Y}(\operatorname{cint} H^{-1}(y) \bigcap D)$,*
(iii) *there exists a nonempty set $X_0 \subset X$ such that for each $N \in \mathcal{F}(X)$, there is a compact contractible subset L_N of X containing $X_0 \bigcup N$ such that for each compactly open subset U of X, the set $\bigcap_{x\in U}(G(x) \bigcap L_N)$ is empty or contractible, and the set $K = \bigcap_{y\in X_0}(\operatorname{cint} H^{-1}(y))^c$ is empty or compact where $(\operatorname{cint} H^{-1}(y))^c$ denotes the complement of the set $\operatorname{cint} H^{-1}(y)$.*

Then there exists a point $\hat{x} \in X$ such that $\hat{x} \in G(\hat{x})$.

Proof. Case (I). We first assume that $K = \bigcap_{y\in X_0}(\operatorname{cint} H^{-1}(y))^c$ is empty. For a given $N_0 \in \mathcal{F}(X_0)$, by the condition (iii), there exists a compact contractible subset L_{N_0} of X containing X_0 such that for each compactly open subset U of X, the set $\bigcap_{x\in U}(G(x) \bigcap L_{N_0})$ is empty or contractible. Now we claim that for each $x \in L_{N_0}$, $H(x) \bigcap L_{N_0} \neq \emptyset$. Indeed, if $H(x_0) \bigcap L_{N_0} = \emptyset$ for some $x_0 \in L_{N_0}$, then $y \notin H(x_0)$ for all $y \in L_{N_0}$ and hence $x_0 \notin H^{-1}(y) \supseteq \operatorname{cint} H^{-1}(y)$ for all $y \in L_{N_0}$. It follows from $X_0 \subseteq L_{N_0}$ that $x_0 \in \bigcap_{y\in L_{N_0}}(\operatorname{cint} H^{-1}(y))^c \subseteq \bigcap_{y\in X_0}(\operatorname{cint} H^{-1}(y))^c = K$ which contradicts the fact $K = \emptyset$. Hence we can define two set-valued mappings $H^*, G^* : L_{N_0} \to 2^{L_{N_0}}$ by

$$H^*(x) = H(x) \bigcap L_{N_0} \text{ and } G^*(x) = G(x) \bigcap L_{N_0}, \qquad \forall x \in L_{N_0},$$

such that $H^*(x) \neq \emptyset$ and $H^*(x) \subseteq G^*(x)$ for each $x \in L_{N_0}$. For each $y \in L_{N_0}$, we have

$$\begin{aligned} (H^*)^{-1}(y) &= \{x \in L_{N_0} : y \in H^*(x)\} = \{x \in L_{N_0} : y \in H(x) \bigcap L_{N_0}\} \\ &= \{x \in L_{N_0} : y \in H(x)\} = H^{-1}(y) \bigcap L_{N_0}. \end{aligned}$$

From $\emptyset = K = \bigcap_{y\in X_0}(\operatorname{cint} H^{-1}(y))^c \supset \bigcap_{y\in L_{N_0}}(\operatorname{cint} H^{-1}(y))^c$, we obtain

$$\begin{aligned} L_{N_0} &= L_{N_0} \setminus \bigcap_{y\in L_{N_0}} (\operatorname{cint} H^{-1}(y))^c = \bigcup_{y\in L_{N_0}} (\operatorname{cint} H^{-1}(y) \bigcap L_{N_0}) \\ &= \bigcup_{y\in L_{N_0}} \operatorname{cint}_{L_{N_0}}(H^{-1}(y) \bigcap L_{N_0}) = \bigcup_{y\in L_{N_0}} \operatorname{cint}(H^*)^{-1}(y). \end{aligned}$$

Note L_{N_0} is a compact subset of X, hence each open subset U^* of L_{N_0} is also a compactly open subset of X. By the condition (iii), we have the set $\bigcap_{x\in U^*} G^*(x) = \bigcap_{x\in U^*}(G(x) \bigcap L_{N_0})$ is empty or compact. Since L_{N_0} is also a compact contractible space, H^* and G^* satisfy all conditions of Corollary 2.1. By Corollary 2.1, there exists a point $\hat{x} \in L_{N_0} \subset X$ such that $\hat{x} \in G^*(\hat{x}) \subseteq G(\hat{x})$.

Case (II). Now assume $K = \bigcap_{y\in X_0}(\operatorname{cint} H^{-1}(y))^c$ is nonempty and compact. By the condition (iii), for each $N \in \mathcal{F}(X)$, there exists a compact contractible subset

L_N of X containing $X_0 \bigcup N \supseteq N$ such that

$$\begin{aligned} L_N \setminus K &= L_N \setminus \bigcap_{y \in X_0} (\operatorname{cint} H^{-1}(y))^c = \bigcup_{y \in X_0} (\operatorname{cint} H^{-1}(y) \bigcap L_N) \\ &\subseteq \bigcup_{y \in L_N} \operatorname{cint} H^{-1}(y). \end{aligned}$$

Hence, by Theorem 2.2, there exists a point $\hat{x} \in X$ such that $\hat{x} \in G(\hat{x})$. This completes the proof.

Corollary 2.2. *Let X be a nonempty convex subset of a topological vector space. Let $H, G\colon X \to 2^X$ be such that*

(i) *for each $x \in X$, $H(x) \subseteq G(x)$, and $G(x)$ is convex,*
(ii) *for each $y \in X$, $H^{-1}(y)$ contains a relatively open subset O_y of X (O_y may be empty for some $y \in X$) such that for each compact subset D of X, $D = \bigcup_{y \in X}(O_y \bigcap D)$,*
(iii) *there exists a nonempty set $X_0 \subset X$ such that X_0 is contained in a compact convex subset X_1 of X and the set $K = \bigcap_{y \in X_0} O_y^c$ is empty or compact.*

Then there exists a point $\hat{x} \in X$ such that $\hat{x} \in G(\hat{x})$.

Proof. By (ii), for each $y \in X$, we have $O_y \subset \operatorname{cint} H^{-1}(y)$ and for each compact subset D of X, $D = \bigcup_{y \in X}(O_y \bigcap D) \subseteq \bigcup_{y \in X}(\operatorname{cint} H^{-1}(y) \bigcap D)$ and hence $D = \bigcup_{y \in X}(\operatorname{cint} H^{-1}(y) \bigcap D)$. Since X_1 is a compact convex subset of X containing X_0, for each $N \in \mathcal{F}(X)$, let $L_N = \operatorname{co}(X_1 \bigcup N)$. Then L_N is also a compact convex subset of X containing $X_0 \bigcup N$. Note that each nonempty convex subset in a topological vector space is contractible and $G(x)$ is convex for each $x \in X$, we have that for each compactly open subset U of X, the set $\bigcap_{x \in U}(G(x) \bigcap L_N)$ is empty or convex, and hence it is empty or contractible. Since $O_y \subset \operatorname{cint} H^{-1}(y)$ for each $y \in X$, we have $K' = \bigcap_{y \in X_0}(\operatorname{cint} H^{-1}(y))^c \subseteq \bigcap_{y \in X_0} O_y^c = K$. So if $K = \emptyset$, then $K' = \emptyset$; if K and K' are both nonempty, then K' is a closed subset of K and hence by (iii), K' is a compact subset of X. Now, the conclusion of Corollary 2.2 follows from Theorem 2.3.

Remark 2.3. Corollary 2.2 is a slightly improving version of Tarafdar's fixed theorem in [22] and hence Theorem 2.3 generalizes this theorem to general noncompact topological spaces. Theorem 2.3 also includes Corollary 2.2 of Ding [8] as a special case.

3. Existence of solutions for GVEP and GIVVIP

In this section, by using Theorem 2.3, we shall prove some new existence theorems of solutions for the GVEP (1) and the GIVVIP (3) without any monotonicity assumptions in general noncompact topological spaces.

Theorem 3.1. *Let X and Z be two topological spaces, $F\colon X \times X \to 2^Z$ and $P\colon X \to 2^Z$ be two set-valued mappings such that*

(i) *the mapping $W\colon X \to 2^Z$ defined by $W(x) = Z \setminus P(x)$ is such that the graph $\operatorname{Gr}(W)$ of W is closed in $X \times Z$,*
(ii) *for each $y \in X$, the mapping $x \mapsto F(x, y)$ is upper semicontinuous with compact values on each compact subset of X,*
(iii) *there exists a set-valued mapping $F^*\colon X \times X \to 2^Z$ such that*
    (a) *for each $x \in X$, $F^*(x, x) \nsubseteq P(x)$,*
    (b) *for each $x, y \in X$, $F(x, y) \subseteq P(x)$ implies $F^*(x, y) \subseteq P(x)$,*

(c) *there exists a nonempty set* $X_0 \subset X$ *such that for each* $N \in \mathcal{F}(X)$, *there is a compact contractible subset* L_N *of* X *containing* $X_0 \bigcup N$ *satisfying that for each compactly open subset* U *of* X, *the set* $\bigcap_{x\in U}\{y \in L_N : F^*(x,y) \subseteq P(x)\}$ *is empty or contractible and the set* $K = \bigcap_{y\in X_0}\{x \in X : F(x,y) \not\subseteq P(x)\}$ *is empty or compact.*

Then the solutions set $S = \bigcap_{y\in X}\{x \in X : F(x,y) \not\subseteq P(x)\}$ *of the GVEP (1) is nonempty and compact in* X.

Proof. We first show that the solutions set S of the GVEP (1) is nonempty. If it is false, then for each $x \in X$, there exists $y \in X$ such that $F(x,y) \subseteq P(x)$. Define set-valued mappings $H, G\colon X \to 2^X$ by

$$H(x) = \{y \in X : F(x,y) \subseteq P(x)\} \text{ and } G(x) = \{y \in X : F^*(x,y) \subseteq P(x)\}$$

for each $x \in X$. Then $H(x) \neq \emptyset$ for each $x \in X$. By (iii)(b), we have $H(x) \subseteq G(x)$. Now we claim that for each $y \in X$, the set $Q(y) = \{x \in X : F(x,y) \not\subseteq P(x)\}$ is compactly closed in X. Indeed, for each fixed $y \in X$ and for any compact subset K of X, let $\{x_\lambda\}_{\lambda\in\Lambda}$ be a net in $Q(y)\bigcap K$ such that $\{x_\lambda\}$ converges to x. Then we have $x \in K$ and $F(x_\lambda, y) \not\subseteq P(x_\lambda)$ and hence there exists $z_\lambda \in F(x_\lambda, y)$ such that $z_\lambda \notin P(x_\lambda)$, or $z_\lambda \in W(x_\lambda)$ for all $\lambda \in \Lambda$. Since K is compact it follows from the condition (ii) and Proposition 3.1.11 of Aubin and Ekeland [4] that the set $\bigcup_{x\in K} F(x,y)$ is compact. Since $\{z_\lambda\} \subseteq \bigcup_{x\in K} F(x,y)$, without loss of generality , we may assume $z_\lambda \to z$. By the upper semicontinuity of $F(\cdot, y)$, we obtain $z \in F(x,y)$. By the condition (i), we have $(x,z) \in \mathrm{Gr}(W)$, i.e., $z \notin P(x)$. Hence $x \in Q(y)\bigcap K$ and $Q(y)$ is compactly closed in X for each $y \in X$. For each $y \in X$, we have

$$H^{-1}(y) = \{x \in X : y \in H(x)\} = \{x \in X : F(x,y) \subseteq P(x)\} = X \setminus Q(y),$$

and hence for each $y \in X$, $H^{-1}(y)$ is compactly open in X. Note $H(x) \neq \emptyset$ for each $x \in X$, we have $X = \bigcup_{y\in X} H^{-1}(y) = \bigcup_{y\in X} \operatorname{cint} H^{-1}(y)$. Hence for each compact subset D of X, we have $D = \bigcup_{y\in X}(\operatorname{cint} H^{-1}(y)\bigcap D)$. The condition (iii)(c) implies that for each compactly open subset U of X, $\bigcap_{x\in U}(G(x)\bigcap L_N)$ is empty or contractible and the set $K = \bigcap_{y\in X_0}\{x \in X : F(x,y) \not\subseteq P(x)\} = \bigcap_{y\in X_0}(H^{-1}(y))^c = \bigcap_{y\in X_0}(\operatorname{cint} H^{-1}(y))^c$ is empty or compact. By Theorem 2.3, there exists a point $\hat{x} \in X$ such that $\hat{x} \in G(\hat{x})$, i.e. , $F^*(\hat{x}, \hat{x}) \subseteq P(\hat{x})$ which contradicts the condition (iii)(a). Hence the solutions set S of the GVEP (1) is nonempty. Since $S = \bigcap_{y\in X}\{x \in X : F(x,y) \not\subseteq P(x)\}$ is a nonempty closed subset of the compact set $K = \bigcap_{y\in X_0}\{x \in X : F(x,y) \not\subseteq P(x)\}$, we must have S is compact. This completes the proof.

By Theorem 3.1, it is easy to see that the following result holds.

Corollary 3.1. *Let* X *and* Z *be two topological spaces,* $f\colon X \times X \to Z$ *be a single-valued mapping and* $P\colon X \to 2^Z$ *be a set-valued mapping such that*

(i) *The mapping* $W\colon X \to 2^Z$ *defined by* $W(x) = Z \setminus P(x)$ *is such that the graph* $\mathrm{Gr}(W)$ *of* W *is closed in* $X \times Z$,

(ii) *for each* $y \in X$, *the mapping* $x \mapsto f(x,y)$ *is continuous on each compact subset of* X,

(iii) *there exists a single-valued mapping* $g\colon X \times X \to Z$ *such that*

(a) *for each* $x \in X$, $g(x,x) \notin P(x)$,

(b) *for each* $x, y \in X$, $f(x,y) \in P(x)$ *implies* $g(x,y) \in P(x)$,

(c) *there exists a nonempty set* $X_0 \subseteq X$ *such that for each* $N \in \mathcal{F}(X)$, *there is a compact contractible subset* L_N *of* X *containing* $X_0 \bigcup N$ *satisfying that for each compactly open subset* U *of* X, *the set* $\bigcap_{x\in U}\{y \in L_N : g(x,y) \in$

$P(x)\}$ is empty or contractible and the set $K = \bigcap_{y\in X_0}\{x \in X : f(x,y) \notin P(x)\}$ is empty or compact.

Then the solutions set $S = \bigcap_{y\in X}\{x \in X : f(x,y) \notin P(x)\}$ of the VEP (2) is nonempty and compact.

Corollary 3.2. *Let X be a nonempty convex subset of a topological vector space E, Z be a topological vector space and $C\colon X \to 2^Z$ be a set-valued mapping such that for each $x \in X$, $C(x)$ is a closed, pointed and convex cone with apex at the origin and $\operatorname{int} C(x) \neq \emptyset$. Let $f\colon X \times X \to Z$ be a single-valued mapping such that*

(i) *the mapping $W\colon X \to 2^Z$ defined by $W(x) = Z \setminus \operatorname{int} C(x)$ for each $x \in X$ is upper semicontinuous on X,*
(ii) *for each $y \in X$, $f(\cdot, y)$ is continuous on each compact subset of X,*
(iii) *there exists a mapping $g\colon X \times X \to Z$ such that*
  (a) *for each $x \in X$, $g(x,x) \notin \operatorname{int}C(x)$,*
  (b) *for all $x, y \in X$, $g(x,y) - f(x,y) \in \operatorname{int} C(x)$,*
  (c) *for each $x \in X$, the set $\{y \in X : g(x,y) \in \operatorname{int} C(x)\}$ is convex,*
(iv) *there exists a nonempty subset X_0 of X which is contained in a compact convex subset X_1 of X such that the set $K = \bigcap_{x\in X_0}\{x \in X : f(x,y) \notin \operatorname{int} C(x)\}$ is empty or compact.*

Then the solutions set $S = \bigcap_{y\in X}\{x \in X : f(x,y) \notin \operatorname{int}C(x)\}$ of the VEP (2) with $P(x) = \operatorname{int} C(x)$ for each $x \in X$ is nonempty and compact.

Proof. By (i), W is upper semicontinuous with closed valued. It follows from Proposition 3.1.7 of Aubin and Ekeland [4], the graph $\operatorname{Gr}(W)$ of W is closed in $X \times Z$. From the condition (iii)(b) it follows that for all $x, y \in X$, $f(x,y) \in \operatorname{int} C(x)$ implies $g(x,y) \in \operatorname{int} C(x)$. Indeed, if $x, y \in X$ such that $f(x,y) \in \operatorname{int} C(x)$, then, by (iii)(b), we have $g(x,y) = g(x,y) - f(x,y) + f(x,y) \in \operatorname{int} C(x) + \operatorname{int}C(x) = \operatorname{int} C(x)$. For each $N \in \mathcal{F}(X)$, let $L_N = \operatorname{co}(X_1 \bigcup N)$, then L_N is a compact convex subset of X containing $(X_0 \bigcup N)$, and by (iii)(c), we have that for any compactly open subset U of X, the set $\bigcap_{x\in U}\{y \in L_N : g(x,y) \in \operatorname{int}C(x)\}$ is empty or convex and hence it is empty or contractible. Now it is easy to see that all conditions of Corollary 3.1 with $P(x) = \operatorname{int}C(x)$ for each $x \in X$ are satisfied. The conclusion of Corollary 3.2 follows from Corollary 3.1.

Remark 3.1. Corollary 3.2 improves Theorem 1 of Ansari [1] and hence Corollary 3.1 and Theorem 3.1 further generalizes Theorem 1 of Ansari [1] to general topological space without linear structure and strengthens its corresponding conclusion.

Theorem 3.2. *Let X, Y and Z be three topological spaces, $T\colon X \to 2^Y$ and $P\colon X \to 2^Z$ be two set-valued mappings and $\phi\colon Y \times X \times X \to Z$ be a single-valued mapping such that*

(i) *the mapping $W\colon X \to 2^Z$ defined by $W(x) = Z \setminus P(x)$ is such that the graph $\operatorname{Gr}(W)$ of W is closed in $X \times Z$,*
(ii) *T is upper semicontinuous with nonempty compact values on each compact subset of X,*
(iii) *for each $y \in X$, $\phi(\cdot,\cdot,y)$ is continuous on each compact subset of $Y \times X$,*
(iv) *there exists a single-valued mapping $\psi\colon Y \times X \times X \to Z$ such that*
  (a) *for each $x \in X$, there exists $s \in T(x)$ such that $\psi(s,x,x) \notin P(x)$,*
  (b) *for each $x, y \in X$ and each $s \in T(x)$, $\phi(s,x,y) \in P(x)$ implies $\psi(s,x,y) \in P(x)$,*
  (c) *there exists a nonempty set $X_0 \subseteq X$ such that for each $N \in \mathcal{F}(X)$, there is a nonempty compact contractible subset L_N of X containing $(X_0 \bigcup N)$*

satisfying that for each compactly open subset U of X, the set

$$K = \bigcap_{x \in U} \{y \in L_N : \phi(T(x), x, y) \subseteq P(x)\}$$

is empty or contractible where $\phi(T(x), x, y) = \bigcup_{s \in T(x)} \phi(s, x, y)$ and the set

$$\bigcap_{y \in X_0} \{x \in X : \psi(T(x), x, y) \nsubseteq P(x)\}$$

is empty or compact where $\psi(T(x), x, y) = \bigcup_{s \in T(x)} \psi(s, x, y)$.

Then the solutions set $S = \bigcap_{y \in X} \{x \in X : \phi(T(x), x, y) \nsubseteq P(x)\}$ of the GIVVIP (3) is nonempty and compact.

Proof. Define a set-valued mapping $F^*, F\colon X \times X \to 2^Z$ by

$$F(x, y) = \phi(T(x), x, y) = \bigcup_{s \in T(x)} \phi(s, x, y),$$

and

$$F^*(x, y) = \psi(T(x), x, y) = \bigcup_{s \in T(x)} \psi(s, x, y)$$

for all $(x, y) \in X \times X$. Then, by the conditions (ii) and (iii), for each $y \in X$, the mapping $x \mapsto F(x, y)$ is upper semicontinuous with nonempty compact values on each compact subset of X. The condition (iv)(a) implies that for each $x \in X$, $F^*(x, x) \nsubseteq P(x)$. The conditions (iv)(b) and (iv)(c) imply that the conditions (iii)(b) and (iii)(c) of Theorem 3.1 hold. By Theorem 3.1, the solutions set $S = \bigcap_{x \in X} \{y \in X : F(x, y) \nsubseteq P(x)\} = \bigcap_{x \in X} \{y \in X : \bigcup_{s \in T(x)} \phi(s, x, y) \nsubseteq P(x)\}$ of the GVEP (1) is nonempty and compact. We note that $\hat{x} \in S$ is a solution of the GVEP (1) if and only if

$$\bigcup_{s \in T(\hat{x})} \phi(s, \hat{x}, y) \nsubseteq P(\hat{x}), \quad \forall y \in X.$$

The above relation holds if and only if for each $y \in X$, there exists $\hat{s} \in T(\hat{x})$ such that $\phi(\hat{s}, \hat{x}, y) \notin P(\hat{x})$, i.e. $\hat{x}$ is a solution of the GIVVIP (3). This completes the proof.

In Theorem 3.2, if T is a single-valued mapping, then we have the following result.

Corollary 3.3. *Let X, Y and Z be three topological spaces and $P\colon X \to 2^Z$ be a set-valued mapping. Let $T\colon X \to Y$ and $\phi\colon Y \times X \times X \to Z$ be single-valued mappings such that*

(i) *the mapping $W\colon X \to 2^Z$ defined by $W(x) = Z \setminus P(x)$ is such that the graph $\mathrm{Gr}(W)$ of W is closed in $X \times Z$,*
(ii) *T is continuous on each compact subset of X,*
(iii) *for each $y \in X$, the mapping $\phi(\cdot, \cdot, y)$ is continuous on each compact subset of $Y \times X$,*
(iv) *there exists a mapping $\psi\colon Y \times X \times X \to Z$ such that*
  (a) *for each $x \in X$, $\psi(Tx, x, x) \notin P(x)$,*
  (b) *for each $x, y \in X$, $\phi(Tx, x, y) \in P(x)$ implies $\psi(Tx, x, y) \in P(x)$,*
  (c) *there exists a nonempty set $X_0 \subseteq X$ such that for each $N \in \mathcal{F}(x)$ there is a compact contractible subset L_N of X containing $(X_0 \bigcup N)$ satisfying that for each compactly open subset U of X, the set $\bigcap_{x \in U} \{y \in L_N : \phi(T(x), x, y) \in P(x)\}$ is empty or contractible and the set $K = \bigcap_{y \in X_0} \{x \in X : \psi(T(x), x, y) \notin P(x)\}$ is empty or compact.*

Then the solutions set $S=\bigcap_{y\in X}\{x\in X : \phi(Tx,x,y)\notin P(x)\}$ *of the IVVIP (4) is nonempty and compact.*

Now let E and Z be two topological vector spaces, X be a nonempty convex subset of E and $L(E,Z)$ be the space of all continuous linear mappings from E into Z. Let $H\colon X\times X\to 2^Z$, $T\colon X\to 2^{L(E,Z)}$ and $C\colon X\to 2^Z$ be three set-valued mappings such that for each $x\in X$, $C(x)$ is a convex cone in Z with $\operatorname{int} C(x)\neq\emptyset$ and $C(x)\neq Z$. Let $\psi\colon L(E,Z)\times X\times X\to Z$ be a single-valued mapping. $H(x,y)$ is said to be C_x-quasiconvex in y if for any $x,y,z\in X$ and $\alpha\in[0,1]$,

$$H(x,\alpha y+(1-\alpha)z)\subseteq \alpha H(x,y)+(1-\alpha)H(x,z)-C(x).$$

$\psi(s,x,y)$ is said to be C_x-quasiconvex with respect to T in y, if for any $x,y,z\in X$, $s\in T(x)$ and $\alpha\in[0,1]$, we have

$$\psi(s,x,\alpha y+(1-\alpha)z)\in \alpha\psi(s,x,y)+(1-\alpha)\psi(s,x,z)-C(x).$$

Clearly, the above notions generalize the corresponding notions of C_x-quasiconvexity-like and P-convexity introduced by Ansari and Yao [3] and Lee and Kim [17], respectively.

Theorem 3.3. *Let X be a nonempty convex subset of a Hausdorff topological vector space E and Z be a topological vector space. Let $F\colon X\times X\to 2^Z$ and $C\colon X\to 2^Z$ be two set-valued mappings such that for each $x\in X$, $C(x)$ is a closed convex cone in Z with $\operatorname{int} C(x)\neq\emptyset$. Assume that*

(i) *the mapping $W\colon X\to 2^Z$ defined by $W(x)=Z\setminus\{-\operatorname{int}C(x)\}$ such that the graph $\operatorname{Gr}(W)$ of W is closed in $X\times Z$,*
(ii) *for each $y\in X$, $x\mapsto F(x,y)$ is upper semicotinuous with nonempty compact values on each compact subset of X,*
(iii) *there exists a set-valued mapping $H\colon X\times X\to 2^Z$ such that*
  (a) *for each $x\in X$, $H(x,x)\not\subseteq -\operatorname{int} C(x)$,*
  (b) *for each $x,y\in X$, $F(x,y)\subseteq -\operatorname{int} C(x)$ implies $H(x,y)\subseteq -\operatorname{int} C(x)$,*
  (c) *$H(x,y)$ is C_x-quasiconvex in y,*
(iv) *there exists a nonempty subset X_0 of X which is contained in a nonempty compact convex subset X_1 of X such that for each $x\in X\setminus X_1$, there is $y\in X_0$ with $F(x,y)\subseteq -\operatorname{int} C(x)$.*

Then the solutions set $S=\bigcap_{y\in X}\{x\in X : F(x,y)\not\subseteq -\operatorname{int}C(x)\}$ of the GVEP (1) with $P(x)=-\operatorname{int} C(x)$ for each $x\in X$ is a nonempty and compact subset of X_1.

Proof. We first show that for each $x\in X$, the set

$$A=\{y\in X : H(x,y)\subseteq -\operatorname{int} C(x)\}$$

is convex. Indeed, for any $y,z\in A$ and $\alpha\in[0,1]$, by the condition (iii)(c), we have

$$\begin{aligned} H(x,\alpha y+(1-\alpha)z) &\subseteq \alpha H(x,y)+(1-\alpha)H(x,z)-C(x)\\ &\subseteq \alpha(-\operatorname{int} C(x))+(1-\alpha)(-\operatorname{int} C(x))-C(x)\\ &\subseteq -\operatorname{int} C(x)-C(x)\subseteq -\operatorname{int} C(x),\end{aligned}$$

and hence $\alpha y+(1-\alpha)z\in A$ and the set A is convex. Now for any $N\in\mathcal{F}(X)$, let $L_N=\operatorname{co}(X_1\bigcup N)$, then L_N is a compact convex subset of X containing $X_0\bigcup N$. Hence, we obtain that for each compactly open subset U of X, the set

$$\bigcap_{x\in U}\{y\in L_N : H(x,y)\subseteq -\operatorname{int} C(x)\}=L_N\bigcap\{y\in X : F(x,y)\subseteq -\operatorname{int} C(x)\}$$

is empty or convex and so it is empty or contractible. By the condition (iv), for each $x\in X\setminus X_1$ there exists $y\in X_0$ such that $F(x,y)\subseteq -\operatorname{int} C(x)$, that is $x\notin \{z\in X : F(z,y)\not\subseteq -\operatorname{int} C(z)\}$. This implies that $K=\bigcap_{y\in X_0}\{x\in X : F(x,y)\not\subseteq$

$-\operatorname{int} C(x)\} \subseteq X_1$. By the proof of Theorem 3.1 with $P(x) = -\operatorname{int} C(x)$ for each $x \in X$, we see that K is a closed subset of X_1 and hence K is empty or compact. It is easy to check that all conditions of theorem 3.1 with $P(x) = -\operatorname{int} C(x)$ for each $x \in X$ are satisfied. By Theorem 3.1, the solutions set $S = \bigcap_{y \in X}\{x \in X : F(x,y) \nsubseteq -\operatorname{int} C(x)\}$ of the GVEP (1) with $P(x) = -\operatorname{int} C(x)$ for each $x \in X$ is a nonempty and compact subset of X_1.

Remark 3.2. Theorem 3.3 improves Theorem 2.1 of Ansari and Yao [3], since the conditions (ii) and (iii)(c) are weaker than the conditions (iii) and (iv)(c) of Theorem 2.1 in [3]. Hence Theorem 3.1 generalizes Theorem 2.1 of Ansari and Yao [3] from topological vector space to general noncompact topological space without linear structure.

Theorem 3.4. *Let E be a Hausdorff topological vector space on which the topological dual space E^* of E separates points, and let X be a nonempty convex subset of E. Let Z be another Hausdorff topological vector space on which the topological dual space Z^* of Z separates points. Let $C\colon X \to 2^Z$ be a set-valued mapping such that for each $x \in X$, $C(x)$ is a convex cone in Z with $-\operatorname{int} C(x) \neq \emptyset$ and $C(x) \neq Z$. Let $L(E,Z)$ be equipped with either the topology of pointwise convergence or the topology of bounded convergence. Let $T\colon X \to 2^{L(E,Z)}$ be a set-valued mappings and let $\phi\colon L(E,Z) \times X \times X \to Z$ be single-valued mapping. Suppose that*

(i) *the mapping $W\colon X \to 2^Z$ defined by $W(x) = Z \setminus -\operatorname{int} C(x)$ is such that the graph $\operatorname{Gr}(W)$ of W is weakly closed in $X \times Z$,*
(ii) *T is upper semicontinuous with nonempty compact values on each compact subset of X where X is equipped with the weak topology,*
(iii) *for each $y \in X$, $\phi(\cdot,\cdot,y)$ is continuous on each compact subset of $L(E,Z) \times X$, where both X and Z are endowed with the weak topologies,*
(iv) *there exists a mapping $\psi\colon L(E,Z) \times X \times X \to Z$ such that*
  (a) *for each $x \in X$, there is a $s \in T(x)$ such that $\psi(s,x,x) \notin -\operatorname{int} C(x)$,*
  (b) *for each $x, y \in X$ and $s \in T(x)$, $\phi(s,x,y) \in -\operatorname{int} C(x)$ implies $\psi(s,x,y) \in -\operatorname{int} C(x)$,*
  (c) *$\psi(s,x,y)$ is C_x-quasiconvex with respect to T in y,*
(v) *there exists a nonempty set $X_0 \subseteq X$ which is contained in a weakly compact convex subset X_1 of X such that the set $K = \bigcap_{y \in X_0}\{x \in X : \phi(T(x),x,y) \subseteq -\operatorname{int} C(x)\}$ is empty or weakly compact.*

Then the solutions set $S = \bigcap_{y \in X}\{x \in X : \phi(T(x),x,y) \nsubseteq -\operatorname{int} C(x)\}$ of the GIVVIP (3) with $P(x) = -\operatorname{int} C(x)$ for each $x \in X$ is nonempty and weakly compact.

Proof. We first prove that for each $x \in X$ the set

$$A = \{y \in X : \psi(T(x),x,y) \subseteq -\operatorname{int} C(x)\}$$

is convex. Indeed, for any $y, z \in A$, $s \in T(x)$ and $\alpha \in [0,1]$, by (iii)(c), we have

$$\begin{aligned} \psi(s,x,\alpha y + (1-\alpha)z) &\in \alpha\psi(s,x,y) + (1-\alpha)\psi(s,x,z) - C(x) \\ &\subseteq \alpha(-\operatorname{int} C(x)) + (1-\alpha)(-\operatorname{int} C(x)) - C(x) \\ &\subseteq -\operatorname{int} C(x). \end{aligned}$$

It follows that $\psi(T(x),x,\alpha y + (1-\alpha)z) \subseteq -\operatorname{int} C(x)$ and $\alpha y + (1-\alpha)z \in A$. Hence A is a convex subset of X. Let $Y = L(E,Z)$ and $P(x) = -\operatorname{int} C(x)$ for each $x \in X$. It is easy to check that the conditions (i)–(iv) imply that the conditions (i)-(iii), (iv)(a) and (iv)(b) of Theorem 3.2 are satisfied. For each $N \in \mathcal{F}(X)$, let $L_N = \operatorname{co}(X_1 \bigcup N)$, then L_N is a weakly compact convex subset of X containing

$(X_0 \bigcup N)$. Since for each $x \in X$, the set $A = \{y \in X : \phi(Tx, x, y) \subseteq -\operatorname{int} C(x)\}$ is convex, therefore for each weakly compactly open subset U of X, the set

$$\bigcap_{x \in U} \{y \in L_N : \phi(T(x), x, y) \subseteq -\operatorname{int} C(x)\}$$

is empty or convex and so it is empty or contractible. Finally, the condition (v) implies the condition (iv)(c) of Theorem 3.2 holds. The conclusion of Theorem 3.4 holds from Theorem 3.2.

Putting $\phi(s, x, y) = \psi(s, x, y)$ for all $(s, x, y) \in L(E, Z) \times X \times X$ in Theorem 3.4, we obtain the following result.

Corollary 3.4. *Let E, X, E^*, Z, Z^*, $L(E, Z)$, $C\colon X \to 2^Z$, $T\colon X \to 2^{L(E,Z)}$ and $\phi\colon L(E, Z) \times X \times X \to Z$ be same as in Theorem 3.4 such that the conditions (i)–(iii) and (v) of theorem 3.4 hold. Further assume that*

(iv) *for each $x \in X$, there is a point $s \in T(x)$ such that $\phi(s, x, x) \notin -\operatorname{int} C(x)$, and $\phi(s, x, y)$ is C_x-quasiconvex with respect to T in y.*

Then the solutions set $S = \bigcap_{y \in X} \{x \in X : \phi(T(x), x, y) \not\subseteq -\operatorname{int} C(x)\}$ of the GIVVIP (3) is nonempty and weakly compact.

Remark 3.3. Theorem 3.4 and Corollary 3.4 both improve Theorem 3.2 and Corollary 3.2 of Lee and Kum [17] and hence Theorem 3.2 generalizes these results to general noncompact topological space without linear structure under much weaker assumptions and strengthen the corresponding conclusion which shows that the solutions set of the GIVVIP (3) is weakly compact.

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Received October 29, 2002.

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