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A NEW LOOK AT FINSLER CONNECTIONS AND SPECIAL FINSLER MANIFOLDS

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ABSTRACT. Continuing GRIFONE's pioneering work, we present a systematic treatment of some distinguished Finsler connections and some special Finsler manifolds, built on three pillars: the theory of horizontal endomorphisms, the calculus of vector-valued forms and a "tangent bundle version" of the method of moving frames.

CONTENTS

INTRODUCTION

The forthcoming investigations are partly based on GRIFONE's theory of nonlinear connections (whose role is played in our presentation by the so-called *horizontal endomorphisms*) and the coordinate-free, "intrinsic" calculus of the vector-valued differential forms established by A. FRÖLICHER and A. NIJENHUIS. The main purpose of the present work is to insert the theory of Finsler connections and the foundations of special Finsler manifolds in the new approach to Finsler geometry built on the above two pillars. The first, epoch-making steps in this direction were done by J. GRIFONE himself, our work can be considered as a systematic continuation of the program initiated by him. Technically, we enlarged and – at the same time – simplified the apparatus by using the tools of tangent bundle differential geometry. This means first of all the consistent use of a special frame field, constituted by vertically and completely (or vertically and horizontally) lifted vector fields. Thus the third pillar of our

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approach is the method of moving frames. It has a decisive superiority in calculations over coordinate methods: the formulation of the concepts and results becomes perfectly transparent, and the proofs have a purely intrinsic character. Nevertheless, coordinate-calculations will not be avoided completely. In Section 6 we felt that an indication of the connections to the traditional theory will be useful, while in Section 7 the nature of the problem forced us to use a suitable coordinate-system.

The paper is organized as follows: In the first two sections we present a quite detailed exposition of the conceptual and calculative background to make the paper self-contained as far as possible. In Section 3 basic facts on Finsler manifolds can be found. An important novelty is a generalization and a careful investigation of the so-called second Cartan tensor. It plays a crucial role in the theory of Finsler connections, as well as in the tensorial characterization of some special Finsler manifolds. Section 4 is devoted to an invariant, axiomatic description of three distinguished Finsler connections: the Berwald, the Cartan, and the Chern – Rund connection. We hope that our approach helps in better understanding the role of the different axioms, and also opens a path for further, essential generalizations. In Section 5 we derive some important (partly well-known) relations between the curvature data of the different Finsler connections, these will be indispensable for a tensorial description of the special Finsler manifolds studied in the last two sections. Section 6 concentrates on the characterizations of Berwald manifolds; some of them (e.g. 6.5 and 6.7) are new, and all of the proofs are original. We believe that the compact, elegant and efficient formulations presented here demonstrate the power of our approach. In the concluding Section 7 the key observation is given in Proposition 7.2; this provides a very simple proof of the classical characterization of locally Minkowski manifolds.

1. Notation and some basic facts

1.1. M is a connected, paracompact, smooth (i.e., C^∞) manifold of dimension n, where $n \in \mathbb{N} \setminus \{0,1\}$. $C^\infty(M)$ is the ring of real-valued smooth functions on M, the $C^\infty(M)$-module of *vector fields* on M is denoted by $\mathfrak{X}(M)$.

1.2. $\forall k \in \{0, \dots, n\}$: $\Omega^k(M)$ is the module of *differential k-forms* on M; by convention $\Omega^0(M) := C^\infty(M)$. $\Omega(M) := \oplus_{i=0}^n \Omega^k(M)$ is the graded algebra of differential forms with multiplication given by the wedge product. To each vector field $X \in \mathfrak{X}(M)$ correspond two derivations of $\Omega(M)$: *the substitution operator* $\imath_X$ of degree -1, and the *Lie derivative* $\mathcal{L}_X$, of degree 0. These are related to the operator d of the *exterior derivative* through H. Cartan's magic formula

$$\mathcal{L}_X = [\imath_X, d] := \imath_X \circ d + d \circ \imath_X.$$

1.3. A *vector k-form* on M is a skew-symmetric k-multilinear map $[\mathfrak{X}(M)]^k \to \mathfrak{X}(M)$ if $k \in \mathbb{N} \setminus \{0\}$, and a vector field on M, if $k = 0$. They constitute a $C^\infty(M)$-module, denoted by $\Psi^k(M)$. In particular, the elements of $\Psi^1(M)$ are just the $(1,1)$ tensor fields on M.

1.4. The *Frölicher - Nijenhuis bracket* of a vector 1-form $K \in \Psi^1(M)$ and a vector field $Y \in \mathfrak{X}(M)$ is the vector 1-form $[K,Y]$ defined by

$$[K,Y](X) = [K(X),Y] - K[X,Y], \quad X \in \mathfrak{X}(M). \tag{1.4a}$$

The Frölicher - Nijenhuis bracket of the vector 1-forms $K, L \in \Psi(M)$ is the vector 2-form $[K, L] \in \Psi^2(M)$, given by

$$\begin{aligned}[K,L](X,Y) = {} & [K(X),L(Y)] + [L(X),K(Y)] + K\circ L[X,Y] + \\ & + L\circ K[X,Y] - K[X,L(Y)] - K[L(X),Y] - \\ & - L[X,K(Y)] - L[K(X),Y]; \qquad X,Y\in\mathfrak{X}(M).\end{aligned} \tag{1.4b}$$

In particular,

$$\frac{1}{2}[K,K](X,Y) = [K(X),K(Y)] + K^2[X,Y] - K[X,K(Y)] - K[K(X),Y]. \tag{1.4c}$$

$$N_K := \frac{1}{2}[K,K] \tag{1.4d}$$

is said to be the *Nijenhuis torsion* of K.

1.5. The *adjoint operator* $K^*\colon \Omega(M)\to\Omega(M)$, $\omega\mapsto K^*\omega$ of a vector 1-form $K\in\Psi^1(M)$ is defined by the value of $K^*\omega$ on k vector fields $X_1,\dots,X_k\in\mathfrak{X}(M)$ through the formula

$$K^*\omega(X_1,\dots,X_k) := \omega(K(X_1),\dots,K(X_k)) \tag{1.5}$$

if $k\neq 0$, and $\forall f\in C^\infty(M):\ K^*f := f$.

1.6. $\pi\colon TM\to M$ is the *tangent bundle* of M; it is also denoted by τ_M. $\pi_0\colon \mathcal{T}M\to M$ is the subbundle of τ_M constituted by the nonzero tangent vectors to M. The kernel of the tangent map $T\pi$ (or $T\pi_0$) is a canonical subbundle of τ_{TM} (or $\tau_{\mathcal{T}M}$), called the *vertical subbundle* and denoted by τ^v_{TM} (and $\tau^v_{\mathcal{T}M}$, resp.). The sections of the bundles τ^v_{TM} and $\tau^v_{\mathcal{T}M}$ are called *vertical vector fields*; the $C^\infty(TM)$-modules of vertical vector fields are denoted by $\mathfrak{X}^v(TM)$ and $\mathfrak{X}^v(\mathcal{T}M)$, respectively.

1.7. Tangent bundle geometry is dominated by two canonical objects: the *Liouville vector field* $C\in\mathfrak{X}^v(TM)$, and the *vertical endomorphism* $J\in\Psi^1(TM)$. We shall frequently use the following properties:

$$\operatorname{Im} J = \operatorname{Ker} J = \mathfrak{X}^v(TM). \tag{1.7a}$$

$$J^2 = 0. \tag{1.7b}$$

$$N_J := \frac{1}{2}[J,J] = 0. \tag{1.7c}$$

$$[J,C] = J. \tag{1.7d}$$

1.8. A differential form $\omega\in\Omega^k(TM)$ $(k\neq 0)$ is *semibasic*, if for each vector field X on TM, $\imath_{JX}\omega = 0$. Analogously, a vector k-form $K\in\Psi^k(TM)$ is said to be semibasic, if

$$J\circ K = 0 \quad \text{and} \quad \forall X\in\mathfrak{X}(TM):\ i_{JX}K = 0. \tag{1.8}$$

1.9. The vertical endomorphism J determines two derivations of $\Omega(TM)$. The *vertical derivation* $\imath_J$ is defined as follows:

$$\forall f\in C^\infty(TM):\ \imath_J f = 0; \tag{1.9a}$$

$$\imath_J\omega(X_1,\dots,X_k) := \sum_{i=1}^{k}\omega(X_1,\dots,J(X_i),\dots,X_k) \tag{1.9b}$$

$$(\omega\in\Omega^k(TM);\ X_i\in\mathfrak{X}(TM),\ 1\le i\le k).$$

$\imath_J$ is a derivation of degree 0. The *vertical differentiation* d_J is the mapping

$$d_J := [\imath_J, d] := \imath_J \circ d - d \circ \imath_J; \tag{1.9c}$$

it is of degree 1. In particular, we get

$$\forall f \in C^\infty(TM): \ d_J f = \imath_J df = J^* df. \tag{1.9d}$$

The following formulas are easy consequences of the definitions:

$$d \circ d_J = -d_J \circ d. \tag{1.9e}$$

$$\imath_C \circ d_J + d_J \circ \imath_C = \imath_J. \tag{1.9f}$$

1.10. A *semispray* on the manifold M is a mapping

$$S\colon TM \to TTM, \ v \mapsto S(v) \in T_v TM,$$

satisfying the following conditions:

$$S \textit{ is smooth on } \mathcal{T}M; \tag{1.10a}$$

$$JS = C. \tag{1.10b}$$

The semispray S is called a *spray*, if

$$S \textit{ is of class } C^1 \textit{ on } TM \tag{1.10c}$$

and

$$[C, S] = S. \tag{1.10d}$$

1.11. The *vertical lift* of a function $f \in C^\infty(M)$ is $f^v := f \circ \pi \in C^\infty(TM)$, the complete lift of f is the function

$$f^c\colon TM \to \mathbb{R}, \ v \mapsto f^c(v) := df(v) = v(f).$$

Vector fields on TM are determined by their action on $\{ f^c \mid f \in C^\infty(M) \}$ (see [20]). Thus for any vector field $X \in \mathfrak{X}(M)$ there exist vector fields $X^v, X^c \in \mathfrak{X}(TM)$ such that

$$\forall f \in C^\infty(M): \ X^v f^c = Xf \circ \pi = (Xf)^v; \tag{1.11a}$$

$$\forall f \in C^\infty(M): \ X^c f^c = (Xf)^c. \tag{1.11b}$$

X^v and X^c are called the *vertical* and the *complete lift* of X, respectively.

1.12. Lemma. *If S is a semispray on M, then*

$$\forall Z \in \mathfrak{X}(\mathcal{T}M): \ J[JZ, S] = JZ. \tag{1.12}$$

For a proof, see [10], p. 295.

1.13. Lemma. *For each vector fields $X, Y \in \mathfrak{X}(M)$ we have*

$$[X^v, Y^v] = 0, \ [X^v, Y^c] = [X, Y]^v, \ [X^c, Y^c] = [X, Y]^c; \tag{1.13a}$$

$$[C, X^v] = -X^v, \ [C, X^c] = 0; \tag{1.13b}$$

$$JX^c = X^v, \ [J, X^c] = 0. \tag{1.13c}$$

1.14. Lemma (1st local basis property)**.** *If $(X_i)_{i=1}^n$ is a local basis for the module $\mathfrak{X}(M)$, then $(X_i^v, X_i^c)_{i=1}^n$ is a local basis for $\mathfrak{X}(TM)$.*

1.15. Lemma. *A vector field $Y \in \mathfrak{X}(TM)$ is*

(a) *a vertical lift* $\iff JY = 0$ *and* $[J, Y] = 0$;
(b) *a complete lift* $\iff JY \neq 0$ *and* $[J, Y] = 0$.

1.16. Lemma. *A vertical vector field $Y \in \mathfrak{X}^v(TM)$ is a vertical lift if and only if for each vector field X on M $[X^v, Y] = 0$.*

Proof. We first remark that the vector 1-form $[J, Y]$ is semibasic. From this by Lemma 1.14 it follows that $[J, Y]$ is completely determined by its action on $\{\, X^c \mid X \in \mathfrak{X}(M) \,\}$.
Necessity. Assume $Y \in \mathfrak{X}^v(TM)$ is a vertical lift. Then by Lemma 1.15(a) $[J, Y] = 0$. If $X \in \mathfrak{X}(M)$, then $[X^c, Y]$ is vertical since

$$(X^c \underset{\pi}{\sim} X \text{ and } Y \underset{\pi}{\sim} 0) \implies [X^c, Y] \underset{\pi}{\sim} 0.$$

Now we conclude that

$$\forall X \in \mathfrak{X}(M): \; 0 = [J, Y]X^c \overset{(1.4a)}{=} [J(X^c), Y] - J[X^c, Y] \overset{(1.13c)}{=} [X^v, Y].$$

Sufficiency. Suppose that for each vector field X on M we have $[X^v, Y] = 0$. Then $[J, Y]X^c = [X^v, Y] = 0$, hence by Lemma 1.14 $[J, Y] = 0$. This means by Lemma 1.15(a) that Y is a vertical lift. □

2. Horizontal endomorphisms and Finsler connections

2.1. In the sequel we shall introduce some important vector forms over TM. In conformity with the demands of Finsler geometry, their smoothness will *not* be required or assured *a priori* on the whole tangent manifold TM.

2.2. Definitions.
(i) A vector 1-form $h \in \Psi^1(TM)$, smooth only on $\mathcal{T}M$, is said to be a *horizontal endomorphism* on M, if it is a *projector* (i.e. $h^2 = h$) and $\operatorname{Ker} h = \mathfrak{X}^v(TM)$.
(ii) Assume $h \in \Psi^1(TM)$ is a horizontal endomorphism. The mapping

$$X \in \mathfrak{X}(M) \mapsto X^h := hX^c \in \mathfrak{X}(\mathcal{T}M) \tag{2.2a}$$

is called the *horizontal lifting* determined by h. The vector 1-form

$$H := [h, C] \in \Psi^1(TM) \tag{2.2b}$$

is said to be the *tension* of h. If $H = 0$, then h is called *homogeneous.* The vector 2-form

$$t := [J, h] \in \Psi^2(TM) \tag{2.2c}$$

is said to be the *torsion* of h. The *curvature* of the horizontal endomorphism h is the vector 2-form

$$\Omega := -N_h = -\frac{1}{2}[h, h]. \tag{2.2d}$$

2.3. Remark. We emphasize again the condition of differentiability about a horizontal endomorphism is prescribed only on $\mathcal{T}M$. As a consequence, the smoothness of the tension, the torsion and the curvature is also guaranteed only over $\mathcal{T}M$.

2.4. Remark. The following relations are easy consequences of the definitions:

$$h \circ J = 0, \quad J \circ h = J. \tag{2.4a}$$

$$\forall X \in \mathfrak{X}(M): \ JX^h = X^v. \tag{2.4b}$$

$$\forall X, Y \in \mathfrak{X}(M): \ J[X^h, Y^h] = [X, Y]^v. \tag{2.4c}$$

2.5. Lemma and definition. *Suppose that h is a horizontal endomorphism on M.*
(a) If $\mathfrak{X}^h(TM) := \operatorname{Im} h$, then $\mathfrak{X}(TM) = \mathfrak{X}^v(TM) \oplus \mathfrak{X}^h(TM)$ (direct sum). $\mathfrak{X}^h(TM)$ is called the module of horizontal vector fields. *$v := 1_{\mathfrak{X}(TM)} - h$ is also a projector, the* vertical projection *on $\mathfrak{X}^v(TM)$ along $\mathfrak{X}^h(TM)$.*

(b) (2nd local basis property.) *If $(X_i)_{i=1}^n$ is a local basis of $\mathfrak{X}(M)$, then $\left(X_i^v, X_i^h\right)_{i=1}^n$ is a local basis of $\mathfrak{X}(TM)$.*

Proof. Trivial. □

2.6. Lemma. *The curvature form Ω of a horizontal endomorphism h is a semibasic vector 2-form, consequently*

$$\forall X, Y \in \mathfrak{X}(TM): \ \Omega(X, Y) = \Omega(hX, hY) = -v[hX, hY]; \tag{2.6a}$$

$$\forall X, Y \in \mathfrak{X}(M): \Omega(X^h, Y^h) = \Omega(X^c, Y^c) = -v[X^h, Y^h]. \tag{2.6b}$$

Proof. Straightforward calculation. □

2.7. Lemma. *If h is a horizontal endomorphism on M, then there exists a unique almost complex structure F ($F^2 = -1_{\mathfrak{X}(TM)}$) on TM, smooth over $\mathcal{T}M$, such that*

$$F \circ J = h, \quad F \circ h = -J. \tag{2.7}$$

For a proof, see [10], p. 314.

2.8. The following formulas can be obtained easily:

$$J \circ F = v, \quad F \circ v = h \circ F. \tag{2.8a}$$

$$v \circ F = F - F \circ v = F - h \circ F = -J. \tag{2.8b}$$

$$J[C, F] = v - [C, h]. \tag{2.8c}$$

2.9. Definitions. Suppose that h is a horizontal endomorphism on M and consider the almost complex structure F characterized by Lemma 2.7. Let, furthermore, D be a linear connection on the manifold TM or $\mathcal{T}M$. The pair (D, h) is said to be a *Finsler connection* on M if it satisfies the following conditions:

$$Dh = 0 \quad (D \textit{ is reducible}); \tag{2.9a}$$

$$DF = 0 \quad (D \textit{ is almost complex}). \tag{2.9b}$$

The mappings

$$D^h\colon (X, Y) \in \mathfrak{X}(TM) \times \mathfrak{X}(TM) \mapsto D^h_X Y := D_{hX} Y \in \mathfrak{X}(TM), \tag{2.9c}$$

$$D^v\colon (X, Y) \in \mathfrak{X}(TM) \times \mathfrak{X}(TM) \mapsto D^v_X Y := D_{vX} Y \in \mathfrak{X}(TM) \tag{2.9d}$$

are called the *h-covariant* and the *v-covariant differentiation* with respect to (D, h), respectively. The *h-deflection* of (D, h) is the mapping

$$h^*(DC)\colon X \in \mathfrak{X}(TM) \mapsto DC(hX) = D_{hX} C, \tag{2.9e}$$

while the *v-deflection* is $v^*(DC)$. The covariant differential DC is called the *deflection map.*

2.10. Remark. Assume that (D, h) is a Finsler connection on M. Applying (2.9a) we get immediately that

$$Y \in \mathfrak{X}^v(TM) \implies \forall X \in \mathfrak{X}(TM): \; D_X Y \in \mathfrak{X}^v(TM), \tag{2.10a}$$

$$Y \in \mathfrak{X}^h(TM) \implies \forall X \in \mathfrak{X}(TM): \; D_X Y \in \mathfrak{X}^h(TM). \tag{2.10b}$$

Owing to the condition (2.9b) it follows that D is completely determined by its action on $\mathfrak{X}(TM) \times \mathfrak{X}^v(TM)$. Namely, for each vector fields X, Y on TM,

$$D_{vX} hY = F D_{vX} JY, \tag{2.10c}$$

$$D_{hX} hY = F D_{hX} JY. \tag{2.10d}$$

We show that D is almost tangent as well:

$$DJ = 0 \tag{2.10e}$$

To see this, let $X, Y \in \mathfrak{X}(TM)$ be arbitrary. If Y is vertical, then $JY = 0$, $D_X Y \in \mathfrak{X}^v(TM)$ (by (2.10a)), hence $JD_X Y = 0$ and

$$DJ(Y, X) = D_X JY - JD_X Y = 0.$$

Now suppose that Y is horizontal. Then $hY = Y$ and

$$DJ(Y, X) = D_X JY - JD_X Y \overset{(2.7)}{=} -D_X FhY + FhD_X Y \overset{(2.10b)}{=}$$
$$= -D_X FY + FD_X Y = -DF(Y, X) \overset{(2.9b)}{=} 0.$$

2.11. Lemma and definition. *Let (D, h) be a Finsler connection on M. Then the torsion tensor field $\mathbb{T}$ of D is completely determined by the following mappings:*

$$\mathbb{A}(X, Y) := h\mathbb{T}(hX, hY) \text{ – } (h)h\text{-}torsion, \tag{2.11a}$$

$$\mathbb{B}(X, Y) := h\mathbb{T}(hX, vY) \text{ – } (h)hv\text{-}torsion, \tag{2.11b}$$

$$\mathbb{R}^1(X, Y) := v\mathbb{T}(hX, hY) \text{ – } (v)h\text{-}torsion, \tag{2.11c}$$

$$\mathbb{P}^1(X, Y) := v\mathbb{T}(hX, vY) \text{ – } (v)hv\text{-}torsion, \tag{2.11d}$$

$$\mathbb{S}^1(X, Y) := v\mathbb{T}(vX, vY) \text{ – } (v)v\text{-}torsion. \tag{2.11e}$$

2.12. Lemma and definition. *If (D, h) is a Finsler connection on M, then curvature tensor field $\mathbb{K}$ of D is uniquely determined by the following three mappings:*

$$\mathbb{R}(X, Y)Z := \mathbb{K}(hX, hY)JZ \text{ – } h\text{-}curvature, \tag{2.12a}$$

$$\mathbb{P}(X, Y)Z := \mathbb{K}(hX, JY)JZ \text{ – } hv\text{-}curvature, \tag{2.12b}$$

$$\mathbb{Q}(X, Y)Z := \mathbb{K}(JX, JY)JZ \text{ – } v\text{-}curvature. \tag{2.12c}$$

2.13. Example 1: horizontal lift of a linear connection. Suppose that ∇ is a *linear connection* on the manifold M. It is well-known that ∇ induces a homogeneous horizontal structure $h \in \Psi^1(TM)$, which is smooth on the whole tangent manifold TM. In this case

$$\forall X, Y \in \mathfrak{X}(M): \; \left(\nabla_X Y\right)^v = [X^h, Y^v].$$

It is also known (see e.g. [8]), that there exists a unique linear connection $\overset{h}{\nabla}$ on the manifold TM, characterized by the following rules of calculation:

$$\begin{aligned} &\overset{h}{\nabla}_{X^v} Y^v = 0, \quad \overset{h}{\nabla}_{X^h} Y^v = \left(\nabla_X Y\right)^v = [X^h, Y^v], \\ &\overset{h}{\nabla}_{X^v} Y^h = 0, \quad \overset{h}{\nabla}_{X^h} Y^h = \left(\nabla_X Y\right)^h; \quad X, Y \in \mathfrak{X}(M). \end{aligned} \tag{2.13}$$

$\overset{h}{\nabla}$ is called the *horizontal lift* of the linear connection ∇. Now it is easy to check that $(\overset{h}{\nabla}, h)$ satisfies the conditions (2.9a), (2.9b); therefore $(\overset{h}{\nabla}, h)$ is a Finsler-connection on M.

2.14. Example 2: Berwald-type connections. Let a horizontal endomorphism h on the manifold M be given. Define the mapping

$$\overset{\circ}{D} \colon \mathfrak{X}(\mathfrak{T}M) \times \mathfrak{X}(\mathfrak{T}M) \to \mathfrak{X}(\mathfrak{T}M), \ (X, Y) \mapsto \overset{\circ}{D}_X Y$$

as follows:

$$\overset{\circ}{D}_{JX} JY := J[JX, Y], \tag{2.14a}$$

$$\overset{\circ}{D}_{hX} JY := v[hX, JY], \tag{2.14b}$$

$$\overset{\circ}{D}_{JX} hY := h[JX, Y], \tag{2.14c}$$

$$\overset{\circ}{D}_{hX} hY := hF[hX, JY] \tag{2.14d}$$

and

$$\overset{\circ}{D}_X Y := \overset{\circ}{D}_{vX} vY + \overset{\circ}{D}_{hX} vY + \overset{\circ}{D}_{vX} hY + \overset{\circ}{D}_{hX} hY.$$

Then $\overset{\circ}{D}$ is obviously a linear connection on $\mathfrak{T}M$. It is easy to check that $(\overset{\circ}{D}, h)$ satisfies (2.9a) and (2.9b). For example:

$$\begin{aligned} \overset{\circ}{D} F(vY, vX) &= \overset{\circ}{D}_{vX} FvY - F\overset{\circ}{D}_{vX} vY \overset{(2.8a),(2.14a)}{=} \\ &= \overset{\circ}{D}_{vX} hFY - FJ[vX, FY] \overset{(2.14c),(2.7)}{=} \\ &= h[vX, FY] - h[vX, FY] = 0. \end{aligned}$$

Thus $(\overset{\circ}{D}, h)$ is a Finsler connection on M, the *Finsler connection of Berwald-type* induced by h.

2.15. Proposition. *Suppose that $(\overset{\circ}{D}, h)$ is a Finsler connection of Berwald-type on M. Then the h-curvature $\overset{\circ}{\mathbb{R}}$, the hv-curvature $\overset{\circ}{\mathbb{P}}$ and the v-curvature $\overset{\circ}{\mathbb{Q}}$ of $\overset{\circ}{D}$ are semibasic and*

$$\forall X, Y, Z \in \mathfrak{X}(M) : \ \overset{\circ}{\mathbb{R}}(X^c, Y^c)Z^c = [J, \Omega(X^c, Y^c)]Z^h; \tag{2.15a}$$

$$\forall X, Y, Z \in \mathfrak{X}(M) : \ \overset{\circ}{\mathbb{P}}(X^c, Y^c)Z^c = \overset{\circ}{\mathbb{P}}(X^h, Y^h)Z^h = \big[[X^h, Y^v], Z^v\big]; \tag{2.15b}$$

$$\overset{\circ}{\mathbb{Q}} = 0. \tag{2.15c}$$

Proof. It is obvious from (2.12a)–(2.12c) that $\overset{\circ}{\mathbb{R}}$, $\overset{\circ}{\mathbb{P}}$ and $\overset{\circ}{\mathbb{Q}}$ are indeed semibasic. Hence, in view of Lemma 1.14 and Lemma 2.5(b), they are completely determined by their action on the triplets of form

$$(X^c, Y^c, Z^c) \text{ or } (X^h, Y^h, Z^h); \quad X, Y, Z \in \mathfrak{X}(M).$$

(a)Taking into account that for any vector fields $X, Y, Z \in \mathfrak{X}(M)$,

$$[X^h, Y^v],\ \big[X^h, [Y^h, Z^v]\big],\ \dots \text{ are vertical; } h[X^h, Y^h] = [X, Y]^h,$$
$$JX^h = JX^c = X^v \text{ (see (2.4b) and (1.13c)), } J \circ F = v \text{ (from (2.8a));}$$

and applying the rules of calculation (2.14a), (2.14b) and the Jacobi identity for the Lie bracket of vector fields, we obtain:

$$\begin{aligned}
\overset{\circ}{\mathbb{R}}(X^c, Y^c)Z^c &:= \overset{\circ}{\mathbb{K}}(hX^c, hY^c)JZ^c = \overset{\circ}{\mathbb{K}}(X^h, Y^h)Z^v = \overset{\circ}{D}_{X^h}\overset{\circ}{D}_{Y^h}Z^v - \\
&- \overset{\circ}{D}_{Y^h}\overset{\circ}{D}_{X^h}Z^v - \overset{\circ}{D}_{[X^h,Y^h]}Z^v = \overset{\circ}{D}_{X^h}[Y^h, Z^v] - \overset{\circ}{D}_{Y^h}[X^h, Z^v] - \\
&- \overset{\circ}{D}_{[X,Y]^h}Z^v - \overset{\circ}{D}_{v[X^h,Y^h]}Z^v = v\big[X^h, [Y^h, Z^v]\big] - \\
&- v\big[Y^h, [X^h, Z^v]\big] - \big[[X, Y]^h, Z^v\big] - J\big[v[X^h, Y^h], Z^h\big] = \\
&= \big[X^h, [Y^h, Z^v]\big] - \big[Y^h, [X^h, Z^v]\big] - \big[[X, Y]^h, Z^v\big] - \\
&- J\big[v[X^h, Y^h], Z^h\big] = -\big[Y^h, [Z^v, X^h]\big] - \big[Z^v, [X^h, Y^h]\big] + \\
&+ \big[Y^h, [Z^v, X^h]\big] - \big[[X, Y]^h, Z^v\big] - J\big[v[X^h, Y^h], Z^h\big] = \\
&= -\big[Z^v, [X, Y]^h\big] - \big[Z^v, v[X^h, Y^h]\big] + \big[Z^v, [X, Y]^h\big] - \\
&- J\big[v[X^h, Y^h], Z^h\big] = -\big[JZ^h, v[X^h, Y^h]\big] + J\big[Z^h, v[X^h, Y^h]\big] = \\
&= -\big[J, v[X^h, Y^h]\big]Z^h \overset{(2.6b)}{=} [J, \Omega(X^c, Y^c)]Z^h.
\end{aligned}$$

Since $\overset{\circ}{\mathbb{R}}$ is semibasic, this proves (2.15a).

(b) Computing and arguing as before, we find that

$$\begin{aligned}
\overset{\circ}{\mathbb{P}}(X^c, Y^c)Z^c &:= \overset{\circ}{\mathbb{K}}(hX^c, JY^c)JZ^c = \overset{\circ}{D}_{hX^c}\overset{\circ}{D}_{JY^c}JZ^c - \overset{\circ}{D}_{JY^c}\overset{\circ}{D}_{hX^c}JZ^c - \\
&- \overset{\circ}{D}_{[X^h,Y^v]}JZ^c = \overset{\circ}{D}_{hX^c}J[Y^v, Z^h] - \overset{\circ}{D}_{JY^c}v[X^h, Z^v] - \\
&- \overset{\circ}{D}_{JF[X^h,Y^v]}JZ^c = -\overset{\circ}{D}_{JY^c}JF[X^h, Z^v] - J\big[[X^h, Y^v], Z^c\big] = \\
&= -J\big[Y^v, F[X^h, Z^v]\big] - J\big[[X^h, Y^v], Z^c\big].
\end{aligned}$$

Since by (1.13c) $[J, Z^c] = 0$, we can write

$$\begin{aligned}
0 = [J, Z^c][X^h, Y^v] &= \big[J[X^h, Y^v], Z^c\big] - J\big[[X^h, Y^v], Z^c\big] = \\
&= -J\big[[X^h, Y^v], Z^c\big],
\end{aligned}$$

and so

$$\overset{\circ}{\mathbb{P}}(X^c, Y^c)Z^c = J\big[F[X^h, Z^v], Y^v\big].$$

Now applying Lemma 1.15(a), we obtain that

$$0 = [J, Y^v]F[X^h, Z^v] = \big[JF[X^h, Z^v], Y^v\big] - J\big[F[X^h, Z^v], Y^v\big],$$

and thus

$$J\big[F[X^h, Z^v], Y^v\big] = \big[[X^h, Z^v], Y^v\big] = -\big[[Z^v, Y^v], X^h\big] - \\ -\big[[Y^v, X^h], Z^v\big] = \big[[X^h, Y^v], Z^v\big];$$

hence our assertion.

(c) A quick and easy calculation yields the relation (2.15c) and we omit it. □

2.16. We recall that if h is a horizontal endomorphism on M,then there always exists a semispray on M, which is horizontal with respect to h. To see this, consider an arbitrary semispray S' on M. If $S = hS'$, then $JS = J(hS') \overset{(2.4a)}{=} JS' = C$ and hence S is a semispray. Since $hS = h^2S' = hS' = S$, S is horizontal with respect to h. It is also clear that S does not depend on the choice of S'.

The spray S constructed in this way is called *the semispray associated with* h (c.f. [10], p.306).

2.17. Proposition. *Suppose that h is a homogeneous horizontal endomorphism on M and let $(\overset{\circ}{D}, h)$ be the Berwald-type Finsler connection induced by h. If S is a semispray on M, then*

$$\forall X, Y \in \mathfrak{X}(\mathcal{T}M): \quad \overset{\circ}{\mathbb{R}}(X,Y)S = \Omega(X,Y). \tag{2.17}$$

Proof. Since $\overset{\circ}{\mathbb{R}}$ and Ω are semibasic, it is enough to check that

$$\forall X, Y \in \mathfrak{X}(M): \quad \overset{\circ}{\mathbb{R}}(X^h, Y^h)S = \Omega(X^h, Y^h). \tag{2.17a}$$

We can also assume that S is the semispray associated with h. Then $hS = S$ and (2.15a) yields the relation

$$\begin{aligned}\overset{\circ}{\mathbb{R}}(X^h, Y^h)S &= [J, \Omega(X^h, Y^h)]S = [C, \Omega(X^h, Y^h)] - J[S, \Omega(X^h, Y^h)] \\ &\overset{(2.6b),(2.8a)}{=} \big[v[X^h, Y^h], C\big] + J\big[JF\Omega(X^h, Y^h), S\big] \\ &\overset{\text{lemma 1.12}}{=} \big[v[X^h, Y^h], C\big] + \Omega(X^h, Y^h).\end{aligned}$$

It remains only to show that the first term on the right hand side vanishes. In view of the homogeneity,

$$[v, C] = 0 \text{ and } [X^h, C] = 0,$$

thus, in particular,

$$0 = [v, C][X^h, Y^h] = \big[v[X^h, Y^h], C\big] - v\big[[X^h, Y^h], C\big].$$

Finally, using the Jacobi identity and the homogeneity of h over again, we obtain that

$$0 = \big[[X^h, Y^h], C\big] + \big[[Y^h, C], X^h\big] + \big[[C, X^h], Y^h\big] = \big[[X^h, Y^h], C\big]$$

which completes the proof. □

2.18. Corollary. *Hypothesis as in Proposition 2.17. Then*

$$\overset{\circ}{\mathbb{R}} = 0 \iff \Omega = 0.$$

Proof. If Ω vanishes, then $\overset{\circ}{\mathbb{R}}$ also vanishes by 2.15(a). Conversely, it follows from (2.17) that $\overset{\circ}{\mathbb{R}} = 0 \implies \Omega = 0$. □

3. Finsler manifolds. The Cartan tensors

3.1. Definitions. Let a function $E\colon TM \to \mathbb{R}$ be given. The pair (M, E), or simply M, is said to be a *Finsler manifold*, if the following conditions are satisfied:

(3.1a) $\forall a \in \mathcal{T}M: \quad E(a) > 0;\ E(0) = 0.$

(3.1b) E is of class C^1 on TM and smooth over $\mathcal{T}M$.

(3.1c) $CE = 2E$; i.e. E is homogeneous of degree 2.

(3.1d) The *fundamental form* $\omega := dd_J E \in \Omega^2(\mathcal{T}M)$ is nondegenerate.

The function E is called the *energy function* of the Finsler manifold. A horizontal endomorphism on M is said to be *conservative* if $d_h E = 0$.

3.2. Metrics. Assume (M, E) is a Finsler manifold with fundamental form ω.
(a) The mapping

(3.2a) $$\bar{g}\colon \mathfrak{X}^v(\mathcal{T}M) \times \mathfrak{X}^v(\mathcal{T}M) \to C^\infty(\mathcal{T}M), (JX, JY) \mapsto \bar{g}(JX, JY) := \omega(JX, Y)$$

is a well-defined, nondegenerate, symmetric bilinear form which is said to be the *Riemann–Finsler metric* of (M, E). The Finsler manifold is called *positive definite* if $\bar{g}$ is positive definite.
(b) Suppose that h is a horizontal endomorphism on M, $v = 1 - h$. Then

(3.2b) $$\begin{gathered} g\colon \mathfrak{X}(\mathcal{T}M) \times \mathfrak{X}(\mathcal{T}M) \to C^\infty(\mathcal{T}M), \\ (X, Y) \mapsto g(X, Y) := \bar{g}(JX, JY) + \bar{g}(vX, vY) \end{gathered}$$

is a pseudo-Riemannian metric on $\mathcal{T}M$, called the *prolongation of* $\bar{g}$ along h.

3.3. Remark. It follows at once from (3.2b) that

(3.3a) $$\forall X, Y \in \mathfrak{X}(\mathcal{T}M): \ g(hX, JY) = 0.$$

3.4. Lemma. *Hypothesis as in 3.2. (b).*

(3.4a) $$g(C, C) = \bar{g}(C, C) = 2E.$$

(3.4b) $$\forall X, Y \in \mathfrak{X}(M): \quad g(X^v, Y^v) = \bar{g}(X^v, Y^v) = X^v(Y^v E).$$

Proof. (a) Let S be an arbitrary semispray on M. Then

$$\begin{aligned} g(C, C) &:= \bar{g}(JC, JC) + \bar{g}(vC, vC) = \bar{g}(C, C) = \bar{g}(JS, JS) = \\ &\overset{(3.2a)}{=} \omega(C, S) = (\imath_C \omega)(S) \overset{(3.1d)}{=} \imath_C dd_J E(S) = \\ &\overset{(1.9e)}{=} -\imath_C d_J dE(S) \overset{(1.9f)}{=} (d_J \imath_C dE - \imath_J dE)(S) = \\ &\overset{(3.1c)}{=} (2d_J E - \imath_J dE)(S) \overset{(1.9d)}{=} (\imath_J dE)(S) = \\ &= (dE)(C) = CE \overset{(3.1c)}{=} 2E. \end{aligned}$$

(b)

$$\begin{aligned} g(X^v, Y^v) = \bar{g}(X^v, Y^v) &\overset{(1.13c)}{=} \bar{g}(JX^c, JY^c) \overset{(3.2a)}{=} \omega(X^v, Y^c) = \\ &\overset{(3.1d)}{=} d(d_J E)(X^v, Y^c) = X^v\big(d_J E(Y^c)\big) - Y^c\big(d_J E(X^v)\big) \end{aligned}$$

$$- d_J E[X^v, Y^c] \overset{(1.9d)}{=} X^v\big(dE(JY^c)\big) - Y^c\big(dE(JX^v)\big)$$
$$- dE(J[X^v, Y^c]) \overset{(1.13a,c)}{=} X^v\big(dE(Y^v)\big) = X^v(Y^v E).$$

□

3.5. The Cartan tensors. Let a Finsler manifold (M, E) be given. Suppose that h is a horizontal endomorphism on M and let g be the prolongation of $\bar{g}$ along h.
(i) There exists a unique tensor

$$\mathcal{C}\colon \mathfrak{X}(\mathfrak{T}M) \times \mathfrak{X}(\mathfrak{T}M) \to \mathfrak{X}(\mathfrak{T}M)$$

such that $J \circ \mathcal{C} = 0$ and

$$\forall X, Y, Z \in \mathfrak{X}(\mathfrak{T}M): \quad g(\mathcal{C}(X,Y), JZ) = \frac{1}{2}\left(\mathcal{L}_{JX} J^* g\right)(Y, Z). \tag{3.5a}$$

The tensor $\mathcal{C}$, as well as its lowered tensor $\mathcal{C}_\flat$ defined by

$$\forall X, Y, Z \in \mathfrak{X}(\mathfrak{T}M): \quad \mathcal{C}_\flat(X, Y, Z) := g(\mathcal{C}(X,Y), JZ), \tag{3.5b}$$

is called the *first Cartan tensor* of the Finsler manifold.
(ii) Analogously, we introduce a tensor

$$\mathcal{C}'\colon \mathfrak{X}(\mathfrak{T}M) \times \mathfrak{X}(\mathfrak{T}M) \to \mathfrak{X}(\mathfrak{T}M)$$

by the conditions $J \circ \mathcal{C}' = 0$ and

$$\forall X, Y, Z \in \mathfrak{X}(\mathfrak{T}M): \quad g(\mathcal{C}'(X,Y), JZ) = \frac{1}{2}\left(\mathcal{L}_{hX} g\right)(JY, JZ). \tag{3.5c}$$

Then $\mathcal{C}'$ is well-defined; it is called the *second Cartan tensor* of the Finsler manifold, belonging to the horizontal endomorphism h. We use the same terminology also for the lowered tensor $\mathcal{C}'_\flat$.

3.6. Remark. $\mathcal{C}$ and $\mathcal{C}'$ are clearly semibasic. We shall see soon that $\mathcal{C}$ is independent of the choice of the horizontal endomorphism h, it depends on the energy function alone (i.e., on the Finsler structure). On the other hand, the second Cartan tensor $\mathcal{C}'$ strongly depends on the horizontal endomorphism!

3.7. Lemma. *Let (M, E) be a Finsler manifold and $(\overset{\circ}{D}, h)$ a Berwald-type Finsler connection on M. Then for each vector fields X, Y, Z on M,*

$$2\mathcal{C}_\flat(X^c, Y^c, Z^c) = \left(\overset{\circ}{D}_{X^v} g\right)(Y^v, Z^v) = X^v[Y^v(Z^v E)]; \tag{3.7a}$$

$$2\mathcal{C}'_\flat(X^c, Y^c, Z^c) = \left(\overset{\circ}{D}_{X^h} g\right)(Y^v, Z^v) = \big[Y^v, [X^h, Z^v]\big]E + {} \\ + Y^v[Z^v(X^h E)] \tag{3.7b}$$

($\mathcal{C}'$ is the second Cartan tensor belonging to h).

Proof. (a) On the one hand

$$2\mathcal{C}_\flat(X^c, Y^c, Z^c) \overset{(3.5b)}{=} 2g(\mathcal{C}(X^c, Y^c), Z^v) = \left(\mathcal{L}_{X^v} J^* g\right)(Y^c, Z^c) =$$
$$= X^v g(Y^v, Z^v) \overset{(3.4b)}{=} X^v[Y^v(Z^v E)],$$

since $J[X^v, Y^c] = J[X, Y]^v = 0$, $J[X^v, Z^c] = 0$; on the other hand

$$\left(\overset{\circ}{D}_{X^v} g\right)(Y^v, Z^v) = X^v g(Y^v, Z^v) - g(\overset{\circ}{D}_{X^v} Y^v, Z^v) - g(Y^v, \overset{\circ}{D}_{X^v} Z^v) = \\ = X^v g(Y^v, Z^v),$$

since e.g. $\overset{\circ}{D}_{X^v} Y^v = \overset{\circ}{D}_{JX^c} JY^c \overset{(2.14a)}{=} J[X^v, Y^c] = 0$. Thus (3.7a) is proved.
(b) Computing as before, we find that

$$\begin{aligned} 2\mathcal{C}'_\flat(X^c, Y^c, Z^c) = \left(\overset{\circ}{D}_{X^h} g\right)(Y^v, Z^v) &= X^h g(Y^v, Z^v) - g([X^h, Y^v], Z^v) - \\ - g(Y^v, [X^h, Z^v]) &\overset{(3.4b)}{=} X^h[Y^v(Z^v E)] - [X^h, Y^v](Z^v E) - \\ - g([X^h, Z^v], Y^v) &= Y^v[X^h(Z^v E)] - [X^h, Z^v](Y^v E) = \\ &= Y^v([X^h, Z^v]E + Z^v(X^h E)) - [X^h, Z^v](Y^v E) = \\ &= \big[Y^v, [X^h, Z^v]\big]E + Y^v[Z^v(X^h E)]. \end{aligned}$$

□

3.8. Corollary. *The first Cartan tensor is symmetric, the lowered first Cartan tensor is totally symmetric.*

Proof. We infer this immediately from (3.7a): since the Lie bracket of vertically lifted vector fields vanishes, $X^v[Y^v(Z^v E)]$ does not depend on the order of the vector fields. □

3.9. Lemma. *If S is an arbitrary semispray, then $\imath_S \mathcal{C} = \imath_S \mathcal{C}_\flat = 0$.*

Proof. Due to symmetry, it is enough to check that for each vector fields X, Y on M we have $\mathcal{C}_\flat(S, X, Y) = 0$. From (3.7a)

$$2\mathcal{C}_\flat(S, X^c, Y^c) = \left(\overset{\circ}{D}_{JS} g\right)(X^v, Y^v) = Cg(X^v, Y^v) - g(\overset{\circ}{D}_C X^v, Y^v) - \\ - g(X^v, \overset{\circ}{D}_C Y^v) = C[X^v(Y^v E)],$$

since e.g.

$$\overset{\circ}{D}_C X^v = \overset{\circ}{D}_{JS} JX^c \overset{(2.14a)}{=} J[C, X^c] \overset{(1.13b)}{=} 0$$

We next show that $C[X^v(Y^v E)]$ vanishes. Use for a moment the abbreviation $\tilde{E} := Y^v E$. Then

$$\begin{aligned} C[X^v(Y^v E)] = C(X^v \tilde{E}) &= [C, X^v]\tilde{E} + X^v C\tilde{E} \overset{(1.13b)}{=} -X^v \tilde{E} + X^v C\tilde{E} \\ &= -X^v(Y^v E) + X^v[C(Y^v E)] = -X^v(Y^v E) + \\ &\quad + X^v\left([C, Y^v]E + Y^v(CE)\right) \overset{(1.13b),(3.1c)}{=} \\ &= -X^v(Y^v E) + X^v(-Y^v E + 2Y^v E) = 0, \end{aligned}$$

which ends the proof. □

3.10. Proposition. *Let (M, E) be a Finsler manifold. If h is a conservative, torsion-free horizontal endomorphism then the lowered second Cartan tensor is totally symmetric.*

Proof. Since h is conservative, for each vector field X on M,

$$0 = (d_h E)(X^c) \overset{(1.9d)}{=} (i_h dE)(X^c) = (dE)X^h = X^h E,$$

so it follows from (3.7b) that

$$\forall X, Y, Z \in \mathfrak{X}(M) : \quad 2\mathcal{C}'_\flat(X^c, Y^c, Z^c) = \big[Y^v, [X^h, Z^v]\big]E.$$

We easily see from Definition (2.2c) of the torsion

$$\forall X, Y \in \mathfrak{X}(M) : \quad t(X^c, Y^c) = [X^h, Y^v] - [Y^h, X^v] - [X, Y]^v. \tag{3.10}$$

Now, using the Jacobi identity and the condition $t = 0$, we obtain

$$\begin{aligned} 0 &= \big[Y^v, [X^h, Z^v]\big] + \big[X^h, [Z^v, Y^v]\big] + \big[Z^v, [Y^v, X^h]\big] \overset{(1.13a)}{=} \\ &= \big[Y^v, [X^h, Z^v]\big] + \big[Z^v, [Y^v, X^h]\big] = \big[Y^v, [X^h, Z^v]\big] + \\ &\quad + \big[Z^v, -[Y^h, X^v]\big] + \big[Z^v, -[X, Y]^v\big] = \big[Y^v, [X^h, Z^v]\big] - \big[Z^v, [Y^h, X^v]\big]; \end{aligned}$$

therefore $\big[Y^v, [X^h, Z^v]\big] = \big[Z^v, [Y^h, X^v]\big]$. This means that

$$\mathcal{C}'_\flat(X^c, Y^c, Z^c) = \mathcal{C}'_\flat(Y^c, Z^c, X^c).$$

The other symmetries of $\mathcal{C}'_\flat$ can be shown in the same manner. □

4. Notable Finsler connections

4.1. Lemma and definition. *On any Finsler manifold (M, E) there is a spray $S \colon TM \to TTM$, uniquely determined on $\mathcal{T}M$ by the relation*

$$\imath_S \omega = -dE \tag{4.1}$$

and prolonged to a C^1-mapping of TM such that $S \restriction TM \setminus \mathcal{T}M = 0$. The spray S is called the canonical spray *of the Finsler manifold.*

For a proof, see [10], p.323 and [7], p.60.

4.2. Theorem (Fundamental lemma of Finsler geometry)**.** *Let (M, E) be a Finsler manifold. There exists a* unique *horizontal endomorphism h on M,called the* Barthel endomorphism*, such that*

(a) *h is conservative (i.e., $d_h E = 0$),*

(b) *h is homogeneous (i.e., $H = [h, C] = 0$),*

(c) *h is torsion-free (i.e., $t = [J, h] = 0$).*

Explicitly,

$$h = \frac{1}{2}\left(1 + [J, S]\right),$$

where S is the canonical spray.

The result is due to J. Grifone [10].

4.3. Theorem. *Let (M, E) be a Finsler manifold and let h be a horizontal endomorphism on M. There exists a unique Finsler connection $(\overset{\circ}{D}, h)$ on M such that*

(4.3a) *the $(v)hv$-torsion $\overset{\circ}{\mathbb{P}}{}^1$ of $\overset{\circ}{D}$ vanishes;*

(4.3b) *the $(h)hv$-torsion $\overset{\circ}{\mathbb{B}}$ of $\overset{\circ}{D}$ vanishes.*

This Finsler connection is of Berwald-type, so the covariant derivatives with respect to $\overset{\circ}{D}$ can be calculated by (2.14a)–(2.14d). If $(\overset{\circ}{D}, h)$ satisfies the further conditions

(4.3c) *h is conservative;*

(4.3d) *the h-deflection h^*DC vanishes,*

(4.3e) *the $(h)h$-torsion $\overset{\circ}{\mathbb{A}}$ of $\overset{\circ}{D}$ vanishes,*

then h is just the Barthel endomorphism of the Finsler manifold.

4.4. Remark. The Finsler connection determined by (4.3a)–(4.3e) is said to be the *Berwald connection* of the Finsler manifold. The axioms presented here were at first formulated by T. Okada [15], with a slight difference. The novelty of our approach consists in drawing a distinction between the roles of the first group (4.3a)–(4.3b) and of the second group (4.3c)–(4.3e) of axioms. For an intrinsic proof of the theorem, see [17].

4.5. Theorem and definition. *Let (M, E) be a Finsler manifold and suppose that h is a conservative torsion-free horizontal endomorphism on M. Let g be the prolongation of $\bar{g}$ along h and $\mathcal{C}'$ the second Cartan tensor belonging to h. There exists a unique Finsler connection (D, h) on M such that*

(4.5a) *D is metrical (i.e. $Dg=0$);*

(4.5b) *the $(v)v$-torsion $\mathbb{S}^1$ of D vanishes;*

(4.5c) *the $(h)h$-torsion $\mathbb{A}$ of D vanishes.*

The covariant derivatives with respect to D can be explicitly calculated by the following formulas: for each vector fields X, Y on $\mathfrak{T}M$,

$$D_{JX}JY = J[JX, Y] + \mathcal{C}(X, Y) = \overset{\circ}{D}_{JX}JY + \mathcal{C}(X, Y); \tag{4.5d}$$

$$D_{hX}JY = v[hX, JY] + \mathcal{C}'(X, Y) = \overset{\circ}{D}_{hX}JY + \mathcal{C}'(X, Y); \tag{4.5e}$$

$$D_{JX}hY = h[JX, Y] + F\mathcal{C}(X, Y) = \overset{\circ}{D}_{JX}hY + F\mathcal{C}(X, Y); \tag{4.5f}$$

$$D_{hX}hY = hF[hX, JY] + F\mathcal{C}'(X, Y) = \overset{\circ}{D}_{hX}hY + F\mathcal{C}'(X, Y). \tag{4.5g}$$

Then

$$h^*DC = \frac{1}{2}H$$

where H is the tension of h (2.2b). Therefore, if in addition to (4.5a)–(4.5c)

$$h^*DC = 0 \tag{4.5h}$$

is also satisfied, then h is the Barthel endomorphism of the Finsler manifold. In this case (D,h) is called the (classical) Cartan connection *of the Finsler manifold (M,E).*

Proof. The idea of the *existence proof* is immediate. We start from a conservative, torsion-free horizontal endomorphism h (whose existence is clearly guaranteed) and build the second Cartan tensor $\mathcal{C}'$ belonging to h. Then we define a rule of covariant differentation by the formulas (4.5d)–(4.5g). It can be checked by a straightforward calculation that the pair (D,h) obtained in this way is indeed a Finsler connection, and the axioms (4.5a)–(4.5c) are satisfied.

In our subsequent considerations we are going to prove the *unicity* statement. Assume (D,h) is a Finsler connection on M, satisfying (4.5a)–(4.5c). We show that the rules of calculation (4.5d)–(4.5g) are valid.

1st step. Applying the "Christoffel process", we derive (4.5d). We can restrict ourselves to vertically lifted vector fields. From condition (4.5a), for any vector fields $X,Y,Z\in\mathfrak{X}(M)$,

$$\begin{aligned} X^v g(Y^v,Z^v) &= g(D_{X^v}Y^v,Z^v)+g(Y^v,D_{X^v}Z^v),\\ Y^v g(Z^v,X^v) &= g(D_{Y^v}Z^v,X^v)+g(Z^v,D_{Y^v}X^v),\\ -Z^v g(X^v,Y^v) &= -g(D_{Z^v}X^v,Y^v)-g(X^v,D_{Z^v}Y^v).\end{aligned}$$

Since from (4.5b) $D_{X^v}Y^v - D_{Y^v}X^v = [X^v,Y^v]=0$ and so on, adding the above three equations we get the relation

$$\begin{aligned} g(2D_{X^v}Y^v,Z^v) &= X^v g(Y^v,Z^v)+Y^v g(Z^v,X^v)-Z^v g(X^v,Y^v) =\\ &\overset{(3.4b),(3.7a),3.8}{=} 2\mathcal{C}_\flat(X^c,Y^c,Z^c)=2g(\mathcal{C}(X^c,Y^c),Z^v).\end{aligned}$$

Hence

$$D_{X^v}Y^v=\mathcal{C}(X^c,Y^c)=\mathcal{C}(hX^c+vX^c,hY^c+vY^c)=\mathcal{C}(X^h,Y^h),$$

from which (4.5d) easily follows. In view of (2.10c) this implies (4.5f).

2nd step. Now we apply the Christoffel process to the h-covariant derivatives of g. Then (4.5a) yields the relations

$$\text{(a)}\qquad \begin{cases} X^h g(Y^v,Z^v)=g(D_{X^h}Y^v,Z^v)+g(Y^v,D_{X^h}Z^v)\\ Y^h g(Z^v,X^v)=g(D_{Y^h}Z^v,X^v)+g(Z^v,D_{Y^h}X^v),\\ -Z^h g(X^v,Y^v)=-g(D_{Z^h}X^v,Y^v)-g(X^v,D_{Z^h}Y^v)\end{cases}$$

$(X,Y,Z\in\mathfrak{X}(M))$. From condition (4.5c), i.e., from the vanishing of the $(h)h$-torsion we conclude that e.g.

$$D_{X^h}Y^h-D_{Y^h}X^h=h[X^h,Y^h]=[X,Y]^h=h[X,Y]^c,$$

hence

$$FD_{X^h}Y^h-FD_{Y^h}X^h=Fh[X,Y]^c\overset{(2.7),(1.13c)}{=}-[X,Y]^v.$$

So, taking into account (2.10d), we obtain the relations

$$\text{(b)}\qquad \begin{cases} D_{X^h}Y^v-D_{Y^h}X^v=[X,Y]^v,\\ D_{Y^h}Z^v-D_{Z^h}Y^v=[Y,Z]^v,\\ -D_{Z^h}X^v+D_{X^h}Z^v=-[Z,X]^v.\end{cases}$$

Adding now both sides of (a) and using (b), it follows that

$$\text{(c)}\qquad \begin{aligned} g(2D_{X^h}Y^v, Z^v) &= X^h g(Y^v, Z^v) + Y^h g(Z^v, X^v) - Z^h g(X^v, Y^v) + \\ &+ g([X,Y]^v, Z^v) - g([Y,Z]^v, X^v) + g([Z,X]^v, Y^v). \end{aligned}$$

<u>3rd step.</u> We apply the Christoffel process to the tensor $\mathcal{C}'$ belonging to h. For any vector fields $X, Y, Z \in \mathfrak{X}(M)$, we get:

$$\begin{aligned} 2g(\mathcal{C}'(X^h, Y^h), Z^v) &= X^h g(Y^v, Z^v) - g([X^h, Y^v], Z^v) - g(Y^v, [X^h, Z^v]), \\ 2g(\mathcal{C}'(Y^h, Z^h), X^v) &= Y^h g(Z^v, X^v) - g([Y^h, Z^v], X^v) - g(Z^v, [Y^h, X^v]), \\ -2g(\mathcal{C}'(Z^h, X^h), Y^v) &= -Z^h g(X^v, Y^v) + g([Z^h, X^v], Y^v) + g(X^v, [Z^h, Y^v]). \end{aligned}$$

Adding these three equations, in view of the symmetry of $\mathcal{C}'$ (assured by Proposition 3.10) we obtain:

$$\text{(d)}\qquad \begin{aligned} g(2\mathcal{C}'(X^h, Y^h), Z^v) &= X^h g(Y^v, Z^v) + Y^h g(Z^v, X^v) - Z^h g(X^v, Y^v) - \\ &- g([X^h, Y^v] + [Y^h, X^v], Z^v) + g([Z^h, X^v] - [X^h, Z^v], Y^v) + \\ &+ g([Z^h, Y^v] - [Y^h, Z^v], X^v). \end{aligned}$$

From (c) and (d) it follows that

$$\text{(e)}\qquad \begin{aligned} g(2D_{X^h}Y^v, Z^v) &= g(2\mathcal{C}'(X^h, Y^h), Z^v) + \\ &+ g([X^h, Y^v] + [Y^h, X^v] + [X,Y]^v, Z^v) + \\ &+ g([X^h, Z^v] - [Z^h, X^v] - [X,Z]^v, Y^v) + \\ &+ g([Y^h, Z^v] - [Z^h, Y^v] - [Y,Z]^v, X^v). \end{aligned}$$

Since h is torsion-free, the last two terms on the right hand side of (e) vanish (c.f. (3.10)), while in the second term

$$[X^h, Y^v] + [Y^h, X^v] + [X,Y]^v = 2[X^h, Y^v].$$

Hence

$$g(2D_{X^h}Y^v, Z^v) = g(2\mathcal{C}'(X^h, Y^h) + 2[X^h, Y^v], Z^v),$$

which yields the formula

$$D_{X^h}Y^v = [X^h, Y^v] + \mathcal{C}'(X^h, Y^h).$$

This proves (4.5e) and, by (2.10d), (4.5g). We have thus established the unicity assertion.

<u>4th step.</u> We show that

$$h^* DC = \frac{1}{2} H.$$

Let X and Y be arbitrary vector fields on M. Then $h^* DC(X^h) = DC(X^h) = D_{X^h}C$ and

$$\begin{aligned} g(D_{X^h}C, Y^v) \quad &= \quad g(D_{X^h}JS, Y^v) \overset{(4.5e)}{=} g\big([X^h, C], Y^v\big) + \\ &+ g(\mathcal{C}'(X^h, S), Y^v) \overset{(3.5c)}{=} g\big([X^h, C], Y^v\big) + \tfrac{1}{2} X^h g(C, Y^v) - \\ &- \tfrac{1}{2} g\big([X^h, C], Y^v\big) - \tfrac{1}{2} g\big([X^h, Y^v], C\big) = \tfrac{1}{2} g\big([X^h, C], Y^v\big) + \\ &+ \tfrac{1}{2}\left(X^h(Y^v E) - [X^h, Y^v]E\right) = \tfrac{1}{2} g\big([X^h, C], Y^v\big) + \tfrac{1}{2} Y^v(X^h E) = \end{aligned}$$

$$\overset{d_hE=0}{=} g\big(\tfrac{1}{2}[X^h, C], Y^v\big),$$

from which follows that

$$\frac{1}{2}[X^h, C] = D_{X^h}C.$$

This means that $\frac{1}{2}H(X^h) = h^*DC(X^h)$, proving our assertion. □

4.6. Remarks.
(a) Observe that axioms (4.5a) and (4.5h) imply for any Finsler connection (D, h) that h is a conservative horizontal endomorphism. Indeed, for each vector field X on M,

$$\begin{aligned} 0 &\overset{(4.5a)}{=} (D_{X^h}g)(C, C) = X^h g(C, C) - 2g(D_{X^h}C, C) = \\ &\overset{(3.4a)}{=} 2X^hE - 2g(D_{X^h}C, C) \overset{(4.5h)}{=} 2X^hE = 2(d_hE)(X^h), \end{aligned}$$

which means that $d_hE = 0$.
(b) Axioms to characterize the classical Cartan connection were first formulated by M. Matsumoto; for an instructive historical remark see [14], p.112.

4.7. Corollary. *Let (M, E) be a Finsler manifold and h the Barthel endomorphism. If $\mathcal{C}'$ is the second Cartan tensor belonging to h, then $\imath_S\mathcal{C}' = 0$, for any semispray S.*

Proof. We consider the classical Cartan connection (D, h). Then for each vector field $X \in \mathfrak{X}(\mathcal{T}M)$:

$$\begin{aligned} 0 &\overset{(4.5h)}{=} D_{hX}C = D_{hX}JS \overset{(4.5e)}{=} v[hX, C] + \mathcal{C}'(X, S) = \\ &= H(hX) + \mathcal{C}'(X, S) = 2(h^*DC)(hX) + \mathcal{C}'(X, S) \overset{(4.5h)}{=} \mathcal{C}'(X, S). \end{aligned}$$

□

4.8. Lemma. *Let (M, E) be a Finsler manifold, (D, h) the classical Cartan connection and S the canonical spray. Then $D_S\mathcal{C} = -\mathcal{C}'$ ($\mathcal{C}'$ is the second Cartan tensor belonging to h).*

For a proof see [11], pp. 331–332.

4.9. Theorem and definition. *Let (M, E) be a Finsler manifold and h a conservative torsion-free horizontal endomorphism on M. Assume g is the prolongation of $\bar{g}$ along h and C' is the second Cartan tensor belonging to h. There exists a unique Finsler connection $(\overset{R}{D}, h)$ on M such that*

(4.9a) $\qquad J^*\overset{R}{D} = J^*\overset{\circ}{D}$;

(4.9b) $\qquad \overset{R}{D}$ *is h-metrical, i.e. $\forall X \in \mathfrak{X}(\mathcal{T}M)$: $\overset{R}{D}_{hX}g = 0$;*

(4.9c) $\qquad$ *the $(h)h$-torsion of $\overset{R}{D}$ vanishes.*

The covariant derivatives with respect to $\overset{R}{D}$ can be calculated by the following formulas: for each vector fields X, Y on $\mathcal{T}M$,

(4.9d) $\qquad \overset{R}{D}_{JX}JY = J[JX, Y] = \overset{\circ}{D}_{JX}JY$;

$$\overset{R}{D}_{hX} JY = v[hX, JY] + \mathcal{C}'(X,Y) = D_{hX} JY; \tag{4.9e}$$

$$\overset{R}{D}_{JX} hY = h[JX, Y] = \overset{\circ}{D}_{JX} hY; \tag{4.9f}$$

$$\overset{R}{D}_{hX} hY = hF[hX, JY] + F\mathcal{C}'(X,Y) = D_{hX} hY. \tag{4.9g}$$

Then

$$h^* \overset{R}{D} C = \frac{1}{2} H.$$

Therefore, if in addition to (4.5a)–(4.5c)

$$h^* \overset{R}{D} C = 0 \tag{4.9h}$$

is also satisfied, then h is the Barthel endomorphism of the Finsler manifold. In this case $(\overset{R}{D}, h)$ *is called the* (classical) Rund *(or the* Chern – Rund) connection.

The proof of this theorem is completely analogous to that of Theorem 4.5.

4.10. Remark. The classical Chern – Rund connection was first constructed by S. S. Chern in 1948, using a local coframe field. Three years later H. Rund also discovered an important, seemingly different Finsler connection. Almost fifty years had passed until M. Anastasiei [2] realized that both constructions result the *same* Finsler connection.

4.11. Remark. Using vertically and horizontally lifted vector fields, the rules of calculation for covariant derivatives take a somewhat simpler form. Namely, if $\widetilde{D}$ stands for $\overset{\circ}{D}$, D or $\overset{\mathrm{R}}{D}$, we get the following table:

	Berwald ($\overset{\circ}{D}$)	Cartan (D)	Rund ($\overset{\mathrm{R}}{D}$)
$\widetilde{D}_{X^v} Y^v$	0	$\mathcal{C}(X^h, Y^h)$	0
$\widetilde{D}_{X^h} Y^v$	$[X^h, Y^v]$	$[X^h, Y^v] + \mathcal{C}'(X^h, Y^h)$	$[X^h, Y^v] + \mathcal{C}'(X^h, Y^h)$
$\widetilde{D}_{X^v} Y^h$	0	$F\mathcal{C}(X^h, Y^h)$	0
$\widetilde{D}_{X^h} Y^h$	$F[X^h, Y^v]$	$F[X^h, Y^v] + F\mathcal{C}'(X^h, Y^h)$	$F[X^h, Y^v] + F\mathcal{C}'(X^h, Y^h)$

5. Basic curvature identities

5.1. Convention. Throughout this section (M, E) is a Finsler manifold and h is the Barthel endomorphism on M. $\overset{\circ}{D}$, D and $\overset{\mathrm{R}}{D}$ denote the Berwald, the (classical) Cartan and the Rund connection, respectively.

5.2. Proposition. *Let $\overset{\circ}{\mathbb{R}}$, $\mathbb{R}$ and $\overset{R}{\mathbb{R}}$ be the h-curvature of the Berwald, the Cartan and the Rund connection, respectively. We have the following relations:*

$$\begin{aligned} \mathbb{R}(X,Y)Z = \overset{\circ}{\mathbb{R}}(X,Y)Z &+ (D_{hX}\mathcal{C}')(Y,Z) - (D_{hY}\mathcal{C}')(X,Z) - \\ &- \mathcal{C}'(X, F\mathcal{C}'(Y,Z)) + \mathcal{C}'(Y, F\mathcal{C}'(X,Z)) + \\ &+ \mathcal{C}(F\Omega(X,Y), Z); \end{aligned} \tag{5.2a}$$

$$\overset{R}{\mathbb{R}}(X,Y)Z = \mathbb{R}(X,Y)Z - \mathcal{C}(F\Omega(X,Y),Z) \qquad (X,Y,Z \in \mathfrak{X}(\mathcal{T}M)). \tag{5.2b}$$

Proof. The first formula has been obtained by J. Grifone; see [11], pp.333–334. Since the tensors $\mathbb{R}$ and $\overset{\mathrm{R}}{\mathbb{R}}$ are semibasic, it is enough to check (5.2b) for triplets of the form (X^c, Y^c, Z^c); $X,Y,Z \in \mathfrak{X}(M)$. Taking into account that $\overset{\mathrm{R}}{D}{}^h = D^h$ from (4.9e) and (4.9g) while $\overset{\mathrm{R}}{D}{}^v = \overset{\circ}{D}{}^v$ from (4.9d) and (4.9g), we get:

$$\begin{aligned}
\overset{\mathrm{R}}{\mathbb{R}}(X^c,Y^c)Z^c &= \overset{\mathrm{R}}{\mathbb{K}}(hX^c,hY^c)JZ^c = \overset{\mathrm{R}}{D}_{X^h}\overset{\mathrm{R}}{D}_{Y^h}Z^v - \overset{\mathrm{R}}{D}_{Y^h}\overset{\mathrm{R}}{D}_{X^h}Z^v - \overset{\mathrm{R}}{D}_{[X^h,Y^h]}Z^v = \\
&= D_{X^h}D_{Y^h}Z^v - D_{Y^h}D_{X^h}Z^v - \overset{\mathrm{R}}{D}_{h[X^h,Y^h]}Z^v - \overset{\mathrm{R}}{D}_{v[X^h,Y^h]}Z^v = \\
&= \mathbb{R}(X^c,Y^c)Z^c + D_{v[X^h,Y^h]}Z^v - \overset{\mathrm{R}}{D}_{v[X^h,Y^h]}Z^v = \\
&= \mathbb{R}(X^c,Y^c)Z^c + \mathcal{C}(F[X^h,Y^h],Z^c) = \mathbb{R}(X^c,Y^c)Z^c + \mathcal{C}(Fv[X^h,Y^h],Z^c) + \\
&\quad + \mathcal{C}(Fh[X^h,Y^h],Z^c) \overset{(2.7),(2.6b)}{=} \mathbb{R}(X^c,Y^c)Z^c - \mathcal{C}(F\Omega(X^c,Y^c),Z^c).
\end{aligned}$$

□

5.3. Proposition. *The hv-curvature tensors of the Cartan, the Berwald and the Rund connection are related as follows:*

$$\begin{aligned}
\mathbb{P} = \overset{\circ}{\mathbb{P}}(X,Y)Z &+ (D_{hX}\mathcal{C})(hY,hZ) - (D_{JY}\mathcal{C}')(hX,hZ) + \\
&+ \mathcal{C}(F\mathcal{C}'(hX,hY),hZ) + \mathcal{C}(F\mathcal{C}'(hX,hZ),hY) - \\
&- \mathcal{C}'(F\mathcal{C}(hX,hY),hZ) - \mathcal{C}'(F\mathcal{C}(hY,hZ),hX).
\end{aligned} \tag{5.3a}$$

$$\overset{R}{\mathbb{P}}(X^c,Y^c)Z^c = \overset{\circ}{\mathbb{P}}(X^c,Y^c)Z^c + [\mathcal{C}'(X^h,Z^h),Y^v]. \tag{5.3b}$$

Proof. In the same manner as above, let us consider a triplet X^c, Y^c, Z^c;

(a)
$$\begin{aligned}
\mathbb{P}(X^c,Y^c)Z^c &\overset{(2.12b)}{=} \mathbb{K}(hX^c,JY^c)JZ^c \overset{(1.13c),(2.2a)}{=} \mathbb{K}(X^h,Y^v)Z^v = \\
&= D_{X^h}D_{Y^v}Z^v - D_{Y^v}D_{X^h}Z^v - D_{[X^h,Y^v]}Z^v \overset{4.11}{=} D_{X^h}\mathcal{C}(Y^h,Z^h) - \\
&\quad - D_{Y^v}([X^h,Z^v] + \mathcal{C}'(X^h,Z^h)) - D_{JF[X^h,Y^v]}JZ^h = \\
&= (D_{X^h}\mathcal{C})(Y^h,Z^h) + \mathcal{C}(D_{X^h}Y^h,Z^h) + \mathcal{C}(Y^h,D_{X^h}Z^h) - \\
&\quad - D_{Y^v}JF[X^h,Z^v] - D_{Y^v}\mathcal{C}'(X^h,Z^h) - J\big[[X^h,Y^v],Z^h\big] - \\
&\quad - \mathcal{C}(F[X^h,Y^v],Z^h) \overset{4.11}{=} (D_{X^h}\mathcal{C})(Y^h,Z^h) + \mathcal{C}(F[X^h,Y^v],Z^h) + \\
&\quad + \mathcal{C}(F\mathcal{C}'(X^h,Y^h),Z^h) + \mathcal{C}(Y^h,F[X^h,Z^v]) + \mathcal{C}(Y^h,F\mathcal{C}'(X^h,Z^h)) - \\
&\quad - J\big[Y^v,F[X^h,Z^v]\big] - \mathcal{C}(Y^h,F[X^h,Z^v]) - (D_{Y^v}\mathcal{C}')(X^h,Z^h) - \\
&\quad - \mathcal{C}'(F\mathcal{C}(Y^h,X^h),Z^h) - \mathcal{C}'(X^h,F\mathcal{C}(Y^h,Z^h)) - \mathcal{C}(F[X^h,Y^v],Z^h) = \\
&= \overset{\circ}{\mathbb{P}}(X^c,Y^c)Z^c + (D_{X^h}\mathcal{C})(Y^h,Z^h) - (D_{Y^v}\mathcal{C}')(X^h,Z^h) + \\
&\quad + \mathcal{C}(F\mathcal{C}'(X^h,Y^h),Z^h) + \mathcal{C}(F\mathcal{C}'(X^h,Z^h),Y^h) - \mathcal{C}'(F\mathcal{C}(X^h,Y^h),Z^h) - \\
&\quad - \mathcal{C}'(F\mathcal{C}(Y^h,Z^h),X^h),
\end{aligned}$$

since from the proof of (2.15b) $-J\big[Y^v, F[X^h, Z^v]\big] = \overset{\circ}{\mathbb{P}}(X^c, Y^c)Z^c$. Thus (5.3a) is verified.
(b) A similar reasoning, but a shorter calculation shows that

$$\begin{aligned}
\overset{\mathrm{R}}{\mathbb{P}}(X^c, Y^c)Z^c = \overset{\mathrm{R}}{\mathbb{K}}(X^h, Y^v)Z^v &= \overset{\mathrm{R}}{D}_{X^h}\overset{\mathrm{R}}{D}_{Y^v}Z^v - \overset{\mathrm{R}}{D}_{Y^v}\overset{\mathrm{R}}{D}_{X^h}Z^v - \overset{\mathrm{R}}{D}_{[X^h,Y^v]}Z^v = \\
&= -\overset{\mathrm{R}}{D}_{Y^v}\left([X^h, Z^v] + \mathcal{C}'(X^h, Z^h)\right) - \overset{\mathrm{R}}{D}_{JF[X^h,Y^v]}JZ^h = \\
&= -\overset{\mathrm{R}}{D}_{JY^h}JF[X^h, Z^v] - \overset{\mathrm{R}}{D}_{JY^h}JF\mathcal{C}'(X^h, Z^h) - J\big[[X^h, Y^v], Z^h\big] = \\
&= -J\big[Y^v, F[X^h, Z^v]\big] - J[Y^v, F\mathcal{C}'(X^h, Z^h)] = \overset{\circ}{\mathbb{P}}(X^c, Y^c)Z^c + [\mathcal{C}'(X^h, Z^h), Y^v],
\end{aligned}$$

since

$$0 \overset{1.15}{=} [J, Y^v]F\mathcal{C}'(X^h, Z^h) \overset{(2.8a)}{=} [\mathcal{C}'(X^h, Z^h), Y^v] - J[F\mathcal{C}'(X^h, Z^h), Y^v].$$

Thus we obtain our second assertion. □

5.4. Proposition.

(a) The v-curvature tensor of the Berwald and of the Rund connection vanishes, i.e., $\overset{\circ}{\mathbb{Q}} = 0$ *and* $\overset{R}{\mathbb{Q}} = 0$.

(b) For the v-curvature tensor of the Cartan connection we have the expression

(5.4) $$\mathbb{Q}(X, Y)Z = \mathcal{C}\big(F\mathcal{C}(Z, X), Y\big) - \mathcal{C}\big(X, F\mathcal{C}(Y, Z)\big) \qquad (X, Y, Z \in \mathfrak{X}(\mathcal{T}M)).$$

Proof. The first assertion can be verified by an immediate calculation. Formula (5.4) was derived by J. Grifone [11], we recall only the key observation, the identity

$$\left(D_{JX}\mathcal{C}\right)(Y, Z) = \left(D_{JY}\mathcal{C}\right)(X, Z) \qquad (X, Y, Z \in \mathfrak{X}(\mathcal{T}M)).$$

□

5.5. Example. Suppose that (M, E) is a positive definite two-dimensional Finsler manifold. Let S be the canonical spray, h the Barthel endomorphism and F the almost complex structure induced by h. Consider the prolongation g of the Riemann – Finsler metric along h (3.2b). Let $C_0 := \frac{1}{\sqrt{2E}}C$ be the normalized Liouville vector field; then $g(C_0, C_0) = 1$. The vector field $S_0 := \frac{1}{\sqrt{2E}}S = FC_0$ is g-orthogonal to C_0, i.e.,

$$g(S_0, C_0) = g(hS_0, vC_0) = g(hS_0, JFC_0) \overset{(3.3a)}{=} 0,$$

and $g(S_0, S_0) = 1$. Next, using the Gram – Schmidt process, we can construct—at least locally— a g-orthonormal basis (C_0, X_0) of $\mathfrak{X}^v(\mathcal{T}M)$, where the vector field X_0 is uniquely determined up to sign. Using the almost complex structure once more, we arrive at a (local) g-orthonormal basis

(5.5a) $$(C_0, X_0, FX_0, S_0)$$

for the module $\mathfrak{X}(\mathcal{T}M)$. The quadruple (5.5a) is called the *Berwald frame* of the Finsler manifold after L. Berwald, see his posthumous paper [4]. As for the details of the coordinate-free construction, we refer to [19].

Now, since the first Cartan tensor $\mathcal{C}$ is semibasic, it follows by 3.9 that $\mathcal{C}$ is completely determined by its value on the pair (FX_0, FX_0). Taking into account (5.4), we infer immediately that *in two dimensions the v-curvature of the Cartan connection vanishes.* This proposition was first proved by D. Laugwitz [12] with the machinery of classical tensor calculus.

5.6. Remark. If $n \geq 3$, (M, E) is a positive-definite n-dimensional Finsler manifold, and the energy function is symmetric ($E(-v) = E(v)$ for any tangent vector $v \in \mathcal{T}M$), then the vanishing of $\mathbb{Q}$ implies that (M, E) is a Riemannian manifold. This far from trivial result (conjectured by D. Laugwitz [13]) was first proved by F. Brickell [5].

6. Berwald manifolds

6.1. Convention. Retaining the notations introduced above, in our following discussion h will denote the Barthel endomorphism and $\mathcal{C}'$ the second Cartan tensor belonging to h, unless otherwise stated.

6.2. To begin with, we recall an important classical result, first formulated and proved *intrinsically* by J. G. Diaz; see [9].

6.3. Theorem and definition. *Let (M, E) be a Finsler manifold. The following assertions are equivalent:*

(a) *The h-curvature $\mathbb{P}$ of the Cartan connection vanishes.*
(b) *The second Cartan tensor $\mathcal{C}'$ vanishes.*
(c) $\forall X, Y, Z \in \mathfrak{X}(\mathcal{T}M)$: $\quad (D_{hX}\mathcal{C})(Y, Z) = (D_{hY}\mathcal{C})(X, Z)$.
(d) $\forall X, Y, Z \in \mathfrak{X}(\mathcal{T}M)$: $\quad \overset{\circ}{\mathbb{P}}(X, Y)Z = -(D_{hX}\mathcal{C})(Y, Z)$.

If one, and therefore all, of the conditions (a)–(d) are satisfied, then (M, E) is called a Landsberg manifold.

6.4. Other characterizations.

(a) The property $\mathcal{C}' = 0$ implies immediately (see e.g. 4.11) that in a Landsberg manifold the h-covariant derivatives with respect to the Cartan, the Berwald and the Rund connection coincide. Conversely, if $D^h = \overset{\circ}{D}{}^h$ or $\overset{\circ}{D}{}^h = \overset{\mathrm{R}}{D}{}^h$, then $\mathcal{C}' = 0$ and the Finsler manifold is a Landsberg manifold.

(b) Now let us have a look at the $(v)hv$-torsion of the Cartan connection. For each vector fields X, Y on M,

$$\begin{aligned}\mathbb{P}^1(X^h, Y^v) &:= v\mathbb{T}(X^h, Y^v) = vD_{X^h}Y^v - vD_{Y^v}X^h - v[X^h, Y^v] = \\ &\overset{4.11}{=} v[X^h, Y^v] + v\mathcal{C}'(X^h, Y^h) - vF\mathcal{C}(X^h, Y^h) - \\ &\quad - v[X^h, Y^v] \overset{(2.8b)}{=} \mathcal{C}'(X^h, Y^h) + J\mathcal{C}(X^h, Y^h) = \mathcal{C}'(X^h, Y^h).\end{aligned}$$

It follows that the *vanishing of the $(v)hv$-torsion of the Cartan connection characterizes the Landsberg manifolds.*

(c) We infer immediately from (3.7b) that *a Finsler manifold is a Landsberg manifold if and only if the Berwald connection is h-metrical* (i.e., $\overset{\circ}{D}{}^h g = 0$). In Matsumoto's monograph [14] Landsberg manifolds are defined by this property.

6.5. Definition. A Finsler manifold (M, E) is said to be a *Berwald manifold* if there is a linear connection ∇ on M such that for each vector fields X, Y on M,

$$(\nabla_X Y)^v = [X^h, Y^v],$$

where the horizontal lifting is taken with respect to the Barthel endomorphism.

6.6. Remarks.

(a) The linear connection ∇ in definition 6.5 is clearly unique, so it will be mentioned as *the linear connection of the Berwald manifold.* One can see also at once that the horizontal endomorphism induced by ∇ is just the Barthel endomorphism. We immediately infer that *the Barthel endomorphism and the canonical spray of a Berwald manifold are smooth on the whole tangent manifold TM.* The converse is also true: if the canonical spray of the Finsler manifold (M, E) is smooth on TM, then (M, E) is a Berwald manifold. For another reasoning see [9].

(b) By a clever observation of Z. I. Szabó the linear connection ∇ of a positive definite Berwald manifold is *Riemann-metrizable*: there always exists a Riemannian metric g_M on M whose Levi–Civita connection is ∇. This is the first step toward the classification of positive definite Berwald manifolds achieved by him in [16].

6.7. Lemma. *A Finsler manifold (M, E) is a Berwald manifold if and only if*

$$\forall X, Y \in \mathfrak{X}(M) : [X^h, Y^v] \text{ is a vertical lift.} \tag{6.7}$$

Proof. The necessity of (6.7) is evident. To see the sufficiency, we consider the mapping

$$\nabla : (X, Y) \in \mathfrak{X}(M) \times \mathfrak{X}(M) \mapsto \nabla_X Y \in \mathfrak{X}(M), (\nabla_X Y)^v := [X^h, Y^v],$$

where the horizontal lifting is taken with respect to the Barthel endomorphism, of course. Then ∇ is well-defined and an easy calculation shows that it is a linear connection, indeed. □

6.8. Lemma. *Suppose that (M, E) is a Berwald manifold and let ∇ be its linear connection. Then the pair $(\overset{h}{\nabla}, h)$, where $\overset{h}{\nabla}$ is the horizontal lift of ∇ and h is the Barthel endomorphism, is just the Berwald connection.*

Proof. We have already learnt from 6.6(a) and from 2.13 that $(\overset{h}{\nabla}, h)$ is a Finsler connection. Our only task is to check (4.3a) and (4.3b). But this is easy: for any vector fields X, Y, Z on M, we have

$$\mathbb{P}^1(X^h, Y^v) = v\left(\overset{h}{\nabla}_{X^h} Y^v - \overset{h}{\nabla}_{Y^v} X^h - [X^h, Y^v]\right) \overset{(2.13)}{=} v\left((\nabla_X Y)^v - [X^h, Y^v]\right) = 0,$$

$$\mathbb{B}(X^h, Y^v) = h\left(\overset{h}{\nabla}_{X^h} Y^v - \overset{h}{\nabla}_{Y^v} X^h - [X^h, Y^v]\right) \overset{(2.13)}{=} 0.$$

□

6.9. A coordinate view. Let $(\mathcal{U}, (u^i)_{i=1}^n)$ be a chart on M. With the help of the induced chart

$$\left(\pi^{-1}(\mathcal{U}), (x^i, y^i)_{i=1}^n\right); \quad x^i := u^i \circ \pi,$$
$$y^i : v \in \pi^{-1}(\mathcal{U}) \mapsto y^i(v) := v(u^i) \qquad (1 \leq i \leq n)$$

we review some important coordinate expressions. Einstein's summation convention will be used.

(i) We get from (3.4b) that the components of the Riemann – Finsler metric $\bar{g}$ are

$$\bar{g}_{ij} := \bar{g}\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right) = \frac{\partial^2 E}{\partial y^i \partial y^j}.$$

(3.7a) implies that the components of the first Cartan tensors $\mathcal{C}_\flat$ and $\mathcal{C}$ are

$$(\mathcal{C}_\flat)_{ijk} = \tfrac{1}{2} C_{ijk}, \qquad \mathcal{C}_{ij}^\ell = \tfrac{1}{2} C_{ij}^\ell$$

where

$$C_{ijk} := \frac{\partial^3 E}{\partial y^i \partial y^j \partial y^k}, \quad C_{ij}^\ell := \bar{g}^{\ell k} C_{ijk}, \quad \left(\bar{g}^{\ell k}\right) := \left(\bar{g}_{\ell k}\right)^{-1}.$$

The coordinate expression of the canonical spray S is

$$S \restriction \pi^{-1}(\mathcal{U}) = y^k \frac{\partial}{\partial x^k} - 2G^k \frac{\partial}{\partial y^k},$$

where

$$G^k = \bar{g}^{jk} G_j, \quad G_j := \frac{1}{2}\left(y^k \frac{\partial^2 E}{\partial x^k \partial y^j} - \frac{\partial E}{\partial x^j}\right). \tag{6.9a}$$

The Barthel endomorphism can be represented in the form

$$h \restriction \mathfrak{X}\big(\pi^{-1}(\mathcal{U})\big) = \left(\frac{\partial}{\partial x^i} - G_i^k \frac{\partial}{\partial y^k}\right) \otimes dx^i, \quad G_i^k := \frac{\partial G^k}{\partial y^i} \qquad (1 \le i, k \le n). \tag{6.9b}$$

The Berwald connection $(\overset{\circ}{D}, h)$ of (M, E) is completely determined by the functions

$$G_{ij}^k := \frac{\partial G_i^k}{\partial y^j} = \frac{\partial^2 G^k}{\partial y^i \partial y^j} \qquad (1 \le i, j, k \le n). \tag{6.9c}$$

The functions G_{ij}^k are called the *connection parameters* for the Berwald connection.

(ii) Traditionally Berwald manifolds are defined as follows: *"the connection parameters for the Berwald connection depend only on the position"* , i.e. by the condition

$$G_{ij\ell}^k := \frac{\partial G_{ij}^k}{\partial y^\ell} = 0 \qquad (1 \le i, j, k, \ell \le n).$$

Finsler manifolds with this property were called *affinely connected spaces* by L. Berwald himself, see [3]. Now we show that our definition is equivalent to the classical one. To see this, let ∇ be a linear connection on M, locally given by the functions $\Gamma_{ij}^k \in C^\infty(\mathcal{U})$ $(1 \le i, j, k \le n)$, such that

$$\nabla_{\frac{\partial}{\partial u^i}} \frac{\partial}{\partial u^j} = \Gamma_{ij}^k \frac{\partial}{\partial u^k}.$$

Then

$$\left(\nabla_{\frac{\partial}{\partial u^i}} \frac{\partial}{\partial u^j}\right)^v = \left(\Gamma_{ij}^k \circ \pi\right) \frac{\partial}{\partial y^k},$$

$$\left[\left(\frac{\partial}{\partial u^i}\right)^h, \left(\frac{\partial}{\partial u^j}\right)^v\right] = \left[h\left(\frac{\partial}{\partial u^i}\right)^c, \frac{\partial}{\partial y^j}\right] \overset{(6.9b)}{=} \left[\frac{\partial}{\partial x^i} - G_i^k \frac{\partial}{\partial y^k}, \frac{\partial}{\partial y^j}\right] =$$
$$= \frac{\partial G_i^k}{\partial y^j} \frac{\partial}{\partial y^k} \overset{(6.9c)}{=} G_{ij}^k \frac{\partial}{\partial y^k},$$

thus

$$\forall i, j \in \{1, \dots, n\} : \left(\nabla_{\frac{\partial}{\partial u^i}} \frac{\partial}{\partial u^j}\right)^v = \left[\left(\frac{\partial}{\partial u^i}\right)^h, \left(\frac{\partial}{\partial u^j}\right)^v\right]$$

$$\begin{aligned} &\iff \Gamma_{ij}^k \circ \pi = G_{ij}^k \qquad (1 \leq k \leq n) \\ &\iff G_{ij\ell}^k = 0 \qquad (1 \leq k, \ell \leq n). \end{aligned}$$

(iii) Notice that the components of the hv-curvature tensor $\overset{\circ}{\mathbb{P}}$ of the Berwald connection are just the functions $-G_{ij\ell}^k$ (hence the classical definition in (ii) has a tensorial character). Indeed,

$$\begin{aligned} \overset{\circ}{\mathbb{P}}\left(\left(\frac{\partial}{\partial u^i}\right)^h, \left(\frac{\partial}{\partial u^j}\right)^h\right)\left(\frac{\partial}{\partial u^\ell}\right)^h &\overset{(2.15\mathrm{b})}{=} \left[\left[\left(\frac{\partial}{\partial u^i}\right)^h, \left(\frac{\partial}{\partial u^j}\right)^v\right], \left(\frac{\partial}{\partial u^\ell}\right)^v\right] = \\ &= \left[G_{ij}^k \frac{\partial}{\partial y^k}, \frac{\partial}{\partial y^\ell}\right] = -\frac{\partial G_{ij}^k}{\partial y^\ell} = -G_{ij\ell}^k. \end{aligned}$$

It follows that *a Finsler manifold is a Berwald manifold if and only if the hv-curvature of the Berwald connection vanishes.* We shall see soon how this follows intrinsically at once.

6.10. Proposition. *The second Cartan tensor $\mathcal{C}'$ vanishes in any Berwald manifold, consequently the Berwald manifolds are at the same time Landsberg manifolds.*

Proof. We have already seen in the proof of 3.10 that for any vector fields X, Y, Z on M,

$$2\mathcal{C}'_\flat(X^c, Y^c, Z^c) = \left[Y^v, [X^h, Z^v]\right]E$$

(due to the fact that h is homogeneous and conservative). Since (M, E) is a Berwald manifold, $[X^h, Z^v]$ is a vertical lift by Lemma 6.7. Hence $[Y^v, [X^h, Z^v]] = 0$ by Lemma 1.16, whence our conclusion. □

6.11. Corollary. *The Berwald connection of a Berwald manifold is h-metrical (i.e., $\overset{\circ}{D}{}^h g = 0$).*

6.12. Theorem. *Let (M, E) be a Finsler manifold. Then the following assertions are equivalent:*

(a) *(M, E) is a Berwald manifold.*
(b) *The hv-curvature tensor $\overset{\circ}{\mathbb{P}}$ of the Berwald connection vanishes.*
(c) *The hv-curvature tensor $\overset{R}{\mathbb{P}}$ of the Rund connection vanishes.*
(d) *With respect to the Cartan connection, the h-covariant derivative of the first Cartan tensor vanishes (i.e., $D^h\mathcal{C} = 0$).*

Proof.
(a) $\iff$ (b) This is an immediate consequence of (2.15b), Lemma 6.7 and Lemma 1.16.
(a) $\implies$ (c) We infer this at once from (5.3b), from Proposition 6.10 ($\mathcal{C}' = 0$) and from the equivalence (a) $\iff$ (b).
(c) $\implies$ (a) To prove the implication, note first that

$$\forall X, Y \in \mathfrak{X}(M): \quad \overset{\mathrm{R}}{\mathbb{P}}(X^h, Y^h)S = \mathcal{C}'(X^h, Y^h),$$

where S is an arbitrary semispray. Indeed,

$$\overset{\mathrm{R}}{\mathbb{P}}(X^h, Y^h)S = \overset{\mathrm{R}}{\mathbb{K}}(X^h, Y^v)C = \overset{\mathrm{R}}{D}_{X^h}\overset{\mathrm{R}}{D}_{Y^v}C - \overset{\mathrm{R}}{D}_{Y^v}\overset{\mathrm{R}}{D}_{X^h}C - \overset{\mathrm{R}}{D}_{[X^h, Y^v]}C,$$

where $\overset{\mathrm{R}}{D}_{X^h}C = 0$ by (4.9h), while

$$\begin{aligned}\overset{\mathrm{R}}{D}_{X^h}\overset{\mathrm{R}}{D}_{Y^v}C &= \overset{\mathrm{R}}{D}_{X^h}\overset{\mathrm{R}}{D}_{JY^h}JS \overset{(4.9\mathrm{d})}{=} \overset{\mathrm{R}}{D}_{X^h}J[Y^v,S] \overset{(1.12)}{=} \overset{\mathrm{R}}{D}_{X^h}Y^v = \\ &\overset{4.11}{=} [X^h,Y^v] + \mathcal{C}'(X^h,Y^h), \\ \overset{\mathrm{R}}{D}_{[X^h,Y^v]}C &= \overset{\mathrm{R}}{D}_{JF[X^h,Y^v]}JS \overset{(4.9\mathrm{d})}{=} J\big[JF[X^h,Y^v],S\big] = [X^h,Y^v],\end{aligned}$$

thus we obtain the desired relation. We conclude that $\overset{\mathrm{R}}{\mathbb{P}} = 0$ implies that $\mathcal{C}' = 0$ and, therefore, that $\overset{\circ}{\mathbb{P}} = 0$ (by (5.3b)); hence our assertion.

(a) $\Longrightarrow$ (d) In view of the property $\mathcal{C}' = 0$ it follows that in Berwald manifolds $D^h = \overset{\circ}{D}{}^h$ (c.f. 6.4(a)). Therefore it is enough to check that

$$\forall X \in \mathfrak{X}(M): \quad \overset{\circ}{D}_{X^h}\mathcal{C} = 0.$$

Let $X,Y,Z,U \in \mathfrak{X}(M)$ be arbitrary. $\overset{\circ}{D}$ is h-metrical by Corollary 6.11, so

$$\begin{aligned}0 = \big(\overset{\circ}{D}_{X^h}g\big)\big(\mathcal{C}(Y^h,Z^h),U^v\big) &= X^h g\big(\mathcal{C}(Y^h,Z^h),U^v\big) - g\big(\overset{\circ}{D}_{X^h}\mathcal{C}(Y^h,Z^h),U^v\big) - \\ - g\big(\mathcal{C}(Y^h,Z^h),\overset{\circ}{D}_{X^h}U^v\big) &\overset{(3.5\mathrm{a}),4.11}{=} \tfrac{1}{2}X^h(\mathcal{L}_{JY^h}J^*g)(Z^h,U^h) - \\ - g\big(\overset{\circ}{D}_{X^h}\mathcal{C}(Y^h,Z^h),U^v\big) &- \tfrac{1}{2}\big(\mathcal{L}_{JY^h}J^*g\big)(Z^h,F[X^h,U^v])\end{aligned}$$

and hence

$$\begin{aligned}&2g\big(\overset{\circ}{D}_{X^h}\mathcal{C}(Y^h,Z^h),U^v\big) = \\ &= X^h\big(Y^v g(Z^v,U^v) - g(J[Y^v,Z^h],U^v) - g(Z^v,J[Y^v,U^h])\big) - \\ &- Y^v g(Z^v,[X^h,U^v]) + g(J[Y^v,Z^h],[X^h,U^v]) + g\big(Z^v,J\big[Y^v,F[X^h,U^v]\big]\big) = \\ &\overset{(1.7\mathrm{a})}{=} X^h\big(Y^v g(Z^v,U^v)\big) - Y^v g(Z^v,[X^h,U^v]),\end{aligned}$$

taking into account that from the proof of 2.15(b)

$$J\big[Y^v,F[X^h,U^v]\big] = -J\big[F[X^h,U^v],Y^v\big] = -\overset{\circ}{\mathbb{P}}(X^c,Y^c)U^c \overset{(\mathrm{a})\Longleftrightarrow(\mathrm{b})}{=} 0.$$

Now we evaluate the term $\overset{\circ}{D}_{X^h}\mathcal{C}$. For each vector fields Y,Z on M,

$$\begin{aligned}\big(\overset{\circ}{D}_{X^h}\mathcal{C}\big)(Y^h,Z^h) &= \overset{\circ}{D}_{X^h}\mathcal{C}(Y^h,Z^h) - \mathcal{C}(\overset{\circ}{D}_{X^h}Y^h,Z^h) - \mathcal{C}(Y^h,\overset{\circ}{D}_{X^h}Z^h) = \\ &\overset{4.11}{=} \overset{\circ}{D}_{X^h}\mathcal{C}(Y^h,Z^h) - \mathcal{C}(F[X^h,Y^v],Z^h) - \mathcal{C}(Y^h,F[X^h,Z^v]).\end{aligned}$$

Applying the last two results, we obtain:

$$\begin{aligned}&2g\Big(\big(\overset{\circ}{D}_{X^h}\mathcal{C}\big)(Y^h,Z^h),U^v\Big) = 2g\big(\overset{\circ}{D}_{X^h}\mathcal{C}(Y^h,Z^h),U^v\big) - \\ &2g\big(\mathcal{C}(F[X^h,Y^v],Z^h),U^v\big) - 2g\big(\mathcal{C}(Y^h,F[X^h,Z^v]),U^v\big) = \\ &= X^h\big(Y^v g(Z^v,U^v)\big) - Y^v g(Z^v,[X^h,U^v]) - \big(\mathcal{L}_{[X^h,Y^v]}J^*g\big)(Z^h,U^h) - \\ &(\mathcal{L}_{Y^v}J^*g)(F[X^h,Z^v],U^h) = X^h\big(Y^v g(Z^v,U^v)\big) - Y^v g(Z^v,[X^h,U^v]) -\end{aligned}$$

$$- [X^h, Y^v]g(Z^v, U^v) + g\big(J\big[[X^h, Y^v], Z^h\big], U^v\big) + g\big(Z^v, J\big[[X^h, Y^v], U^h\big]\big)$$
$$- Y^v g([X^h, Z^v], U^v) + g\big(J\big[Y^v, F[X^h, Z^v]\big], U^v\big) +$$
$$+ g([X^h, Z^v], J[Y^v, U^h]) = -Y^v g(Z^v, [X^h, U^v]) + Y^v\big(X^h g(Z^v, U^v)\big) -$$
$$- Y^v g([X^h, Z^v], U^v) - g\big(\overset{\circ}{\mathbb{P}}(X^c, Y^c)Z^c, U^v\big) = Y^v\big(X^h g(Z^v, U^v)\big) -$$
$$- Y^v g([X^h, Z^v], U^v) - Y^v g(Z^v, [X^h, U^v]) \overset{4.11}{=} Y^v \left[\left(\overset{\circ}{D}_{X^h} g\right)(Z^v, U^v)\right] = 0,$$

since $\overset{\circ}{D}$ is h-metrical. Hence $\overset{\circ}{D}_{hX}\mathcal{C} = 0$.

(d) $\Longrightarrow$ (a) Assume $D^h\mathcal{C} = 0$. Then for each vector field X on $\mathcal{T}M$, $D_{hX}\mathcal{C} = 0$. In particular, taking the canonical spray S, we obtain:

$$0 = D_{hS}\mathcal{C} = D_S\mathcal{C} \overset{\text{lemma 4.8}}{=} -\mathcal{C}'.$$

The vanishing of $\mathcal{C}'$ implies by Theorem 6.3 that

$$\forall X, Y, Z \in \mathfrak{X}(\mathcal{T}M) : \overset{\circ}{\mathbb{P}}(X, Y)Z = -\left(D_{hX}\mathcal{C}\right)(Y, Z) \overset{\text{(d)}}{=} 0,$$

so (M, E) is a Berwald manifold. With this we reach the end of the proof of 6.12. □

7. Locally Minkowski manifolds

7.1. Proposition and definition. *Let (M, E) be a Berwald manifold. Then the following conditions are equivalent:*

$$\text{(a) } \Omega := -\tfrac{1}{2}[h, h] = 0, \quad \text{(b) } \mathbb{R} = 0,$$
$$\text{(c) } \overset{\circ}{\mathbb{R}} = 0, \qquad\qquad \text{(d) } \overset{R}{\mathbb{R}} = 0.$$

If one, and therefore all, of these conditions are satisfied, then (M, E) is called a *locally Minkowski manifold.*

Proof.

(a) $\Longleftrightarrow$ (b) Since the second Cartan tensor $\mathcal{C}'$ vanishes in any Berwald manifold, formula (5.2a) reduces to

$$\text{(7.1)} \qquad \forall X, Y, Z \in \mathfrak{X}(\mathcal{T}M) : \quad \mathbb{R}(X, Y)Z = \overset{\circ}{\mathbb{R}}(X, Y)Z + \mathcal{C}(F\Omega(X, Y), Z)$$

in this case.We infer at once from (7.1) and 2.18 that (a) $\Longrightarrow$ (b). Conversely, suppose that $\mathbb{R} = 0$ and let S be the canonical spray. Then for each vector fields X, Y on $\mathcal{T}M$,

$$0 \overset{(7.1)}{=} \overset{\circ}{\mathbb{R}}(X, Y)S + \mathcal{C}(F\Omega(X, Y), S) \overset{3.9}{=} \overset{\circ}{\mathbb{R}}(X, Y)S \overset{(2.17)}{=} \Omega(X, Y),$$

so (b) $\Longrightarrow$ (a).

(b) $\Longleftrightarrow$ (c) We have just seen that $\mathbb{R} = 0$ implies $\Omega = 0$. Then, by (7.1), $\overset{\circ}{\mathbb{R}} = 0$; thus we get the desired implication (b) $\Longrightarrow$ (c). The converse is obvious from (7.1) and (2.17).

(b) $\Longleftrightarrow$ (d) If $\mathbb{R} = 0$, then $\Omega = 0$ by the equivalence (a) $\Longleftrightarrow$ (b) and we deduce from (5.2b) that $\overset{\mathrm{R}}{\mathbb{R}} = 0$. Thus (b) $\Longrightarrow$ (d). Assume $\overset{\mathrm{R}}{\mathbb{R}} = 0$. If S is the canonical spray again, then from (5.2b)

$$\forall X, Y \in \mathfrak{X}(\mathcal{T}M) : \quad 0 = \overset{\mathrm{R}}{\mathbb{R}}(X, Y)S = \mathbb{R}(X, Y)S - \mathcal{C}(F\Omega(X, Y), S) =$$

$$\overset{3.9}{=} \mathbb{R}(X,Y)S \overset{(7.1)}{=} \overset{\circ}{\mathbb{R}}(X,Y)S \overset{(2.17)}{=} \Omega(X,Y),$$

so $\Omega = 0$ and, therefore, $\mathbb{R} = 0$. □

7.2. Proposition. *A Finsler manifold (M,E) is a locally Minkowski manifold if and only if there exists a torsion-free, flat linear connection ∇ on M whose horizontal lift $\overset{h}{\nabla}$ is h-metrical with respect to the horizontal endomorphism arising from ∇.*

Proof.

Necessity. Assume (M,E) is a locally Minkowski manifold. Then (M,E) is a Berwald manifold as well; let ∇ be its linear connection (in the sense of 6.6(a)). Then ∇ is torsion-free and, by Proposition 7.1, it is flat. In view of Lemma 6.8, the horizontal lift $\overset{h}{\nabla}$ of ∇ is the Berwald connection of (M,E), which is h-metrical by Corollary 6.11.

Sufficiency. Suppose that ∇ is a torsion-free, flat linear connection on M, satisfying the condition

$$\forall X \in \mathfrak{X}(\mathcal{T}M): \quad \overset{h}{\nabla}_{hX} g = 0.$$

Let h be the horizontal endomorphism arising from ∇. Clearly, the tension, the torsion and the curvature of h vanish. We claim that the Finsler connection $(\overset{h}{\nabla}, h)$ is of Berwald-type. To show this, by Theorem 4.3 it is enough to check (4.3a) and (4.3b). For each vector fields $X, Y \in \mathfrak{X}(M)$,

$$\begin{aligned} \mathbb{P}^1(X^h, Y^v) &:= v\mathbb{T}(X^h, Y^v) = v\left(\overset{h}{\nabla}_{X^h} Y^v - \overset{h}{\nabla}_{Y^v} X^h - [X^h, Y^v]\right) = \\ &\overset{(2.13)}{=} [X^h, Y^v] - [X^h, Y^v] = 0; \end{aligned}$$

$$\mathbb{B}(X^h, Y^v) := h\mathbb{T}(X^h, Y^v) = 0,$$

hence our statement. Let S be the geodesic spray of ∇ ([6], p. 173). Then $hS = S$ and for any vector field X on M

$$\overset{h}{\nabla}_{X^h} S = \overset{h}{\nabla}_{hX^h} hS \overset{(2.14d)}{=} hF[X^h, JS] = hF[X^h, C] = 0,$$

since the tension of h vanishes. Now we show that h is conservative. For each vector field X on M, we have

$$\begin{aligned} 0 = \left(\overset{h}{\nabla}_{X^h} g\right)(S,S) &= X^h g(S,S) - 2g\left(\overset{h}{\nabla}_{X^h} S, S\right) = X^h g(S,S) = \\ &\overset{(3.2b)}{=} X^h \bar{g}(JS, JS) = X^h \bar{g}(C,C) \overset{(3.4a)}{=} 2X^h E = 2d_h E(X^c), \end{aligned}$$

which gives the result. Finally we check that the $(h)h$-torsion $\mathbb{A}$ of $\overset{h}{\nabla}$ vanishes. For any two vector fields X, Y on M,

$$\begin{aligned} \mathbb{A}(X^c, Y^c) &= h\mathbb{T}(hX^c, hY^c) = h\mathbb{T}(X^h, Y^h) = h\left(\overset{h}{\nabla}_{X^h} Y^h - \overset{h}{\nabla}_{Y^h} X^h - [X^h, Y^h]\right) = \\ &\overset{(2.13)}{=} h\left(\left(\nabla_X Y\right)^h - \left(\nabla_Y X\right)^h - [X^h, Y^h]\right) = \left(\nabla_X Y\right)^h - \left(\nabla_Y X\right)^h - [X,Y]^h = \end{aligned}$$

$$= \Big(\nabla_X Y - \nabla_Y X - [X,Y]\Big)^h = 0,$$

since ∇ is torsion-free. Now we infer from Theorem 4.3 that h is the Barthel endomorphism and, therefore, $(\overset{h}{\nabla}, h)$ is the Berwald connection. Hence (M,E) is a Berwald manifold satisfying the condition $\Omega = -\frac{1}{2}[h,h] = 0$; i.e., (M,E) is a locally Minkowski manifold. □

7.3. Locally affine structures. Recall that an atlas $\mathcal{A} = (\mathcal{U}_\alpha, (u^i_\alpha)_{i=1}^n)_{\alpha\in A}$ of a manifold M is said to be a *locally affine structure* on M if the transition functions

$$\mathit{\Gamma}_{\alpha\beta}\colon p \in \mathcal{U}_\alpha \cap \mathcal{U}_\beta \mapsto \mathit{\Gamma}_{\alpha\beta}(p) := \Big(\frac{\partial u^i_\beta}{\partial u^j_\alpha}(p)\Big) \in \mathrm{GL}(\mathbb{R}^n)$$
$$((\alpha,\beta) \in A \times A)$$

are constant. If $X, Y \in \mathfrak{X}(\mathcal{U}_\alpha)$, $Y = Y^i \frac{\partial}{\partial u^i_\alpha}$ and

$$\nabla^\alpha_X Y := X(Y^i)\frac{\partial}{\partial u^i_\alpha} \qquad (\alpha \in A),$$

then the family $(\nabla^\alpha)_{\alpha\in A}$ determines a well-defined linear connection ∇ on M. We say that ∇ *arises from the locally affine structure* $\mathcal{A}$. Clearly, the torsion tensor and the curvature tensor of ∇ vanish. Frobenius' classical theorem on integrable distributions assures that the converse is also true. Namely, we have

7.4. Lemma. *A linear connection on a manifold M which has zero curvature and torsion arises from a locally affine structure on M.*

This is proved e.g. in [6].

7.5. Theorem. *A Finsler manifold (M,E) is a locally Minkowski manifold if and only if there exists an atlas $\big(\mathcal{U}_\alpha, (u^i_\alpha)_{i=1}^n\big)_{\alpha\in A}$ on M such that in the induced atlas $\big(\pi^{-1}(\mathcal{U}_\alpha), (x^i_\alpha, y^i_\alpha)_{i=1}^n\big)_{\alpha\in A}$*

$$\forall\alpha \in A: \quad \frac{\partial E}{\partial x^i_\alpha} = 0, \qquad 1 \le i \le n;$$

i.e. the energy function "does not depend on the position" over the induced charts.

Proof.

Necessity. Assume (M,E) is a locally Minkowski manifold. Then, in particular, (M,E) is a Berwald manifold and the curvature tensor of its (torsion-free) linear connection ∇ vanishes by Proposition 7.2. Applying Lemma 7.4 it follows that ∇ arises from a locally affine structure $\mathcal{A} = \big(\mathcal{U}_\alpha, (u^i_\alpha)_{i=1}^n\big)_{\alpha\in A}$ on M. Choose a chart $\big(\mathcal{U}, (u^i)_{i=1}^n\big) \in \mathcal{A}$ (the lower index α is omitted for simplicity). Then over $\mathcal{U}$

$$\nabla_{\frac{\partial}{\partial u^i}} \frac{\partial}{\partial u^j} = 0, \qquad 1 \le i,j \le n.$$

Since the Berwald connection of (M,E) is just $(\overset{h}{\nabla}, h)$ by Lemma 6.8 (h is the Barthel endomorphism), we have on the one hand

$$\overset{\circ}{D}_{\left(\frac{\partial}{\partial u^i}\right)^h} \Big(\frac{\partial}{\partial u^j}\Big)^v = \overset{h}{\nabla}_{\left(\frac{\partial}{\partial u^i}\right)^h} \Big(\frac{\partial}{\partial u^j}\Big)^v \overset{(2.13)}{=} \Big(\nabla_{\frac{\partial}{\partial u^i}} \frac{\partial}{\partial u^j}\Big)^v = 0 \qquad (1 \le i,j \le n).$$

On the other hand, for each indices $i, j \in \{1, \dots, n\}$

$$\overset{\circ}{D}_{\left(\frac{\partial}{\partial u^i}\right)^h} \left(\frac{\partial}{\partial u^j}\right)^v \overset{4.11}{=} \left[\left(\frac{\partial}{\partial u^i}\right)^h, \left(\frac{\partial}{\partial u^j}\right)^v\right] \overset{6.9\text{(ii)}}{=} G^k_{ij} \frac{\partial}{\partial y^k},$$

consequently

$$G^k_{ij} = 0, \qquad 1 \leq i, j, k \leq n.$$

Notice that the functions G^k_i (introduced in 6.9) are homogeneous of degree 1. Thus, by Euler's theorem,

$$G^k_i = y^j \frac{\partial G^k_i}{\partial y^j} \overset{(6.9\text{c})}{=} y^j G^k_{ij} = 0, \qquad 1 \leq i, k \leq n.$$

Hence

$$\left(\frac{\partial}{\partial u^i}\right)^h = \frac{\partial}{\partial x^i}, \qquad 1 \leq i \leq n.$$

Since the Barthel endomorphism is conservative, i.e. $d_h E = 0$, we deduce

$$0 = d_h E \left(\frac{\partial}{\partial u^i}\right)^c = dE \left(\frac{\partial}{\partial u^i}\right)^h = dE \left(\frac{\partial}{\partial x^i}\right) = \frac{\partial E}{\partial x^i}, \qquad 1 \leq i \leq n.$$

It means that the atlas $\mathcal{A}$ has the desired property.

Sufficiency. Assume the condition holds. Again, choose a chart $\left(\mathcal{U}, (u^i)_{i=1}^n\right) \in \mathcal{A}$. Then we have

$$\frac{\partial E}{\partial x^i} = 0, \qquad 1 \leq i \leq n.$$

Let h be the Barthel endomorphism. The coordinate expression of the property $d_h E = 0$ reduces to the relation

$$G^k_i \frac{\partial E}{\partial y^k} = 0, \qquad 1 \leq i \leq n.$$

Now from (6.9a) we deduce

$$G^k = 0, \qquad 1 \leq k \leq n.$$

Hence the functions G^k_i, G^k_{ij} and G^k_{ijl} also vanish over $\mathcal{U}$. The vanishing of the functions G^k_{ijl} means that $\overset{\circ}{\mathbb{P}} = 0$, therefore (M, E) is a Berwald manifold (6.9(iii)). Finally, the relations

$$G^k_i = 0, \quad G^k_{ij} = 0, \qquad (1 \leq i, j, k \leq n)$$

imply that $\Omega = 0$, and this ends the proof. □

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