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STUDY OF FUZZY ALGEBRAS AND RELATIONS FROM A GENERAL VIEWPOINT

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ABSTRACT. We give definitions for fuzzy subalgebras, for fuzzy relations on fuzzy sets, and for fuzzy compatibility, which generalize, improve and correct the existing ones. The unit interval is replaced here by a partially ordered algebra. Then we study their connection with the corresponding crisp concepts through their newly defined Q-cuts.

1. INTRODUCTION

As it is well known, in the "classical" fuzzy theory established by L. A. Zadeh [5], a fuzzy set A is defined as a map from A to the real unit interval $I = [0, 1]$. The set of all fuzzy sets on A is usually denoted by I^A. It is also known that under the natural ordering I^A is a complete lattice. The order and the lattices as well as other operations on I can be extended "pointwise" to I^A.

In the paper [3] J. A. Goguen replaced I by a complete lattice L in the definition of fuzzy sets introducing the notion of L-fuzzy sets. Later more generalizations were also made using various membership sets and operations.

The definitions given to the concepts of fuzzy substructures, relations and compatibility also involve different membership sets and operations. Now we unify and generalize these definitions recognizing that in each case some kind of an ordered set and an operation having some properties were used. To study the connection between the corresponding crisp and fuzzy concepts the notion of Q-cut introduced by the author in [2] will be used. The theorems proved also highly generalize the existing ones.

2. RESULTS

Let A be a nonvoid set and $P = (P, *, 1, \leq)$ a $(2, 0)$-type ordered algebra, i.e. let

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(i) $(P, *)$ be a monoid,where 1 is the unity for $*$;
(ii) $(P, \leq)$ be a (partially) ordered set with 1 as the greatest element;
(iii) $*$ be isotone in both variables.

Further on, P always denotes such a structure.

A map $\mu : A \to P$ will be called a P-fuzzy subset of A or a P-fuzzy set on A. Denote their family by P^A. The order and the operations on P can also be extended pointwise to P^A. Recall that a subset Q of an ordered set $(P, \leq)$ is called a right segment or an upper set $(P, \leq)$ iff

$$\forall q \in Q, \forall p \in P : \ (q \leq p \implies p \in Q).$$

Clearly any closed interval $[p, 1]$ in P is a right segment of $(P, \leq)$. If P is the unit interval, then only the closed intervals $[\alpha, 1]$, $0 \leq \alpha \leq 1$ form a right segment. If P is a lattice L, then any filter (dual ideal) in L is a right segment. Conversely, a right segment in L that is closed under $*$ (specially under meet) is a filter. Let Q be a right segment of P. Then by the Q-cut μ_Q of some $\mu \in P^A$ we mean the following subset of A:

$$\mu_Q = \{x | x \in A, \mu(x) \in Q\}.$$

In case of $P = I$,the Q-cut reduces to the well known α-cut. A fuzzy relation r on A is usually defined as an element of $I^{A \times A}$. Here we will use (and generalize) the concept of fuzzy relation on fuzzy set, introduced by A. Rosenfeld [4] and not frequently studied in literature.

Definition 1. A P-fuzzy subset r of $P^{A \times A}$ is called a P-fuzzy relation on $\mu \in P^A$, if it satisfies the following property

$$\forall x, y : \ r(x, y) \leq \mu(x) * \mu(y).$$

Their family will be denoted by $R(\mu)$.

Definition 2. An $r \in R(\mu)$ is said to be

(i) reflexive, if

$$\forall x \in A : \ r(x, y) = \mu(x) * \mu(y);$$

(ii) symmetric, if

$$\forall x, y \in A : \ r(x, y) = r(y, x);$$

(iii) transitive, if for any $x, z \in A$

$$\forall y \in A : \ r(x, z) \geq r(x, y) * r(y, z).$$

A reflexive, symmetric and transitive P-fuzzy relation $r \in R(\mu)$ is called a P-fuzzy similarity (on μ).

If A denotes a (universal) algebra, that is if $A = (A, F)$, where A is a nonvoid set and F a specified set of finitary operations on A, then we can introduce the concept of a fuzzy algebra μ on A, and the concept of a fuzzy compatible relation r on μ.

Definition 3. A P-fuzzy set $\mu \in P^A$ is called a P-fuzzy algebra on the algebra A or a P-fuzzy subalgebra of A, if

(i) for any n-ary $(n \geq 1)$ operation $f \in F$

$$\mu(f(x_1 ... x_n)) \geq \mu(x_1) * \ldots * \mu(x_n),\ \forall x_1, \ldots, x_n \in A;$$

(ii) for any constant (nullary operation) c

$$\mu(c) \geq \mu(x),\ \forall x \in A.$$

The type of a P-fuzzy algebra on A is given by that of A.

Definition 4. Let A be an algebra and $\mu \in P^A$ a P-fuzzy algebra on A. An $r \in R(\mu)$ is called a P-fuzzy compatible relation on μ if

(i) for any n-ary $(n \geq 1)$ operation $f \in F$

$$r(f(x_1 \ldots x_n), f(y_1 \ldots y_n)) \geq r(x_1, y_1) * \ldots * r(x_n, y_n)$$

for all $x_1, \ldots, x_n, y_1, \ldots, y_n \in A$;

(ii) for any constant (nullary operation)

$$\forall x, y \in A:\ r(c, c) \geq r(x, y).$$

A compatible P-fuzzy similarity is called a P-fuzzy congruence (on the P-fuzzy algebra μ).

When A is a group, (ii) in Definitions 3 and 4 is a consequence of (i), respectively.

In fuzzy theory the following typical special cases used for the general concepts defined in Definitions 1,3,4:

1. $P = I$, $*$ =minimum ("classical" case);
2. $P = I$, $*$ =some t-norm, e.g. t-fuzzy group [1];
3. $P = L$, $*$ =meet (L-fuzzy case).

Now we establish the connection between these fuzzy concepts and the corresponding crisp ones through their Q-cuts.

Lemma 1. *Let* $J = \{1, \ldots, n\}$ *and* $K = \{j_1, \ldots, j_k\} \subseteq J$, *where* $k \geq 2$ *and* $j_1 < j_2 < \ldots < j_k$. *Then in* $(P, \leq)$

$$p_1 * \cdots * p_n \leq p_{j_1} * \cdots * p_{j_k}\ (k = 2, \ldots, n)$$

for all $p_1, \ldots, p_n \in P$.

Proof. Using the isotonity we have

$$p_1 * \cdots * p_n \leq 1 * \cdots * p_{j_1} * \cdots * p_{j_k} * \cdots * 1 = p_{j_1} * \cdots * p_{j_k}.$$

□

Lemma 2. *Let $\mu \in P^A$. If $r \in R(\mu)$, then*
(i) $\forall x, y \in A: \ r(x,y) \leq \mu(x),\ r(x,y) \leq \mu(y)$;
(ii) $r_Q \subseteq \mu_Q \times \mu_Q$, *where Q is a right segment of $(P, \leq)$.*

Proof. (i) $r(x,y) \leq \mu(x) * \mu(y) \leq \mu(x) * 1 = \mu(x)$,
$r(x,y) \leq \mu(x) * \mu(y) \leq 1 * \mu(y)$.
(ii) $(x,y) \in r_Q \implies r(x,y) \in Q$. Thus by (i):

$$\mu(x) \in Q,\ \mu(y) \in Q \implies x, y \in \mu_Q \implies (x,y) \in \mu_Q \times \mu_Q.$$

□

Theorem 1. *Let A be an algebra and let $\mu \in P^A$. If each non-empty Q-cut μ_Q of μ is a subalgebra of A, then μ is a P-fuzzy algebra on A.*

Proof. Take any elements $x_1, \ldots, x_n$ from A and any n-ary $(n \geq 1)$ operation f from F, and consider the following right segment of P

$$Q = [\mu(x_1) * \cdots * \mu(x_n), 1].$$

By Lemma 1 $\mu(x_i) \in Q$, therefore $x_i \in \mu_Q$ for all i. Since μ_Q is a subalgebra, hence $f(x_1 \ldots x_n) \in \mu_Q$, that is $\mu(f(x_1 \ldots x_n) \in Q$ is also true, which means that

$$\mu(f(x_1 \ldots x_n)) \geq \mu(x_1) * \cdots * \mu(x_n).$$

Now let c be some constant of the algebra A. Then by definition c is an element of all subalgebras of A, specially of any μ_Q. Consequently $\mu(c) \geq \mu(x)$ must hold for all $x \in A$. Suppose namely that $\mu(c) < \mu(x)$ for some $x \in A$. Then c does not belong to the Q-cut μ_Q of μ, where $Q = [\mu(x), 1]$. This contradiction verifies (ii) of Definition 3, too. □

Theorem 2. *Let A be an algebra, $\mu \in P^A$ a P-fuzzy algebra on A, and Q a right segment of P. If Q is closed under $*$, then μ_Q is a subalgebra of A.*

Proof. Consider a right segment Q satisfying the given condition. Then for any elements $x_1, \ldots, x_n \in \mu_Q$

$$\mu(x_1) * \cdots * \mu(x_n) \in Q$$

holds, since $\mu(x_i) \in Q$ $(i = 1, \ldots, n)$ and Q is closed under $*$. From here by Definition 3 we get

$$\mu(f(x_1 \ldots x_n)) \in Q \text{ and } f(x_1 \ldots x_n) \in \mu_Q,$$

which means that μ_Q is closed under f.

Since by (ii) of Definition 3 $\mu(c) \geq \mu(x)$ for any constant c and for all $x \in A$, hence $c \in \mu_Q$ for all nonempty μ_Q. Thus μ_Q is closed under nullary operations, too, completing the proof. □

Theorem 3. *Let $r \in R(\mu)$, where $\mu \in P^A$. If each Q-cut r_Q is an equivalence relation on μ_Q for any right segment Q of P, then r is a P-fuzzy similarity on μ.*

Proof. Let x be an arbitrary element of A. Take $Q = [\mu(x) * \mu(x), 1]$. Then $\mu(x) \in Q$, that is $x \in \mu_Q$ by Lemma 1. Since r_Q is reflexive on μ_Q, so $x \in \mu_Q$ implies $(x, x) \in r_Q$. This means by definition that $r(x, x) \geq \mu(x) * \mu(x)$.

On the other hand, by definition of P-fuzzy relation on μ (Definition 1) $r(x, x) \leq \mu(x) * \mu(x)$. These two inequalities together prove the reflexivity of r.

Now, let x, y be arbitrary elements of A. If $x, y \in r_Q$ for some right segment Q, then by Lemma 2 $x, y \in \mu_Q$. Take $Q = [r(x, y), 1]$. Obviously $r(x, y) \in Q$, that is $(x, y) \in r_Q$. Since r_Q is symmetric, it follows that $(y, x) \in r_Q$ or $r(y, x) \in Q$. Thus $r(y, x) \geq r(x, y)$ holds. Interchanging the role of x and y we similarly get: $r(y, x) \geq r(x, y)$. Consequently $r(x, y) = r(y, x)$ for all $x, y \in A$, verifying the symmetry of r.

To prove the transitivity of r, consider the arbitrary elements x, y, z of A, and choose the following right segment Q of P: $Q = [r(x, y) * r(y, z), 1]$. Then by Lemma 1 $r(x, y) \in Q$ and $r(y, z) \in Q$, that is equivalently $(x, y) \in r_Q$ and $(y, z) \in r_Q$. From here $x, y, z \in \mu_Q$ follows by Lemma 2. Further, since r_Q is transitive on μ_Q, we have $(x, z) \in r_Q$, i.e. $r(x, z) \in Q$. Thus $r(x, z) \geq r(x, y) * r(y, z)$, what we wanted to prove. □

Now we consider the inverse of Theorem 3.

Theorem 4. *Let r be a P-fuzzy similarity on $\mu \in P^A$ and let Q be a right segment of P. If Q is closed under $*$, then r_Q is an equivalence relation on μ_Q.*

Proof. Assume that Q is closed under $*$ and let $x \in \mu_Q$. Then $\mu(x) \in Q$. Since r is reflexive, we have $r(x, x) = \mu(x) * \mu(x) \in Q$, which implies that $(x, x) \in r_Q$ so that r_Q is reflexive on μ_Q. Let $(x, y) \in r_Q$. Then $r(x, y) \in Q$. Using the symmetry of r, we get $r(y, x) \in Q$ or $(y, x) \in r_Q$. This proves that r_Q is symmetric.

To prove transitivity, let $(x, y) \in r_Q$ and $(y, z) \in r_Q$. Then $r(x, y) \in Q$ and $r(y, z) \in Q$, which imply that $r(x, y) * r(y, z) \in Q$ because Q is closed under $*$. Since r is transitive, we have $r(x, z) \geq r(x, y) * r(y, z)$.

Using the fact that Q is a right segment of P, we conclude that $r(x, z) \in Q$ or $(x, z) \in r_Q$. This completes the proof. $\square$

Theorem 5. *Let A be an algebra, $\mu \in P^A$, and $r \in R(\mu)$. If for all non-empty right segment Q of P μ_Q is a subalgebra and r_Q is congruence on μ_Q, then r is P-fuzzy congruence on μ.*

Proof. By Theorem 1 μ is a P-fuzzy algebra on A, so the statement is not meaningless. Moreover, by Theorem 3 r is a P-fuzzy similarity on μ. Thus it is enough to show that r is a P-fuzzy compatible relation by Definition 4. Consider the elements x_i, y_i $(i = 1, \ldots, n)$ from A, and the n-ary $(n \geq 1)$ operation $f \in F$. If $(x_i, y_i) \in r_Q$ for some right segment Q and $i = 1, \ldots, n$, then by Lemma 2 $x_i, y_i \in \mu_Q$. Take the foolowing right segment:

$$Q = [p_1 * \cdots * p_n, 1],$$

where $p_i = r(x_i, y_i)$, $i = 1, \ldots, n$. Then by Lemma 1 $r(x_i, y_i) \in Q$, and consequently $(x_i, y_i) \in r_Q$ for all i. Since r_Q is a compatible relation on the subalgebra μ_Q, therefore $(x_i, y_i) \in r_Q$ $(i = 1, \ldots, n)$ implies that

$$(f(x_1 \ldots x_n), f(y_1 \ldots y_n)) \in r_Q,$$

and consequently

$$r(f(x_1 \ldots x_n), f(y_1 \ldots y_n) \in Q,$$

which means that

$$r(f(x_1 \ldots x_n), f(y_1 \ldots y_n)) \geq p_1 * \cdots * p_n,$$

proving the fulfilment of (i) in Definition 4 for r.

Now, let c be a constant in A. Since μ is a P-fuzzy algebra, so by definition $\mu(c) \geq \mu(x)$ for all $x \in A$. Since r is a P-fuzzy similarity on μ, hence

$$r(c, c) = \mu(c) * \mu(c) \geq \mu(x) * \mu(y) \geq r(x, y).$$

Thus (ii) of Definition 4 also holds for r. $\square$

Theorem 6. *Let A be an algebra, μ a P-fuzzy algebra on A, and r a P-fuzzy congruence on μ. If a right segment Q of P is closed under $*$, then r_Q is congruence on μ_Q.*

Proof. Because of Theorems 2 and 4 it is enough to prove that r_Q is a compatible relation on μ_Q, where Q is some right segment satisfying the given condition.

If $(x_i, y_i) \in r_Q$, that is $r(x_i, y_i) \in Q$, where x_i, y_i $(i = 1, \ldots, n)$ are arbitrary elements of A, then

$$r(x_1, y_1) * \cdots * r(x_n, y_n) \in Q$$

is also true because Q is closed under $*$. Since r is a P-fuzzy congruence, therefore

$$r(f(x_1 \dots x_n), f(y_1 \dots y_n)) \in Q,$$

that is

$$(f(x_1 \dots x_n), f(y_1 \dots y_n)) \in r_Q$$

follows from here for any n-ary $(n \geq 1)$ operation $f \in F$.

Further by Lemma 2, $(x_i, y_i) \in r_Q$ implies $x_i, y_i \in \mu_Q$. Thus r_Q is compatible on μ_Q with any n-ary $(n \geq 1)$ operation f.

If c is an arbitrary constant in A, then by definition $\mu(c) \geq \mu(x)$ for all $x \in A$. Thus $c \in \mu_Q$ for any non-empty μ_Q. But since r is reflexive, $c \in \mu_Q$ if and only if $(c, c) \in r_Q$. This completes the proof of the theorem. □

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