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ALMOST SURE FUNCTIONAL LIMIT THEOREMS IN $L^2(]0,1[)$

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Abstract. The almost sure version of Donsker's theorem is proved in $L^2(]0,1[)$. The almost sure functional limit theorem is obtained for the empirical process in $L^2(]0,1[)$.

1. Introduction

The simplest form of the central limit theorem (CLT) is $\frac{1}{\sigma\sqrt{n}}S_n \Rightarrow \mathcal{C}(0,1)$, as $n \to \infty$, if S_n is the n^{th} partial sum of independent, identically distributed (i.i.d.) random variables with mean zero and variance σ^2. Here $\Rightarrow$ denotes convergence in distribution, while $\mathcal{C}(0,1)$ is the standard normal law. The functional CLT, proved by Donsker, states that the broken line process connecting the points $(\frac{i}{n}, \frac{1}{\sigma\sqrt{n}}S_i)$, $i = 0, 1, \dots, n$, converges weakly to the standard Wiener process W in the space $C([0,1])$, see Billingsley [3].

A relatively new version of the CLT is the so called almost sure (a.s.) CLT, see Brosamler [4], Schatte [11], Lacey and Philipp [7]. The simplest form of the a.s. CLT is the following. Drop $\frac{1}{\log n}\frac{1}{k}$ weight to the point $\frac{1}{\sigma\sqrt{k}}S_k(\omega)$, $k = 1, \dots, n$. Then this discrete measure weakly converges to $\mathcal{C}(0,1)$ for $\mathbb{P}$-almost every $\omega \in \Omega$. (Here $(\Omega, \mathcal{A}, \mathbb{P})$ is the underlying probability space.) The almost sure version of Donsker's theorem is also known, see e.g. Fazekas and Rychlik [6] and the references there.

In this paper our first aim (Theorem 2.1) is to prove the a.s. version of Donsker's theorem in $L^2(]0,1[)$. In this space, despite the case of $C([0,1])$, we can manage without any maximal inequality. Using elementary facts of probability theory, we derive our result from the general a.s. limit theorem in Fazekas and Rychlik [6].

A basic result in statistics is that the uniform empirical process converges to the Brownian bridge B in the space $D([0,1])$, see Billingsley [3]. The almost sure version of this theorem is also known, see e.g. Fazekas and Rychlik [6]. The proof of that theorem is based on a sophisticated inequality of Dvoretzky, Kiefer and Wolfowitz.

In this paper we show that the a.s. version of the limit theorem for the empirical process is valid in $L^2(]0,1[)$, see Theorem 3.1. Our proof relies only on simple facts from probability theory.

To produce a self contained paper, we also prove the (non a.s.) functional limit theorems in $L^2(]0,1[)$. Proposition 2.1 is the Donsker theorem, Proposition 3.1 contains the convergence of the empirical process. The proof of these propositions are straightforward calculations to check the tightness conditions given in Oliveira and Suquet [10].

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2. The almost sure Donsker theorem in $L^2(]0,1[)$

In this part we consider the process

$$Y_n(t) = \frac{1}{\sigma\sqrt{n}} S_{[nt]}, \quad \text{if } t \in [0,1], \tag{1}$$

where $S_0 = 0$, $S_k = X_1 + X_2 + \cdots + X_k$, $k \geq 1$, and $X_1, X_2, \ldots$ are i.i.d. real random variables with $\mathbb{E}X_1 = 0$ and $\mathbb{D}^2 X_1 = \sigma^2$. Here $[\cdot]$ denotes the integer part. We shall prove a.s. limit theorem for $Y_n(t)$ in $L^2(]0,1[)$. For the sake of completeness first we prove the usual limit theorem.

We need the result below due to Oliveira and Suquet [10].

Remark 2.1. Let $(X_n(t), n \geq 1)$ be a sequence of random elements in $L^2(]0,1[)$. Assume that

(i) for some $\gamma > 1$, $\sup_{n\geq 1} \mathbb{E}\|X_n\|_1^\gamma < \infty$,
(ii) $\lim_{h\to 0} \sup_{n\geq 1} \mathbb{E}\|X_n(\cdot + h) - X_n(\cdot)\|_2^2 = 0$.

Then $(X_n(t), n \geq 1)$ is tight in $L^2(]0,1[)$. □

Proposition 2.1. *The sequence of processes* $(Y_n(t), n \geq 1)$ *converges weakly to the standard Wiener process* W *in* $L^2(]0,1[)$.

Proof. It is enough to prove that the finite dimensional distributions of the process $Y_n(t)$ converge to those of the Wiener process and that the family $(Y_n(t),\ n \geq 1)$ is tight in $L^2(]0,1[)$.

The convergence of the finite dimensional distributions to those of the Wiener process is an elementary fact, see [3], so it is enough to prove the tightness.

For this aim, we prove that the conditions (i) and (ii) of Remark 2.1 are satisfied.

First we show that (i) is fulfilled with $\gamma = 2$, i.e. $\sup_{n\geq 1} \mathbb{E}\|Y_n\|_1^2 < \infty$ is satisfied. This is implied by the following calculation.

$$\begin{aligned}
\sup_{n\geq 1} \mathbb{E}\|Y_n\|_1^2 &= \sup_{n\geq 1} \mathbb{E}\left\| \frac{1}{\sigma\sqrt{n}} S_{[nt]} \right\|_1^2 = \sup_{n\geq 1} \mathbb{E}\left(\int_0^1 \left| \frac{1}{\sigma\sqrt{n}} S_{[nt]} \right| dt \right)^2 \\
&= \sup_{n\geq 1} \mathbb{E}\left(\sum_{i=0}^{n-1} \int_{i/n}^{(i+1)/n} \left| \frac{S_i}{\sigma\sqrt{n}} \right| dt \right)^2 = \sup_{n\geq 1} \mathbb{E}\left(\frac{1}{\sigma\sqrt{n}} \frac{1}{n} \sum_{i=0}^{n-1} |S_i| \right)^2 \\
&\leq \sup_{n\geq 1} \frac{1}{\sigma^2 n} \frac{1}{n} \mathbb{E}\left(\sum_{i=0}^{n-1} |S_i|^2 \right) = \sup_{n\geq 1} \frac{1}{\sigma^2 n^2} \sum_{i=0}^{n-1} \mathbb{E}|S_i|^2 \\
&= \sup_{n\geq 1} \frac{1}{\sigma^2 n^2} \sigma^2 \sum_{i=0}^{n-1} i = \sup_{n\geq 1} \left(\frac{1}{n^2} \frac{n(n-1)}{2} \right) = \sup_{n\geq 1} \left(\frac{n-1}{2n} \right) < \infty.
\end{aligned}$$

Now we prove condition (ii). (We mention that in [10] any process outside the interval $[0,1]$ is considered to be 0.) Below $\{\cdot\}$ denotes the fractional part.

$$\mathbb{E}\|Y_n(t+h) - Y_n(t)\|_2^2 = \mathbb{E} \int_0^1 |Y_n(t+h) - Y_n(t)|^2 \, dt$$

$$
\begin{aligned}
&= \mathbb{E}\int_0^{1-h} \left(\frac{1}{\sigma\sqrt{n}} S_{[n(t+h)]} - \frac{1}{\sigma\sqrt{n}} S_{[nt]}\right)^2 dt \\
&\quad + \mathbb{E}\int_{1-h}^{1} \left(\frac{1}{\sigma\sqrt{n}} S_{[nt]}\right)^2 dt \\
&= \int_0^{1-h} \mathbb{E}\left(\frac{1}{\sigma\sqrt{n}} \left(X_{[nt]+1} + \cdots + X_{[n(t+h)]}\right)\right)^2 dt \\
&\quad + \int_{1-h}^{1} \mathbb{E}\left(\frac{1}{\sigma\sqrt{n}} S_{[nt]}\right)^2 dt \\
&= \frac{1}{\sigma^2 n}\int_0^{1-h} \sigma^2([n(t+h)] - [nt])\, dt + \frac{1}{\sigma^2 n}\int_{1-h}^{1} \sigma^2 [nt]\, dt \\
&\le \frac{1}{n}\int_0^1 ([\{nt\} + \{nh\}] + [nh])\, dt + \frac{1}{n} hn \le 2h \to 0,
\end{aligned}
$$

as $h \to 0$. The proof of Proposition 2.1 is complete. □

To prove a.s. Donsker's theorem we shall need the next result due to Fazekas and Rychlik [6] (see also Chuprunov and Fazekas [5]). Let μ_X denote the distribution of X. Let δ_x be the point mass at x.

Remark 2.2. Let (M, ρ) be a complete separable metric space and X_n, $n \in \mathbb{N}$, be a sequence of random elements in M. Assume that there exist $C > 0$, $\varepsilon > 0$ and an increasing sequence of positive numbers C_n with $\lim_{n\to\infty} C_n = \infty$, $C_{n+1}/C_n = O(1)$, and M-valued random elements $X_{k,l}$, $k, l \in \mathbb{N}$, $k < l$, such that the random elements X_k and $X_{k,l}$ are independent for $k < l$ and

$$
\mathbb{E}\rho(X_{k,l}, X_l) \le C\left(\frac{C_k}{C_l}\right)^{\beta} \tag{2}
$$

for $k < l$, where $\beta > 0$. Let $0 \le d_k \le \log(C_{k+1}/C_k)$, assume that $\sum_{k=1}^{\infty} d_k = \infty$. Let $D_n = \sum_{k=1}^{n} d_k$. Then, for any probability distribution μ on the Borel σ-algebra of M, the following two statements are equivalent

$$
\frac{1}{D_n}\sum_{k=1}^{n} d_k \delta_{X_k(\omega)} \Rightarrow \mu, \qquad \text{as } n \to \infty \text{ for almost every } \omega \in \Omega;
$$

$$
\frac{1}{D_n}\sum_{k=1}^{n} d_k \mu_{X_k} \Rightarrow \mu, \qquad \text{as } n \to \infty.
$$

The following result is the a.s. Donsker's theorem in $L^2(]0,1[)$.

Theorem 2.1.

$$
\frac{1}{\log n}\sum_{k=1}^{n} \frac{1}{k}\delta_{Y_k(\cdot,\omega)} \Rightarrow \mu_W,
$$

in $L^2(]0,1[)$, as $n \to \infty$, for almost every $\omega \in \Omega$, where W is the standard Wiener process and $Y_k(t,\omega) = Y_k(t)$ is defined in (1).

Proof. We shall prove that the conditions of Remark 2.2 are fulfilled. The separability and completeness of space $L^2(]0,1[)$ are well-known facts.

Let us define the process

$$
Y_{k,n}(t) = \left(Y_n(t) - \frac{S_k}{\sigma\sqrt{n}}\right)\mathbb{I}_{]k/n,1]}(t), \qquad k = 1, 2, \ldots, n-1,\ t \in [0,1],
$$

where $\mathbb{I}_A$ denotes the indicator function of the set A. Then $Y_{k,n}$ and Y_k are independent for $k < n$.

$$
\begin{aligned}
\mathbb{E}\rho(Y_n, Y_{k,n}) &= \mathbb{E}\sqrt{\int_0^1 \left| Y_n(t) - \left(Y_n(t) - \frac{S_k}{\sigma\sqrt{n}} \right) \mathbb{I}_{]k/n,1]}(t) \right|^2 dt} \\
&\le \sqrt{\mathbb{E}\int_0^1 \left| Y_n(t) - \left(Y_n(t) - \frac{S_k}{\sigma\sqrt{n}} \right) \mathbb{I}_{]k/n,1]}(t) \right|^2 dt} \\
&= \sqrt{\mathbb{E}\sum_{i=0}^{n-1} \int_{i/n}^{(i+1)/n} \left[Y_n(t) - \left(Y_n(t) - \frac{S_k}{\sigma\sqrt{n}} \right) \mathbb{I}_{[k/n,1]}(t) \right]^2 dt} \\
&= \sqrt{\mathbb{E}\left(\left(\frac{S_1}{\sigma\sqrt{n}} \right)^2 \frac{1}{n} + \left(\frac{S_2}{\sigma\sqrt{n}} \right)^2 \frac{1}{n} + \dots + \left(\frac{S_{k-1}}{\sigma\sqrt{n}} \right)^2 \frac{1}{n} + \left(\frac{S_k}{\sigma\sqrt{n}} \right)^2 \frac{n-k}{n} \right)} \\
&= \sqrt{\frac{1}{\sigma^2 n^2}[\sigma^2 + 2\sigma^2 + \dots + (k-1)\sigma^2 + k(n-k)\sigma^2]} \\
&= \sqrt{\frac{1}{n^2}((1+2+\dots+(k-1)) + k(n-k))} = \sqrt{\frac{1}{n^2}\left[\left(\frac{k(k-1)}{2} + k(n-k) \right) \right]} \\
&= \sqrt{\frac{k}{n^2}\frac{2n-k-1}{2}} \le \sqrt{\frac{k}{n}}.
\end{aligned}
$$

So condition (2) of Remark 2.2 holds and the proof of Theorem 2.1 is complete. □

3. The empirical process in $L^2(]0,1[)$

In this section, we consider the empirical process

$$
Z_n(t) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (\mathbb{I}_{[0,t]}(U_i) - t), \qquad t \in [0,1],
$$

where U_i $(i = 1, 2, \dots)$ are independent random variables with uniform distribution on the interval $[0,1]$.

For the sake of completeness we prove the weak convergence of Z_n.

Proposition 3.1. *The process $(Z_n(t),\ n \ge 1)$ weakly converges to the Brownian bridge B in space $L^2(]0,1[)$.*

Proof. It is enough to prove that the finite dimensional distributions of the process $Z_n(t)$ converge to those of the Brownian bridge and that the sequence of the processes is tight in space $L^2(]0,1[)$.

The first fact is elementary and well-known (see for example in [3]) so it is enough to show the tightness.

Now we prove that the condition (i) of Remark 2.1 is fulfilled with $\gamma = 2$. Since $\|\cdot\|_1 \le \|\cdot\|_2$ this will be done if we show $\sup_{n\ge1} \mathbb{E}\|Z_n\|_2^2 < \infty$.

$$
\begin{aligned}
\mathbb{E}\|Z_n\|_2^2 &= \mathbb{E}\left\| \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (\mathbb{I}_{[0,t]}(U_i) - t) \right\|_2^2 = \mathbb{E}\int_0^1 \left| \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (\mathbb{I}_{[0,t]}(U_i) - t) \right|^2 dt \\
&= \frac{1}{n}\mathbb{E}\int_0^1 \left| \sum_{i=1}^{n} (\mathbb{I}_{[0,t]}(U_i) - t) \right|^2 dt \\
&= \frac{1}{n}\int_0^1 \mathbb{E}(\xi - nt)^2\, dt = \frac{1}{n}\int_0^1 nt(1-t)\, dt = \frac{1}{6},
\end{aligned}
$$

where ξ is a binomial random variable with parameters t and n.

Now, we will show that condition (ii) of Remark 2.1 is fulfilled.

$$\begin{aligned}
\mathbb{E}\|Z_n(\cdot+h) - Z_n(t)\|_2^2 = & \\
&= \mathbb{E}\int_0^1 |Z_n(t+h) - Z_n(t)|^2\,dt \\
&= \mathbb{E}\int_0^{1-h} |Z_n(t+h) - Z_n(t)|^2\,dt + \mathbb{E}\int_{1-h}^1 |Z_n(t)|^2\,dt \\
&= \mathbb{E}\int_0^{1-h} \Bigg|\frac{1}{\sqrt{n}}\sum_{i=1}^n \big(\mathbb{I}_{[0,t+h]}(U_i) - (t+h)\big) \\
&\quad -\frac{1}{\sqrt{n}}\sum_{i=1}^n \big(\mathbb{I}_{[0,t]}(U_i) - t\big)\Bigg|^2 \\
&\quad + \mathbb{E}\int_{1-h}^1 \left|\frac{1}{\sqrt{n}}\sum_{i=1}^n \big(\mathbb{I}_{[0,t]}(U_i) - t\big)\right|^2\,dt \\
&= \mathbb{E}\frac{1}{n}\int_0^{1-h} \left|\sum_{i=1}^n \big(\mathbb{I}_{]t,t+h]}(U_i) - h\big)\right|^2\,dt \\
&\quad + \mathbb{E}\frac{1}{n}\int_{1-h}^1 \left|\sum_{i=1}^n \big(\mathbb{I}_{[0,t]}(U_i) - t\big)\right|^2\,dt \\
&= \frac{1}{n}\int_0^{1-h} \mathbb{E}\left(\sum_{i=1}^n \mathbb{I}_{]t,t+h]}(U_i) - nh\right)^2\,dt \\
&\quad + \frac{1}{n}\int_{1-h}^1 \mathbb{E}\left(\sum_{i=1}^n \mathbb{I}_{[0,t]}(U_i) - nt\right)^2\,dt \\
&= \frac{1}{n}\int_0^{1-h} \mathbb{E}(\xi - nh)^2\,dt + \frac{1}{n}\int_{1-h}^1 \mathbb{E}(\eta - nt)^2\,dt \\
&= \frac{1}{n}\int_0^{1-h} nh(1-h)\,dt + \frac{1}{n}\int_{1-h}^1 nt(1-t)\,dt \\
&= h(1-h)^2 + \left(\frac{1}{2} - \frac{1}{3}\right) \\
&\quad - \left(\frac{(1-h)^2}{2} - \frac{(1-h)^3}{3}\right) \to 0, \qquad \text{as } h \to 0,
\end{aligned}$$

where ξ is a binomial random variable with parameters h and n, and η is binomial with parameters t and n.

This completes the proof of the Proposition 3.1. □

Theorem 3.1.

$$\frac{1}{\log n}\sum_{k=1}^n \frac{1}{k}\delta_{Z_k(\cdot,\omega)} \Rightarrow \mu_B,$$

in $L^2(]0,1[)$, as $n \to \infty$, for almost every $\omega \in \Omega$, where B is the Brownian bridge.

Proof. We shall prove that the conditions of Remark 2.2 are fulfilled.

The separability and completeness of $L^2(]0,1[)$ are well-known facts. Let us define the process

$$Z_{k,n}(t) = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}(\mathbb{I}_{[0,t]}(U_i) - t) - \frac{1}{\sqrt{n}}\sum_{i=1}^{k}(\mathbb{I}_{[0,t]}(U_i) - t).$$

Then $Z_{k,n}$ and Z_k are independent for $k < n$.

Condition (2) is valid because

$$\begin{aligned}
\mathbb{E}\rho(Z_n, Z_{k,n}) &= \mathbb{E}\sqrt{\int_0^1 \left|\frac{1}{\sqrt{n}}\sum_{i=1}^{k}(\mathbb{I}_{[0,t]}(U_i) - t)\right|^2 dt} \\
&= \frac{1}{\sqrt{n}}\mathbb{E}\sqrt{\int_0^1 \left(\sum_{i=1}^{k}(\mathbb{I}_{[0,t]}(U_i) - t\right)^2 dt} \\
&= \frac{1}{\sqrt{n}}\sqrt{\int_0^1 \mathbb{E}\left(\sum_{i=1}^{k}(\mathbb{I}_{[0,t]}(U_i) - t)\right)^2 dt} \\
&= \frac{1}{\sqrt{n}}\sqrt{\int_0^1 \mathbb{E}(\xi - kt)^2\,dt} = \frac{1}{\sqrt{n}}\sqrt{\int_0^1 kt(1-t)\,dt} = \frac{1}{\sqrt{6}}\frac{\sqrt{k}}{\sqrt{n}},
\end{aligned}$$

where ξ has binomial distribution with parameters t and k.

This completes the proof of the Theorem 3.1. □

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