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TRANSITION FROM THE DYADIC TO THE REAL NONPERIODIC HARDY SPACE

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Abstract. The dyadic Hardy space plays a special role in Walsh analysis. Namely, it separates the $L^p[0,1)$ $(1 < p \le \infty)$ and the $L^1[0,1)$ spaces, and in many cases the results received for the $(1 < p \le \infty)$ case can be extended for the dyadic Hardy space but not for $L^1[0,1)$. The real nonperiodic Hardy space, which is wider than the dyadic one, is often employed in the trigonometric Fourier analysis. It is natural to ask whether the results proved for the dyadic Hardy space remain true for the real nonperiodic Hardy space. The idea behind this question is to make it possible to compare the corresponding results in the trigonometric and in the Walsh analysis. In this paper we provide a simple method for solving this problem for σ-sublinear functionals. Also, we study two well known sequences of functionals to demonstrate how our method works.

Main Result

Let $L^p = L^p[0,1)$ $(1 \le p \le \infty)$ denote the usual Banach spaces. Then a function $\alpha \in L^\infty$ is called a regular atom if either $\alpha \equiv 1$ or there exists an interval $I \subset [0,1)$ such that

$$\operatorname{supp} \alpha \subset I \, , \quad \int \alpha = 0 \, , \quad \|\alpha\|_\infty \le |I|^{-1},$$

where $|I|$ denotes the length of the interval I.

Let the set of regular atoms be denoted by $\mathbf{A}$. If the interval in the definition of the atom is required to be a dyadic interval, i.e. an interval of the form $[k2^{-n}, (k+1)2^{-n})$ $(n \in \mathbb{N},\ k = 0, \dots, 2^n - 1)$, then the atom is called a dyadic atom. The set of dyadic atoms will be denoted by $\mathcal{A}$.

We will introduce the concept of the real nonperiodic and the dyadic Hardy spaces by means of atomic decomposition. Namely, a function f integrable on $[0,1)$ belongs to the real nonperiodic Hardy space $\mathbf{H}$ if and only if there exist $\alpha_k \in \mathbf{A}$ and $c_k \in \mathbb{R}$ $(k \in \mathbb{N})$ such that

$$\sum_{k=1}^\infty |c_k| < \infty \quad , \text{ and } \quad f = \sum_{k=1}^\infty c_k \alpha_k$$

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(the latter equality is understood in the norm of L^1), and

$$\|f\|_{\mathbf{H}} = \inf \sum_{k=1}^{\infty} |c_k|,$$

where the infimum is taken over all such decompositions.

The dyadic Hardy space $\mathcal{H}$ and the corresponding norm are defined in a similar way, with the only modification that the regular atoms should be replaced by dyadic atoms. Clearly,

$$\mathcal{H} \subset \mathbf{H} \subset L^1,$$

and

$$\|f\|_1 \le \|f\|_{\mathbf{H}} \quad (f \in \mathbf{H}), \qquad \|f\|_{\mathbf{H}} \le \|f\|_{\mathcal{H}} \quad (f \in \mathcal{H}).$$

The following set of step functions will play an important role in our results. Set

$$\Omega = \{\omega_{k,n} \,:\, k, n \in \mathbb{N},\ 0 < k < 2^n\},$$

where

$$\omega_{k,n}(x) = \begin{cases} 2^{n-1} & \text{if} \quad (k-1)2^{-n} \le x < k2^{-n} \\ -2^{n-1} & \text{if} \quad k2^{-n} \le x < (k+1)2^{-n} \end{cases}.$$

Then $\omega_{k,n} \in \mathbf{A}$ for any possible n and k but $\omega_{k,n} \notin \mathcal{A}$ if k is even since in this case the adjacent dyadic intervals $[(k-1)2^{-n}, k2^{-n})$, $[k2^{-n}, (k+1)2^{-n})$ do not form a dyadic interval. The following theorem shows that this is what makes the difference between the real nonperiodic and the dyadic Hardy spaces concerning the boundedness of σ-sublinear functionals.

The functional F defined on $\mathbf{H}$ is called σ-sublinear if

$$|F(cf)| = |c||F(f)| \quad (c \in \mathbb{R},\ f \in \mathbf{H}),$$

and if $f = \sum_{k=1}^{\infty} f_k$ $(f,\ f_k \in \mathbf{H})$, the convergence is understood in the norm of $\mathbf{H}$, then

$$|F(f)| \le \sum_{k=1}^{\infty} |F(f_k)|.$$

Theorem 1. *Let F be a σ-sublinear functional on $\mathbf{H}$, and let $\mathcal{F}$ denote its restriction to $\mathcal{H}$. Then F is bounded if and only if it is bounded on Ω and $\mathcal{F}$ is bounded. Moreover*

$$\max\{\|\mathcal{F}\|, \sup_{\omega \in \Omega} |F(\omega)|\} \le \|\mathbf{F}\| \le 4\|\mathcal{F}\| + 2 \sup_{\omega \in \Omega} |F(\omega)|.$$

Applications

In this section we will use *Theorem 1* to decide the uniform boundedness of two sequences of σ-sublinear functionals. In both of our examples positive results are known for the dyadic Hardy space. We will show that in one of the examples the result can be extended to the real nonperiodic Hardy space while in the other it can not.

In order to formulate our examples we need to introduce some concepts of Walsh-Fourier analysis. For the basic properties of them we refer to [5]. The dyadic expansion of an $x \in [0,1)$ is defined as

$$x = \sum_{k=0}^{\infty} x_k 2^{-(k+1)} \qquad (x_k = 0 \text{ or } 1).$$

For the so-called dyadic rationals there are two expressions of this form. In this case we take the one which terminates in 0's.
The concept of dyadic addition ($\dot{+}$) and dyadic shift (τ) are defined as follows

$$x \dot{+} y = \sum_{k=0}^{\infty} |x_k - y_k| 2^{-(k+1)} \qquad (x,\, y \in [0,1))$$

$$\tau_\delta f(x) = f(x \dot{+} \delta) \qquad (\delta \in [0,1),\, x \in [0,1),\, f : [0,1) \mapsto \mathbb{R}).$$

Let r_k denote the kth Rademacher function, i.e.

$$r_0(x) = \begin{cases} +1 & \text{if} \quad 0 \le x < 1/2, \\ -1 & \text{if} \quad 1/2 \le x < 1 \end{cases}$$

periodic by 1, and

$$r_k(x) = r_0(2^k x) \qquad (0 \le x < 1,\, k \in \mathbb{N}).$$

The Walsh functions can be decomposed into products of Rademacher functions. Namely, if $n = \sum_{k=0}^{\infty} n_k 2^k$ $(n_k = 0 \text{ or } 1,\, n \in \mathbb{N})$ is the binary decomposition of n then the nth Walsh function in the Paley enumeration is defined as

$$w_n = \prod_{k=0}^{\infty} r_k^{n_k}.$$

Then it follows from the definition that

$$w_{j2^n+k} = w_{j2^n} w_k \qquad (j, n, k \in \mathbb{N},\, 0 \le k < 2^n). \tag{1}$$

For any $f \in L^1$ let $\hat{f}(k)$ $(k \in \mathbb{N})$ denote its kth Walsh-Fourier coefficient. If the Walsh-Dirichlet kernels are denoted by $D_k = \sum_{j=0}^{k-1} w_j$ $(k \in \mathbb{N})$ then the Walsh-Fourier partial sums $S_k f$ $(k \in \mathbb{N})$ can be calculated as

$$S_k f(x) = \int_0^1 f(t) D_k(x \dot{+} t)\, dt.$$

It is well known (see e.g. [5]) that

$$D_{2^n}(x) = \begin{cases} 2^n & \text{if} \quad 0 \le x < 2^{-n}, \\ 0 & \text{otherwise}. \end{cases} \tag{2}$$

The functionals in our examples are defined as follows

$$U_n f = \frac{1}{\log n} \sum_{k=1}^{n} \frac{\|S_k f\|_1}{k} \qquad (f \in L^1,\, n \in \mathbb{N},\, n > 1),$$

$$T_n f = \frac{1}{2^n} \| \sum_{k=0}^{2^n-1} S_{2^n} f(k2^{-n}) D_{k+1} \|_1 \qquad (f \in L^1,\, n \in \mathbb{N}).$$

Then the aforementioned result is the following.

Theorem 2.
i) There is an $f \in \mathbf{H}$ *for which* $\lim_{n\to\infty} U_n f = \infty$.
ii) There exists $C > 0$ *such that* $T_n(f) \le C\|f\|_{\mathbf{H}} \quad (f \in \mathbf{H},\, n \in \mathbb{N})$.

Remark 1. Concerning the first part of *Theorem 2* we note that $\lim_{n\to\infty} U_n f = \|f\|_1$ for any $f \in \mathcal{H}$ as it was shown by *Simon* in [6]. For its generalization to Vilenkin systems see *Gát* [3]. On the other hand, *Smith* proved in [7] that $\lim_{n\to\infty} \widetilde{U}_n f = \|f\|_1$ $(f \in \mathbf{H})$, where $\widetilde{U}_n$ stands for the trigonometric version of U_n $(n \in \mathbb{N})$. Our result shows a significant difference between the trigonometric and the Walsh system in this context.

Remark 2. Part ii) of *Theorem 2* was proved by *Schipp* in [4] with the dyadic Hardy norm on the right side. Here we improve this result by taking the norm of the real nonperiodic Hardy space. We also note that similar inequality holds for the trigonometric Dirichlet kernels ([4]). Consequently, the trigonometric and the Walsh systems behave similarly in this context.

Throughout this paper C will denote an absolute positive constant not necessarily the same in different occurrences.

Proofs

Proof of Theorem 1. Let F be bounded on $\mathbf{H}$. Then its restriction $\mathcal{F}$ is obviously bounded, and $\|\mathcal{F}\| \le \|F\|$. Since Ω is a subset of the unit ball of $\mathbf{H}$ we have that F is bounded on Ω, and $\sup_{\omega\in\Omega} \le \|F\|$.

Before proving the other direction we show that for a σ-sublinear functional F the boundedness is equivalent to the existence of an absolute positive constant C such that $F(\alpha) < C$ holds for each regular atom α. The necessity is immediate by $\|\alpha\|_{\mathbf{H}} \le 1$ $(\alpha \in \mathbf{A})$. On the other hand, if $F(\alpha) \le C$ for any $\alpha \in \mathbf{A}$, and $f = \sum_{k=1}^{\infty} c_k \alpha_k$ with $\alpha_k \in \mathbf{A}$ $(k \in \mathbb{N})$ and $\sum_{k=1}^{\infty} |c_k| < \infty$ then we have by the σ-sublinearity of F that

$$|F(f)| \le \sum_{k=1}^{\infty} |c_k||F(f_k)| \le C \sum_{k=1}^{\infty} |c_k|.$$

Consequently, $|F(f)| \le C\|f\|_{\mathbf{H}}$.

Suppose now that $\mathcal{F}$ is bounded and $\sup_{\omega\in\Omega} |F(\omega)|$ is finite. Let α be a regular atom different from the constant 1 function. By definition there exists an interval I for which $\operatorname{supp} \alpha \subset I$, $\|\alpha\|_\infty \le |I|^{-1}$, and $\int_0^1 \alpha = 0$. If $2^{-N} \le |I| < 2^{-N+1}$ then there is a K $(K \in \mathbb{N},\ 0 < K < 2^n)$ such that $I \subset [(K-1)2^{-N}, (K+1)2^{-N}]$. Set

$$a_1(x) = \begin{cases} \frac{1}{2}\Big(\alpha(x) - 2^N \int_{(K-1)2^{-N}}^{K2^{-N}} \alpha\Big) & \text{if} \quad (K-1)2^{-N} \le x < K2^{-N}, \\ 0 & \text{otherwise.} \end{cases}$$

Recall that $\|\alpha\|_\infty \le |I|^{-1} \le 2^N$. Therefore $\int_{(K-1)2^{-N}}^{K2^{-N}} |\alpha| \le 1$, and we have that a_1 is a dyadic atom. Similarly,

$$a_2(x) = \begin{cases} \frac{1}{2}\Big(\alpha(x) - 2^N \int_{K2^{-N}}^{(K+1)2^{-N}} \alpha\Big) & \text{if} \quad K2^{-N} \le x < (K+1)2^{-N}, \\ 0 & \text{otherwise} \end{cases}$$

is a dyadic atom. Moreover, it follows from $\int_I \alpha = 0$ that $\int_{(K-1)2^{-N}}^{K2^{-N}} \alpha + \int_{K2^{-N}}^{(K+1)2^{-N}} \alpha = 0$. Therefore, if

$$a_0(x) = \begin{cases} 2^N \int_{(K-1)2^{-N}}^{K2^{-N}} \alpha & \text{if} \quad (K-1)2^{-N} \le x < K2^{-N}, \\ 2^N \int_{K2^{-N}}^{(K+1)2^{-N}} \alpha & \text{if} \quad K2^{-N} \le x < (K+1)2^{-N}, \\ 0 & \text{otherwise} \end{cases}$$

then there exists $|\delta| \le 2$ such that $a_0 = \delta\omega_{K,N}$. Thus α can be decomposed as

$$\alpha = \delta\omega_{K,N} + 2a_1 + 2a_2.$$

Hence $F(\alpha) \le 2\sup_{\omega\in\Omega} |F(\omega)| + 4\|\mathcal{F}\| \quad (\alpha \in \mathbf{A})$. $\square$

Proof of Theorem 2. For the proof of part i) notice that $\omega_{2^{j-1},j}$ can be decomposed as difference of dyadic shifts of D_{2^j} :

$$\omega_{2^{j-1},j} = \frac{1}{2}(\tau_{1/2-1/2^j} D_{2^j} - \tau_{1/2} D_{2^j}) \qquad (j \in \mathbb{N}).$$

Then

$$U_{2^j}\omega_{2^{j-1},j} = \frac{1}{j}\sum_{k=1}^{2^j} \frac{1}{k}\|S_k\tau_{1/2-1/2^j} D_{2^j} - S_k\tau_{1/2} D_{2^j}\|_1 \qquad (j \in \mathbb{N}).$$

It is easy to see that the operators τ_δ and S_k $(\delta > 0,\ k \in \mathbb{N})$ can be interchanged. Indeed,

$$\begin{aligned} S_k\tau_\delta f(x) &= \int_0^1 f(t \dotplus \delta) D_k(x \dotplus t)\, dt = \int_0^1 f(t) D_k(x \dotplus t \dotplus \delta)\, dt = \sum_{\ell=1}^{k} \hat{f}(\ell) w_\ell(x \dotplus \delta) \\ &= \sum_{\ell=1}^{k} \hat{f}(\ell)\tau_\delta w_\ell(x) \\ &= \tau_\delta S_k f(x) \qquad (f \in L^1,\ \delta > 0,\ k \in \mathbb{N}). \end{aligned}$$

Hence

$$\begin{aligned} U_{2^j}\omega_{2^{j-1},j} = \frac{1}{j}\sum_{k=1}^{2^j}\frac{1}{k}\Big(\int_0^{1/2} &|\tau_{1/2-1/2^j} D_k - \tau_{1/2} D_k| \\ &+ \int_{1/2}^{1} |\tau_{1/2-1/2^j} D_k - \tau_{1/2} D_k|\Big) \qquad (j, n \in \mathbb{N}). \end{aligned}$$

Since

$$(1/2 - 1/2^j) \dotplus x = \frac{x_0}{2} + \sum_{\ell=1}^{j-1} |x_\ell - 1| 2^{-(\ell+1)} + \sum_{\ell=j}^{\infty} x_\ell 2^{-(\ell+1)}$$

we have that $x \to (1/2 - 1/2^j) \dotplus x$ is a one-to-one piecewise linear rearrangement that maps $[0, 1/2)$ and $[1/2, 1)$ onto themselves respectively. Similarly, $x \to 1/2 \dotplus x$

maps $[0,1/2)$ and $[1/2,1)$ piecewise linearly onto $[1/2,1)$ and $[0,1/2)$ respectively. Consequently,

$$\begin{aligned} U_{2^j}\omega_{2^{j-1},j} &\geq \frac{1}{j}\sum_{k=1}^{2^j}\frac{1}{k}\Big(\int_0^{1/2}|D_k| - \int_{1/2}^1 |D_k|\Big) \\ &\geq \frac{1}{j}\sum_{k=0}^{j-1}\frac{1}{2^{k+1}}\sum_{\ell=1}^{2^k}\Big(\int_0^1 |D_{2^k+\ell}| - 2\int_{1/2}^1 |D_{2^k+\ell}|\Big) \qquad (j\in\mathbb{N}). \end{aligned}$$

The pointwise estimation $|D_k(x)| \leq 2/x$ $(0<x<1,\ k\in\mathbb{N})$ ([1]) implies

$$\frac{1}{j}\sum_{k=0}^{j-1}\frac{1}{2^{k+1}}\sum_{\ell=1}^{2^k}\int_{1/2}^1 |D_{2^k+\ell}| \leq 1\,.$$

For the other terms we will use the following inequality (see [1] or [2])

$$\frac{1}{n}\sum_{k=1}^{n}\|D_k\|_1 \geq C\log n \qquad (n\in\mathbb{N},\ n\geq 2). \tag{3}$$

By (1) we have $D_{2^k+\ell} = D_{2^k} + w_{2^k}D_\ell$. Then (2) and (3) imply that

$$\sum_{\ell=1}^{2^k}\int_0^1 |D_{2^k+\ell}| \geq \sum_{\ell=1}^{2^k}\Big(\int_0^1 |D_\ell| - 1\Big) \geq C2^k k.$$

Consequently,

$$U_{2^j}\omega_{2^{j-1},j} \geq Cj \qquad (j\in\mathbb{N}).$$

We proved that the sequence of sublinear functionals U_n $(n\in\mathbb{N})$ is not uniformly bounded on $\mathbf{H}$. Then the existence of a function $f\in\mathbf{H}$ with $\lim_{n\to\infty}U_nf=\infty$ follows from the Banach -Steinhaus theorem.

The proof of ii) will be started by showing that T_n $(n\in\mathbb{N})$ is σ-sublinear on L^1, i.e. on $\mathbf{H}$ as well. To this end let $f=\sum_{j=1}^\infty f_j$ $(f, f_j\in L^1)$. By (2) we have

$$\begin{aligned} \Big|S_{2^n}f(x) - \sum_{j=1}^{\ell} S_{2^n}f_j(x)\Big| &\leq \int_0^1 \Big|f(t) - \sum_{j=1}^{\ell} f_j(t)\Big| D_{2^n}(x\dotplus t)\,dt \\ &\leq 2^n\Big\|f - \sum_{j=1}^{\ell} f_j\Big\|_1 \qquad (0\leq x<1,\ \ell\in\mathbb{N}). \end{aligned}$$

Consequently, $S_{2^n}f(x) = \sum_{j=1}^\infty S_{2^n}f_j(x)$ $(0\leq x<1)$. Then

$$\begin{aligned} T_nf &= \frac{1}{2^n}\Big\|\sum_{k=0}^{2^n-1}\sum_{j=1}^{\infty} S_{2^n}f_j(k2^{-n})D_{k+1}\Big\|_1 \\ &= \frac{1}{2^n}\Big\|\sum_{j=1}^{\infty}\Big(\sum_{k=0}^{2^n-1} S_{2^n}f_j(k2^{-n})D_{k+1}\Big)\Big\|_1 \leq \sum_{j=1}^{\infty} T_nf_j \qquad (n\in\mathbb{N}), \end{aligned}$$

i.e. T_n is σ-sublinear on L^1.

It is known ([4]) that there exists $C > 0$ absolute constant such that $T_n f \le C\|f\|_{\mathcal{H}}$ $(f \in \mathcal{H},\ n \in \mathbb{N})$. Therefore, by *Theorem 1* we only need to show that the T_n's $(n \in \mathbb{N})$ are uniformly bounded on Ω.

It is an immediate consequence of the definition of $\omega_{j,\ell}$ $(j, \ell \in \mathbb{N},\ j = 1, \dots, 2^\ell - 1)$ and of (2) that

$$S_{2^n}\omega_{j,\ell} = \begin{cases} \omega_{k,n} & \text{if } n < \ell, \text{ and } j2^{-\ell} = k2^{-n} \text{ with some } k = 0, \dots, 2^n - 1, \\ 0 & \text{if } n < \ell, \text{ and } j2^{-\ell} \neq k2^{-n},\ k = 0, \dots, 2^n - 1, \\ \omega_{j,\ell} & \text{if } n \ge \ell. \end{cases}$$

Therefore, we may suppose that $\ell \le n$. Then

$$T_n\omega_{j,\ell} = \frac{1}{2^n}\Big\| \sum_{k=0}^{2^n-1} \omega_{j,\ell}(k2^{-n}) D_{k+1} \Big\|_1 = \frac{2^{\ell-1}}{2^n}\Big\| \sum_{k=(j-1)2^{n-\ell}}^{j2^{n-\ell}-1} D_{k+1} - \sum_{k=j2^{n-\ell}}^{(j+1)2^{n-\ell}-1} D_{k+1} \Big\|_1.$$

By (1) we have

$$\sum_{k=(j-1)2^{n-\ell}}^{j2^{n-\ell}-1} D_{k+1} = 2^{n-\ell} D_{(j-1)2^{n-\ell}} + w_{(j-1)2^{n-\ell}} \sum_{k=1}^{2^{n-\ell}} D_k.$$

Similarly,

$$\sum_{k=j2^{n-\ell}}^{(j+1)2^{n-\ell}-1} D_{k+1} = 2^{n-\ell} D_{(j-1)2^{n-\ell}} + 2^{n-\ell} w_{(j-1)2^{n-\ell}} D_{2^{n-\ell}} + w_{j2^{n-\ell}} \sum_{k=1}^{2^{n-\ell}} D_k.$$

Then we have by (2) that

$$\begin{aligned} T_n\omega_{j,\ell} &= \frac{2^{\ell-1}}{2^n}\Big\| (w_{(j-1)2^{n-\ell}} - w_{j2^{n-\ell}}) \sum_{k=1}^{2^{n-\ell}} D_k - 2^{n-\ell} w_{(j-1)2^{n-\ell}} D_{2^{n-\ell}} \Big\|_1 \\ &\le \frac{2^{\ell-1}}{2^n}\Big(2^{n-\ell} + 2\Big\| \sum_{k=1}^{2^{n-\ell}} D_k \Big\|_1 \Big) = \frac{1}{2} + 2\|K_{2^{n-\ell}}\|_1, . \end{aligned}$$

where $K_n = 1/n \sum_{k=1}^n D_k$ $(n \in \mathbb{N},\ n \ge 1)$ denotes the nth Walsh-Fejér kernel. Since ([8]) $\|K_n\|_1 \le 2$ for any n we have

$$T_n\omega_{j,\ell} \le \frac{5}{2} \qquad (n, j, \ell \in \mathbb{N},\ j = 0, \dots 2^\ell - 1).$$

Consequently, the T_n's are uniformly bounded on Ω. □

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