Acta Mathematica Academiae Paedagogicae Nyíregyháziensis
Tomus 14 (1998), 37-39

PROPERTIES OF THE UNIT GROUP OF A NONMODULAR GROUP ALGEBRA

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Abstract. We give the nilpotency class of the group of units of a nonmodular group algebra.

1. Introduction.

Let G be a group and $\mathbb{F}$ a field. We denote by $U(\mathbb{F}G)$ the group of units of the group algebra $\mathbb{F}G$. We shall say that $\mathbb{F}G$ is a nonmodular group algebra if the characteristic of $\mathbb{F}$ does not divide the orders of elements of $T(G)$, the set of torsion elements of G. Group theoretical properties of the group of units are subject to intensive research. Among these nilpotence and the n-Engel property are of great importance. The next result is well-known.

(J.L. Fischer, M.M. Parmenter, S.K. Sehgal; I. Khripta; D.S. Passman [2, Theorem V.3.6]**).** *Suppose $\mathbb{F}G$ is a nonmodular group algebra. Then $U(\mathbb{F}G)$ is nilpotent (n-Engel for some n) if and only if G is nilpotent and one of the following conditions holds:*
(I) *$T(G)$ is a central subgroup;*
(II) *$\mathbb{F} = \mathrm{GF}(p)$ with $p = 2^t - 1$ a prime, $T(G)$ is an abelian group of exponent dividing $p^2 - 1$ and $g^{-1}ag = a^p$ for every $g \in G \setminus C_G(T(G))$ and every $a \in T(G)$.*

Sufficiency of these conditions was proved by showing that the nilpotency class $\mathrm{cl}(U)$ of the unit group does not exceed $\mathrm{cl}(G) + 1$ in the case (I), and $\mathrm{cl}(G) + t + 1$ in the case (II). Our aim is to determine $\mathrm{cl}(U)$.

Theorem. *Let G be a group nilpotent class $\mathrm{cl}(G)$ and $\mathbb{F}$ a field such that $\mathbb{F}G$ is nonmodular. Assume that $U = U(\mathbb{F}G)$ is nilpotent of class $\mathrm{cl}(U)$. Then*
(i) *if $T(G)$ is a central subgroup in G then $\mathrm{cl}(U) = \mathrm{cl}(G)$;*
(ii) *if $T(G)$ is not a central subgroup in G and $\mathbb{F} = \mathrm{GF}(p)$ with $p = 2^t - 1$ a prime then $\mathrm{cl}(U) = \max\{\mathrm{cl}(G), t + 1\}$.*

Modifying the proof of the Theorem we immediately have the next

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1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 16A27; Secondary 16A68.
Key words and phrases. Group algebra, group of units, nilpotency class, Engel.
Supported by Hungarian Ministry of Education grant No. FKFP-0451/1997 .

Corollary. *Let G be an n-Engel group and $\mathbb{F}$ a field such that $\mathbb{F}G$ is nonmodular. Assume that $U = U(\mathbb{F}G)$ is m-Engel for some m. Then*
(i) *if $T(G)$ is a central subgroup in G then U is n-Engel;*
(ii) *if $T(G)$ is not a central subgroup in G and $\mathbb{F} = \mathrm{GF}(p)$ with $p = 2^t - 1$ a prime then U is $\max\{n, t+1\}$-Engel.*

2. Proof of the Theorem.

Let $\mathrm{cl}(G) = n$, $T = T(G)$ and $x_1, x_2, \dots, x_{m+1} \in U$, where $m = n$ in the case (I) and $m = \max\{n, t+1\}$ in the case (II). As the kth term of the lower central series of U is generated by the commutators of weight k, to prove $\mathrm{cl}(U) \le m$ we must show $(x_1, x_2, \dots, x_{m+1}) = 1$.

Since in either cases (I) and (II) all idempotents of $\mathbb{F}T$ are central, the requirements of [1, Lemma 1.2] are satisfied and we can write

$$x_i = \sum_{j=1}^{r} \lambda_{i\,j} g_{i\,j} e_j,$$

where $\lambda_{i\,j} \in U(\mathbb{F}T)$, $g_{i\,j} \in G$ and the e_j are pairwise orthogonal idempotents with sum 1.

First assume (I). Since T is central and $\mathrm{cl}(G) = n = m$

$$(x_1, x_2, \dots, x_{m+1}) = \sum_{j=1}^{r} (g_{1\,j}, g_{2\,j}, \dots, g_{n+1\,j}) e_j = 1.$$

Clearly, $\mathrm{cl}(U) \ge \mathrm{cl}(G)$, hence we have the statement (i) of the Theorem.

Now assume that (II) holds and T is noncentral. Pick $g \in G$ wich does not centralize T. Clearly, $g^{-2} a g^2 = a^{p^2} = a$ for any $a \in T$, and therefore $G/C_G(T)$ is of exponent 2, $G' \subseteq C_G(T)$. We have $a \in \zeta(G) \cap T$ if and only if $g^{-1} a g = a^p = a$ i.e. $p - 1 \equiv 0 (\mathrm{mod}\, |a|)$; consequently $\zeta(G) \cap T = \{a \in T \mid |a| \mid p - 1\}$.

Let $\nu = \sum \alpha_i a_i \in U(\mathbb{F}T)$ with $\alpha_i \in \mathbb{F}$. Obviously,

$$\nu^{p^2-1} = (\sum \alpha^{p^2} a_i^{p^2}) \nu^{-1} = (\sum \alpha_i a_i) \nu^{-1} = 1$$

and $U(\mathbb{F}T)$ has exponent dividing $p^2 - 1$. Let T_1 be a finite subgroup of T noncentral in G. Then $\mathbb{F}T_1$ is a direct sum of copies of $\mathrm{GF}(p)$ and $\mathrm{GF}(p^2)$. Since the exponent of T_1 does not divide $p - 1$, there is, in fact, a direct summand isomorphic to $\mathrm{GF}(p^2)$. Therefore $U(\mathbb{F}T)$ is of exponent $p^2 - 1$ If $g_2, g_3, \dots, g_{k+1} \notin C_G(T)$ then $g_j^{-1} \nu g_j = \sum \alpha_i a_i^p = \nu^p$, $(\nu, g_j) = \nu^{p-1}$ and

$$(\nu, g_2, \dots, g_{k+1}) = \nu^{(p-1)(-2)^{k-1}}, \quad (\nu, g_2, \dots, g_{t+1}) = \nu^{(p-1)(-2)^{t-1}}, \quad (\nu, g_2, \dots, g_{t+2}) = 1. \tag{1}$$

If ν is of order $p^2 - 1$ and $g \notin C_G(T)$ then $(\nu, g, t) \ne 1$, and hence $\mathrm{cl}(U) \ge t + 1$. Obviously, $\mathrm{cl}(U) \ge \mathrm{cl}(G)$, therefore $\mathrm{cl}(U) \ge m$.

Let $g_1, g_2, \dots, g_k \in G$ and $\lambda_1, \lambda_2, \dots, \lambda_k \in U(\mathbb{F}T)$. Note that G normalizes and G' centralizes $U(\mathbb{F}T)$. We have

$$(\lambda_1 g_1, \lambda_2 g_2) = (\lambda_1 g_1, g_2)(\lambda_1 g_1, \lambda_2)^{g_2} = (\lambda_1, g_2)^{g_1}(g_1, g_2)(g_1, \lambda_2)^{g_2} =$$

$$= (\lambda_1, g_2)^{g_1}(g_1, \lambda_2)^{g_2}(g_1, g_2) = \theta(g_1, g_2),$$

where $\theta = (\lambda_1, g_2)^{g_1}(g_1, \lambda_2)^{g_2} \in U(\mathbb{F}T)$. Furthermore, with $k \geq 3$, we have

$$w_k = (\lambda_1 g_1, \lambda_2 g_2, \dots, \lambda_k g_k) = (\theta, g_3, \dots, g_k)(g_1, g_2, \dots, g_k).$$

We prove

$$w_{m+1} = 1. \tag{2}$$

Clearly, $(g_1, g_2, \dots, g_{m+1}) = 1$ as $m \geq n \geq 2$, and $w_{m+1} = (\theta, g_3, \dots, g_{m+1})$. If $g_1 \in C_G(T)$ then $\theta = (\lambda_1, g_2)$ and $w_{m+1} = (\lambda_1, g_2, \dots, g_{m+1}) = 1$ by (1) as $m \geq t+1$. Similarly, if $g_2 \in C_G(T)$ then $\theta = (g_1, \lambda_2)$ and $w_{m+1} = (g_1, \lambda_2, g_3, \dots, g_{m+1}) = 1$ by (1). If $g_j \in C_G(T)$ for some $3 \leq j \leq m+1$ then, clearly, $w_{m+1} = 1$. Suppose that none of the g_j are in $C_G(T)$. Then

$$\theta = (\lambda_1^{p-1})^{g_1}(\lambda_2^{1-p})^{g_2} = \lambda_1^{1-p}\lambda_2^{p-1} = (\lambda_1^{-1}\lambda_2)^{p-1} = (\lambda_1^{-1}\lambda_2, g_2),$$

and, by (1),

$$w_{m+1} = (\theta, g_3, \dots, g_{m+1}) = (\lambda_1^{-1}\lambda_2, g_2, \dots, g_{m+1}) = 1.$$

Clearly,

$$(x_1, x_2, \dots, x_{m+1}) = \sum_{j=1}^{r}(\lambda_{1\,j} g_{1\,j}, \lambda_{2\,j} g_{2\,j}, \dots, \lambda_{m+1\,j} g_{m+1\,j})e_j.$$

By (2) this commutator vanishes, which proves $\mathrm{cl}(U) \leq m$. The statement (ii) of the Theorem is clear, and the proof of the Theorem is complete.

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(Received October 7, 1997)