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ON FINSLER SPACES WHOSE GEODESICS ARE CONIC SECTIONS (REMARK TO A PAPER BY M. MATSUMOTO)

Z. KOVÁCS

Dedicated to Prof. L. Tamássy, on his 75th birthday

Abstract. In [Mat95] M. Matsumoto constructed Finsler metrics whose geodesics are two-parameter families of conic sections: semicircles, parabolas and hyperbolas. In this paper another two-parameter family $\mathcal{C}$ of conic sections is given that contains confocal hyperbolas and ellipses. We also construct Finsler spaces whose family of geodesics is the family $\mathcal{C}$.

1. Preliminaries

Recently Darboux's method for (two dimensional) inverse problem of variation calculus was highlighted by M. Matsumoto and some nice geometrical aspect of two-dimensional Finsler spaces was given through this method: [Mat89]. In [Mat95] M. Matsumoto constructed two-dimensional Finsler metrics in the upper semiplane $\{(x,y)|y>0\}$ whose geodesics are two-parameter families of conic sections: semicircles with centers on the x-axis, parabolas with vertex on the x-axis, and hyperbolas with x-axis as one of the asymptotic lines. In this paper we investigate the two-parameter family $\mathcal{C}$ of conic sections

$$\xi x^2 + y^2 = \mu.$$

Obviously for $\xi > 0$, $\mathcal{C}$ consists of ellipses or it is empty, while for $\xi < 0$, $\mathcal{C}$ consists of hyperbolas (see Fig.1.) The space is projectively flat, the substitutions $\bar{x} = x^2$, $\bar{y} = y^2$ give linear equations.

From geometrical point of view it is interesting to note that $\mathcal{C}$ contains confocal hyperbolas and ellipses. Fix $0 < a < b$ and let

$$\rho = \frac{b - a\xi}{1-\xi} \text{ (i.e. } \xi = \frac{b-\rho}{a-\rho})$$

be a new parameter. If we select from $\mathcal{C}$ a one parameter family with the condition $\mu = b - \rho$, we get the equation of confocal conic sections

$$\frac{x^2}{a-\rho} + \frac{y^2}{b-\rho} = 1.$$

For $\rho < a$ this is an ellipse, and for $a < \rho < b$ a hyperbola (see Fig.2.)

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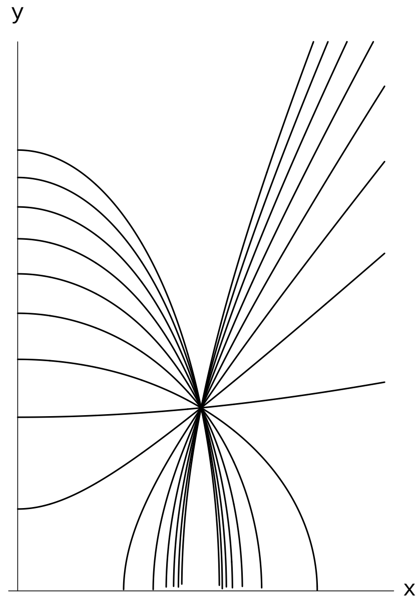

FIGURE 1. Geodesics from one point. (Graphics by *Mathematica*.)

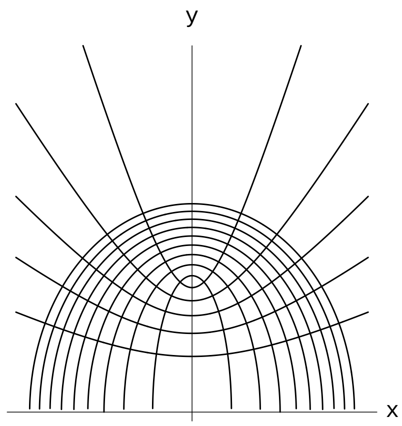

FIGURE 2. Confocal ellipses and hyperbolas. (Graphics by *Mathematica*.)

The essence of Darboux's method is the following. Let

$$y = f(x; a, b)$$

be a two-parameter family $\mathcal{C}$ of curves. In order to find the function $F(x, y, y')$ such that the set of the extremals of the integral

$$\int_{x_1}^{x_2} F(x, y, y')\, dx$$

be the family $\mathcal{C}$, one has to solve the Euler equation:

$$F_y - F_{xz} - F_{yz} z - F_{zz} \bar{Z} = 0,$$

where $z = y'$ and $\bar{Z} = f_{xx}(x; a(x,y,z), b(x,y,z))$. Let $G = F_{zz}$. Substituting this G in the Euler equation we get a first order partial differential equation:

$$G_x + G_y z + G_z \bar{Z} + G\bar{Z}_z = 0.$$

From F we find our Finsler metric as follows:

$$L(x,y,p,q) = F\left(x, y, \frac{q}{p}\right) p,$$

where p is supposed to be positive.

2. The statement

Theorem. *All the Finsler spaces on the underlying manifold* $\{(x,y)|y > 0,\ x > 0\} \subset \mathbb{R}^2$ *with geodesics*

$$\xi x^2 + y^2 = \mu \ (\mu, \xi \in \mathbb{R}) \tag{1}$$

are projective to the Finsler space with fundamental function of the form

$$F(x,y,z) = \frac{y^2}{x}\int_0^z (z-t)H(-yt/x, y^2 - ytx)dt + E_x(x,y) + zE_y(x,y),$$

where H and E are arbitrary functions (with the usual differentiability conditions).

Proof. From (1):

$$\begin{aligned} y &= \sqrt{\mu - \xi x^2} \\ y' = z &= -\frac{\xi x}{y} = -\xi\frac{x}{\sqrt{\mu - \xi x^2}}. \end{aligned} \tag{2}$$

From (2) we express the parameters ξ and μ:

$$\begin{aligned} \xi &= -\frac{yz}{x} \\ \mu &= y^2 - yzx. \end{aligned} \tag{3}$$

Then

$$\bar{Z} = \frac{z}{x} - \frac{z^2}{y},$$

and we obtain the first order P.D.E.

$$G_x + zG_y + G_z\left(\frac{z}{x} - \frac{z^2}{y}\right) + G\left(\frac{1}{x} - 2\frac{z}{y}\right) = 0. \tag{4}$$

Moreover

$$\frac{1}{x} - 2\frac{z}{y} = \frac{1}{x} + 2\frac{\xi x}{\mu - \xi x^2},$$

and the only non-trivial auxiliary equation of (4) is

$$\frac{dG}{dx} = -G\left(\frac{1}{x} + 2\frac{\xi x}{\mu - \xi x^2}\right). \tag{5}$$

One can easily integrate (5):

$$G = c\frac{\mu - \xi x^2}{x}.$$

Substitute ξ from (3):

$$\frac{\mu - \xi x^2}{x} = \frac{y^2}{x}.$$

Therefore we can generate a solution for G in the form

$$G(x,y,z) = \frac{y^2}{x} H(-yz/x, y^2 - yzx),$$

where H is an arbitrary function (with the usual differentiability conditions). Thus a solution for F is

$$F(x,y,z) = \int_0^z (z-t)G(x,y,t)dt + C(x,y) + zD(x,y),$$

where C and D satisfy

$$C_y - D_x = -F^*_y + F^*_{xz} + F^*_{yz}z + F^*_{zz}\bar{Z},$$

here

$$F^* = \int_0^z (z-t)G(x,y,t)dt.$$

From a long but simple computation: $C_y - D_x = 0$ and this gives the form of the statement ($E_x = C$, $E_y = D$). □

For example, let $H(\xi,\eta) = (-\xi)^n$ ($\xi < 0$), $E = 0$. Then

$$L(x,y,p,q) = \frac{1}{(1+n)(2+n)} \frac{y^{n+2}}{x^{n+1}} \frac{q^{n+2}}{p^{n+1}}. \tag{6}$$

This metric is conformal to the locally Minkowski metric q^{n+2}/p^{n+1}. Computing the main scalar of (6) we get the constant

$$\mathcal{I}^2 = \frac{(3+2n)^2}{n^2+3n+2}.$$

If $n = -3/2$, the main scalar is 0, the space is Riemannian.

If $n = 0$ then $\mathcal{I}^2 = \frac{9}{2}$. Applying Berwald's result (see e.g. [AIM93], Theorem 3.5.3.2.) this metric is a Berwald metric of the form

$$L^2 = \beta\gamma \left(\frac{\gamma}{\beta}\right)^{\frac{\mathcal{I}}{r}}, \ r = \sqrt{\mathcal{I}^2 - 4}$$

where β and γ are independent 1-forms in p, q. Namely:

$$\gamma = qy, \ \beta = 2px.$$

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Department of Mathematics, G. Bessenyei College, Sóstói út 31/b, 4400 Nyíregyháza, Hungary
E-mail address: kovacsz@agy.bgytf.hu

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