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ON THE CURVATURES OF $(r+1)$-DIMENSIONAL GENERALIZED TIME-LIKE RULED SURFACE

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Abstract. In this paper, we obtain some relationships between the curvatures of $(r+1)$-dimensional generalized time-like ruled surfaces. We also calculate the drall of a generalized time-like ruled surfaces when the base curve is taken as an orthogonal trajectory of the generated spaces.

1. Introduction

Let R_1^n be n-dimensional Minkowski space with the standard metric given by

$$\langle , \rangle = -dx_0^2 + dx_1^2 + dx_2^2 + \cdots + dx_{n-1}^2$$

where $(x_0, x_1, \ldots, x_{n-1})$ is a rectangular system of R_1^n [2]. Nonzero vectors are classified as time-like, space-like or null, respectively, according to whether

$$\langle v, v \rangle < 0, \quad \langle v, v \rangle > 0, \quad \langle v, v \rangle = 0.$$

Let $\alpha \in R_1^n$ is a curve in Minkowski space. If $\dot{\alpha}$ is a velocity of α and $\langle \dot{\alpha}, \dot{\alpha} \rangle < 0$ then the curve α is called a time-like curve.

Let R_1^n be n-dimensional Minkowski space and M a submanifold of R_1^n Let $\overline{D}$ denote the standard Riemannian connection of R_1^n and let D denote the Riemannian connection of M. For any vector fields X, Y on M we have the Gauss equation [3].

$$\overline{D}_X Y = D_X Y + V(X, Y) \tag{1.1}$$

where $D_X Y$, $V(X, Y)$ are tangential and normal components of $\overline{D}_X Y$, respectively. V is called the second fundamental form of M. We also have the Weingarten equation giving the tangential and normal components of $\overline{D}_X \xi$, where ξ is a normal vector field of M:

$$\overline{D}_X \xi = -A_\xi(X) + D_X^\perp \xi. \tag{1.2}$$

Let X, Y be vector fields on M, ξ be a normal vector field and $\langle , \rangle$ be the Minkowski metric on R_1^n. From (1.1) we have

$$\langle \overline{D}_X Y, \xi \rangle = \langle V(X, Y), \xi \rangle \tag{1.3}$$

and then (1.2) implies

$$\langle V(X, Y), \xi \rangle = \langle A_\xi(X), Y \rangle. \tag{1.4}$$

Let $\{\xi_1, \xi_2, \ldots, \xi_{n-m}\}$ be an orthonormal basis of $\chi^\perp(M)$ and the space of normal vector fields on M. Then there exist smooth functions $V^j(X, Y)$, $j = 1, \ldots, n-m$,

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from M into $\mathbb{R}$ such that

$$V(X,Y)=\sum_{j=1}^{n-m} V^j(X,Y)\xi_j. \tag{1.5}$$

Furthermore we may define the mean curvature vector field H by

$$H=\sum_{j=1}^{n-m} \frac{\operatorname{trace} A_{\xi_j}}{m}\xi_j \tag{1.6}$$

and the mean curvature function as $|H|$. At a point $p\in M$, $H(p)$ is called the mean curvature vector and $|H(p)|$ the mean curvature at p.

If $H(p)=0$ for each $p\in M$, then M is said to be minimal.

Let I be an open interval and $\alpha\colon I\to R_1^n$ be a time-like curve in Minkowski space. For each $t\in I$, let $\{e_1(t),e_2(t),\dots,e_r(t)\}$, $(1\le r\le n-2)$ be an orthonormal set of vectors spanning the r-dimensional space-like subspace $W_r(t)$ of $T_{\alpha(t)}R_1^n$. We have

$$\langle e_i,e_j\rangle=\delta_{ij},\quad (i,j=1,\dots,r) \tag{1.7}$$

and denoting by $\dot e_i$ the derivative of the vector field e_i along the time-like curve α;

$$\langle \dot e_i,e_j\rangle+\langle e_i,\dot e_j\rangle=0. \tag{1.8}$$

We may then define an $(r+1)$-dimensional submanifold of R_1^n as in [5].

Definition 1.1. Let α, $\{e_i\}$ be as above and define $\varphi\colon I\times R_1^r\subseteq R_1^n$ by

$$\varphi(t,u_1,\dots,u_r)=\alpha(t)+\sum_{i=1}^{r}u_ie_i(t) \tag{1.9}$$

for all $(t,u_1,\dots,u_r)\in I\times R_1^r$. Let $M=\varphi(G)$ where $G=I\times R_1^r\subseteq R_1^{r+1}$. Note that

$$\operatorname{rank}(\varphi_t,\varphi_{u_1},\dots,\varphi_{u_r})=\operatorname{rank}\left(\dot\alpha(t)+\sum_{i=1}^{r}u_i\dot e_i(t),e_1(t),e_2(t),\dots,e_r(t)\right)=r+1$$

so M is an $(r+1)$-dimensional submanifold of R_1^n. We call M *an* $(r+1)$*-dimensional generalized time-like ruled surface.* The time-like curve α is called the *base curve* of the generalized time-like ruled surface and the space-like subspace $W_r(t)$ is called the *generating space* (or briefly, *generation*) at the point $\alpha(t)$.

Definition 1.2. The subspace $A(t)$ given by

$$A(t)=\operatorname{Span}\{e_1(t),e_2(t),\dots,e_r(t),\dot e_1(t),\dot e_2(t),\dots,\dot e_r(t)\} \tag{1.10}$$

with dimension $\dim A(t)=r+m$, $0\le m\le r$, is said to be the asymptotic bundle of the generalized time-like ruled surface.

$W_r(t)$ is a subspace of $A(t)$ and using the Gramm−Schmidt orthogonalization process, basis of the form;

$$\{e_1(t),e_2(t),\dots,e_r(t),a_{r+1}(t),\dots,a_{r+m}(t)\} \tag{1.11}$$

may be found. Then there exist b_{ij}, c_{ik} such that

$$\dot e_i=\sum_{j=1}^{r}b_{ij}e_j+\sum_{k=1}^{m}c_{ik}a_{r+k} \tag{1.12}$$

with $b_{ij}=-b_{ji}$ by (1.8). The basis $\{e_1(t),e_2(t),\dots,e_r(t)\}$ is called the *natural carrier basis* of $W_r(t)$.

Now let $\tau_{m+1} = \langle \dot{\alpha}_r(t), a_{r+m+1} \rangle$, and $K_k = \langle \dot{e}_k(t), a_{r+k} \rangle$ for $k = 1, \dots, m$ so that

$$\dot{e}_i = \sum_{j=1}^{r} b_{ij} e_j + K_i a_{r+i}, \quad (1 \leq i \leq m,\ K_i > 0)$$

$$\dot{e}_i = \sum_{j=1}^{r} b_{ij} e_j \ \ (m < i \leq r) \quad [1].$$

We now define the following:

$$\delta_k = \frac{\tau_{m+1}}{K_k}, \quad k = 1, \dots, m \tag{1.13}$$

and note that each δ_k is invariant under a reparametrization $t \to t^*$ with $\frac{dt}{dt^*} > 0$. δ_k is called the kth principle drall (principal distribution parameter) of M lying in $W_r(t)$. The drall (distribution parameter) of M is defined by

$$\delta = |\delta_1 \dots \delta_m|^{\frac{1}{m}} \tag{1.14}$$

as in [1].

2. On the curvatures of generalized time-like surfaces

Let M be an $(r+1)$-dimensional generalized time-like ruled surface and s the arc length parameter of the time-like curve α. Let $\{e_1(s), e_2(s), \dots, e_r(s)\}$ be an orthonormal basis of the generating space-like space $W_r(s)$. Let us choose the base time-like curve α to be an orthogonal trajectory of the generating spaces $W_r(s)$. M is given by

$$\varphi(s, u_1, \dots, u_r) = \alpha(s) + \sum_{i=1}^{r} u_i e_i(s), \quad u_i \in R. \tag{2.1}$$

Let $\{e_0, e_1, \dots, e_r\}$ be a (local) orthonormal basis of the space of vector fields $\chi(M)$ and let us choose $e_0 = \varphi_*(\frac{\partial}{\partial s})$. By (2.1)

$$\varphi_s = \dot{\alpha}(s) + \sum_{i=1}^{r} u_i \dot{e}_i(s), \quad \varphi_{u_i} = e_i(s) \tag{2.2}$$

then

$$\overline{D}_{e_i} e_j = 0, \quad i, j = 1, \dots, r \tag{2.3}$$

and using (1.1),

$$V(e_i, e_j) = 0, \quad i, j = 1, \dots, r. \tag{2.4}$$

Since $\overline{D}_{e_i} e_0 \perp e_j$ and $\overline{D}_{e_i} e_0 \perp e_0$ (for all i, j) then

$$\overline{D}_{e_i} e_0 = V(e_i, e_0), \quad i, j = 1, \dots, r. \tag{2.5}$$

Let $\{\xi_1, \dots, \xi_{n-r-1}\}$ be an orthonormal basis of normal vector fields. Then $\{e_0, e_1, \dots, e_r, \xi_1, \dots, \xi_{n-r-1}\}$ gives a basis of $T_p R_1^n$ for each point $p \in M$. Let us write

$$\begin{aligned} \overline{D}_{e_0} \xi_j &= a_{00}^j e_0 + \sum_{t=1}^{r} a_{0t}^j e_t + \sum_{q=1}^{n-r-1} b_{0q}^j \xi_q, \quad j = 1, \dots, n-r-1 \\ \overline{D}_{e_i} \xi_j &= a_{0i}^j e_0 + \sum_{t=1}^{r} a_{it}^j e_t + \sum_{q=1}^{n-r-1} b_{iq}^j \xi_q, \quad i = 1, \dots, r \end{aligned} \tag{2.6}$$

where the a_{it}^j are coefficients of the matrix of A_{ξ_j}:

$$A_{\xi_j} = -\begin{bmatrix} a_{00}^j & a_{01}^j & \cdots & a_{0r}^j \\ a_{01}^j & a_{11}^j & \cdots & a_{1r}^j \\ \vdots & \vdots & \ddots & \vdots \\ a_{0r}^j & a_{r1}^j & \cdots & a_{rr}^j \end{bmatrix}, \quad (j = 1, \ldots, n-r-1). \tag{2.7}$$

We simplify this matrix using (2.6),

$$\langle \overline{D}_{e_i} e_t, \xi_j \rangle = -a_{it}^j, \quad (i, t = 1, \ldots, r; \quad j = 1, \ldots, n-r-1)$$

and by (2.3), $a_{it}^j = 0$. (2.7) can be written as follows

$$A_{\xi_j} = -\begin{bmatrix} a_{00}^j & a_{01}^j & \cdots & a_{0r}^j \\ a_{01}^j & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{0r}^j & 0 & \cdots & 0 \end{bmatrix}. \tag{2.8}$$

Furthermore, (2.6) and (1.4) respectively lead to the relations:

$$\langle \overline{D}_{e_i} e_0, \xi_j \rangle = -a_{0i}^j, \quad (i = 1, \ldots, r; \, j = 1, \ldots, n-r-1)$$

and

$$\langle V(e_i, e_0), \xi_j \rangle = \langle A_{\xi_j}(e_i), e_0 \rangle = a_{0i}^j, (1 \leq i \leq r, \; 1 \leq j \leq n-r-1)$$

and therefore by (1.5) and (2.5)

$$V(e_i, e_0) = \overline{D}_{e_i} e_0 = \sum_{j=1}^{n-r-1} a_{0i}^j \xi_j, \quad (j = 1, \ldots, r). \tag{2.9}$$

Now, let X, Y be vector fields on the m-dimensional submanifold M whose curvature tensor field is R. As in [4], we have

$$\langle X, R(X,Y)Y \rangle = \langle V(X,X), V(Y,Y) \rangle - \langle V(X,Y), V(X,Y) \rangle \tag{2.10}$$

where V is $2nd$ fundamental form of M embedded in R_1^n.

Definition 2.1. Let M be any m-dimensional Lorentzian manifold with curvature tensor R and $\{e_1, \ldots, e_m\}$ be an orthonormal basis of T_pM for $p \in M$. Then the *Ricci curvature tensor field* Ric is defined by

$$\mathrm{Ric}(X,Y) = \sum_{i=1}^{m} \varepsilon_i \langle R(e_i, Y)X, e_i \rangle \tag{2.11}$$

where

$$\varepsilon_i = \begin{cases} -1, & i = 1 \\ 1, & 2 \leq i \leq m. \end{cases}$$

The *scalar curvature of* M is defined by

$$\mathbf{r} = \sum_{i=1}^{m} \varepsilon_i \, \mathrm{Ric}(e_i, e_i) \tag{2.12}$$

or by (2.11)

$$\mathbf{r} = \sum_{i=1}^{m} \sum_{j=1}^{m} \varepsilon_i \varepsilon_j \langle R(e_j, e_i)e_i, e_j \rangle. \tag{2.13}$$

In order to calculate the Ricci curvature of M in the direction of the vector fields e_t $(t = 1, \ldots, r)$, we use (2.4), (2.9), (2.10), (2.11) and we obtain

$$\operatorname{Ric}(e_t, e_t) = \sum_{j=1}^{n-r-1} (a_{0t}^j)^2, \ (t = 1, \ldots, r), \tag{2.14}$$

and

$$\operatorname{Ric}(e_0, e_0) = \sum_{t=1}^{r} \sum_{j=1}^{n-r-1} (a_{0t}^j)^2. \tag{2.15}$$

From (2.14) and (2.15) we can define

$$\operatorname{Ric}(e_0, e_0) = \sum_{t=1}^{r} \operatorname{Ric}(e_t, e_t). \tag{2.16}$$

Now we have proved the following theorem.

Theorem 2.1. *Let $\{e_1, \ldots, e_r\}$ be an orthonormal basis of the generating space of the (r+1)-dimensional generalized time-like ruled surface M and $\{e_0, e_1, \ldots, e_r\}$ and orthonormal basis of $\chi(M)$. If the base time-like curve of M is chosen as an orthonormal trajectory of generating space then the Ricci curvature in the direction of e_0 is equal to the sum Ricci curvatures in the directions of the vector fields forming a basis of the generating space.*

By (2.12), the scalar curvature of the $(r + 1)$-dimensional generalized time-like ruled surface M may be expressed as

$$\begin{aligned} \mathbf{r} &= -\operatorname{Ric}(e_0, e_0) + \sum_{t=1}^{r} \operatorname{Ric}(e_t, e_t), \\ \mathbf{r} &= 0. \end{aligned}$$

Using (1.3), (1.4), (1.5), (2.6) and (2.8) we have

$$V(e_0, e_0) = -\sum_{i=1}^{n-r-1} (\operatorname{trace} A_{\xi_j})\xi_j. \tag{2.17}$$

Now (1.6) gives

$$H = -\frac{1}{r+1} V(e_0, e_0).$$

If M is minimal then H is zero and so

$$V(e_0, e_0) = 0. \tag{2.18}$$

We say that $X_p, Y_p \in T_pM$ are conjugate if $V(X_p, Y_p) = 0$. We have the following theorem.

Theorem 2.2. *Let $\{e_1, \ldots, e_r\}$ be an orthonormal basis for the generating space of an (r+1)-dimensional generalized time-like ruled surface M and let e_0 be an unit time-like tangent vector field to the base time-like curve, the latter taken to be an orthonormal trajectory of the generating space of M. Then the minimal time-like ruled surface M is totally geodesic iff e_0 is conjugate to each vector e_i, $i = 1, \ldots, r$.*

Proof. $\{e_0, e_1, \ldots, e_r\}$ is an orthonormal basis of $\chi(M)$ and for each $X, Y \in \chi(M)$ we may write

$$X = a_0 e_0 + \sum_{i=1}^{r} a_i e_i, \quad Y = b_0 e_0 + \sum_{i=1}^{r} b_i e_i$$

and then

$$V(X,Y) = a_0 b_0 V(e_0,e_0) + \sum_{i=1}^{r} (a_i b_0 + a_0 b_i) V(e_i,e_0) + \sum_{i=1}^{r} a_i b_i V(e_i,e_i). \tag{2.19}$$

($:\Longrightarrow$) If M is totally geodesic then V is identically zero, so e_0 is certainly conjugate to $e_i, i = 1,\ldots,r$.

($\Longleftarrow:$) $V(e_0,e_i) = 0$ for $i = 1,\ldots,r$ then by (2.4) and (2.18), (2.19) reduces to $V(X,Y) = 0$ and this completes the proof of the theorem. □

If trace $A_{\xi_j} = -a^j_{00}$ from (2.8) is substituted into (1.6) we obtain

$$(r+1)^2 \left|H\right|^2 = \sum_{j=1}^{n-r-1} (a^j_{00})^2. \tag{2.20}$$

Theorem 2.3. *Let M be an $(r+1)$-dimensional generalized time-like ruled surface. Let the base time-like curve α be an orthogonal trajectory of the generating space be parametrized by arc length. Then kth principle distribution parameter is*

$$\delta_k = \frac{\left(1 - \sum_{t=1}^{r} \tau_t^2\right)^{\frac{1}{2}}}{\left| \left\| \overline{D}_{e_0} e_k \right\|^2 - \sum_{j=1}^{r} \langle \overline{D}_{e_0} e_k, e_j \rangle^2 \right|^{\frac{1}{2}}}$$

$(k = 1,\ldots,m)$ and the distribution parameter (drall) is

$$\delta_k = \frac{\left(1 - \sum_{t=1}^{r} \tau_t^2\right)^{\frac{1}{2}}}{\prod_{k=1}^{m} \left| \left\| \overline{D}_{e_0} e_k \right\|^2 - \sum_{j=1}^{r} \langle \overline{D}_{e_0} e_k, e_j \rangle^2 \right|^{\frac{1}{2m}}}.$$

Proof. Using (1.13) and (1.14) we obtain

$$\delta_k = \frac{\left\{ \left\| \dot{\alpha} - \sum_{j=1}^{r} \langle \dot{\alpha}, e_j \rangle e_j - \sum_{t=1}^{r} \langle \dot{\alpha}, a_{r+t} \rangle a_{r+t} \right\| \right\}}{\left\{ \left\| \dot{e}_k - \sum_{j=1}^{r} \langle \dot{e}_k, e_j \rangle e_j \right\| \right\}}. \tag{2.21}$$

The base curve α is an orthogonal trajectory so $\langle \dot{\alpha}, e_j \rangle = 0$ for $j = 1,\ldots,r$. Substituting $\langle \dot{\alpha}, e_j \rangle = 0$, $(j = 1,\ldots,r)$ $\dot{e}_k = \overline{D}_{e_0} e_k$, and $\langle \dot{\alpha}, a_{r+t} \rangle = \tau_t$, $(t = 1,\ldots,m)$ into (2.21), the desired result is obtained. □

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