ARCHIVUM MATHEMATICUM (BRNO)
Tomus 46 (2010), 203–209

A NOTE ON FUSION BANACH FRAMES

S. K. Kaushik and Varinder Kumar

Abstract. For a fusion Banach frame $(\{G_n, v_n\}, S)$ for a Banach space E, if $(\{v_n^*(E^*), v_n^*\}, T)$ is a fusion Banach frame for E^*, then $(\{G_n, v_n\}, S; \{v_n^*(E^*), v_n^*\}, T)$ is called a fusion bi-Banach frame for E. It is proved that if E has an atomic decomposition, then E also has a fusion bi-Banach frame. Also, a sufficient condition for the existence of a fusion bi-Banach frame is given. Finally, a characterization of fusion bi-Banach frames is given.

1. Introduction

Frames for Hilbert spaces were introduced by Duffin and Schaeffer [5] in 1952 and re-introduced in 1986 by Daubechies, Grossmann and Meyer [4]. Casazza [2] and Benedetto and Fickus [1] have studied frames in finite dimensional spaces which attracted more attention due to their use in signal processing. Frames are now used as a tool in many areas like data compression, sampling theory, optics, filter banks, signal detection, time-frequency analysis etc.

The concept of frames in Hilbert spaces was extended to Banach spaces by Feichtinger and Gröchenig [6] who introduced the concept of atomic decompositions in Banach spaces. This concept was further generalized by Gröchenig [7] who introduced the notion of Banach frames for Banach spaces. Jain et al. [9], generalized Banach frames in Banach spaces and introduced frames of subspaces (Fusion Banach frames) for Banach spaces. They gave the following definition of a fusion Banach frame.

Definition 1.1 ([9]). Let E be a Banach space. Let $\{G_n\}$ be a sequence of non-trivial subspaces of E and $\{v_n\}$ be a sequence of bounded linear projections such that $v_n(E) = G_n$, $n \in \mathbb{N}$. We associate a Banach space $\mathcal{A}$ and an operator $S : \mathcal{A} \to E$ with the space E. Then $(\{G_n, v_n\}, S)$ is called a *frame of subspaces* (*fusion Banach frame*) for E with respect to $\mathcal{A}$ if

(i) $\{v_n(x)\} \in \mathcal{A}$, for all $x \in E$,

(ii) there exist constants A, B $(0 < A \le B < \infty)$ such that

$$A\|x\|_E \le \|\{v_n(x)\}\|_{\mathcal{A}} \le B\|x\|_E\,, \quad x \in E\,,$$

2000 *Mathematics Subject Classification*: primary 42C15; secondary 42A38.

Key words and phrases: atomic decompositions, fusion Banach frames, fusion bi-Banach frames.
Received September 24, 2009, revised April 2010. Editor V. Müller.

(iii) S is a bounded linear operator such that

$$S(\{v_n(x)\}) = x\,, \quad x \in E\,.$$

The following lemma, proved in [9], is used in the sequel

Lemma 1.2. *Let $\{G_n\}$ be a sequence of non-trivial subspaces of E and $\{v_n\}$ be a sequence of bounded linear projections with $v_n(E) = G_n$, $n \in \mathbb{N}$. If $\{v_n\}$ is total over E, i.e., $\{x \in E : v_n(x) = 0$, for all $n \in \mathbb{N}\} = \{0\}$, then $\mathcal{A} = \{\{v_n(x)\} : x \in E\}$ is a Banach space with norm $\|\{v_n(x)\}\|_{\mathcal{A}} = \|x\|_E$, $x \in E$.*

For other related notions on frames in Banach spaces one may refer to [3, 8, 10, 11].

In the present paper, we introduce fusion bi-Banach frames for a Banach space E. We prove that if E has an atomic decomposition, then E also has a fusion bi-Banach frame. Also, a sufficient condition for the existence of fusion bi-Banach frames is given. Finally, a characterization of fusion bi-Banach frames is obtained.

2. Main Results

One may observe that, if $(\{G_n, v_n\}, S)$ is a fusion Banach frame for E with respect to some associated Banach space $\mathcal{A}$, then there may not exist a Banach space $\mathcal{A}_1$ associated with E^* together with an operator $T\colon \mathcal{A}_1 \to E^*$ such that $(\{v_n^*(E^*), v_n^*\}, T)$ is a fusion Banach frame for E^* with respect to $\mathcal{A}_1$.

In this regard, we have the following examples

Example 2.1. Consider the Banach space

$$E = \ell^\infty(X) = \big\{\{x_n\} : x_n \in X;\ \sup_{1 \le n < \infty} \|x_n\|_X < \infty\big\}$$

equipped with the norm $\|\{x_n\}\|_E = \sup\limits_{1 \le n < \infty} \|x_n\|_X$, $\{x_n\} \in E$, where $(X, \|\cdot\|)$ is a Banach space. For each $n \in \mathbb{N}$, define $G_n = \{\delta_n^x : x \in X\}$ and $v_n(x) = \delta_n^{x_n}$, $x = \{x_n\} \in E$, where $\delta_n^x = (0, 0, \dots, 0, \underset{\substack{\downarrow \\ n\text{-th place}}}{x}, 0, \dots)$ for all $n \in \mathbb{N}$ and $x \in X$. Then by Lemma 1.2, there exist an associated Banach space $\mathcal{A} = \{\{v_n(x)\} : x \in E\}$ with norm $\|\{v_n(x)\}\|_{\mathcal{A}} = \|x\|_E$, $x \in E$ together with an operator $S\colon \mathcal{A} \to E$ given by $S(\{v_n(x)\}) = x$, $x \in E$ such that $(\{G_n, v_n\}, S)$ is a fusion Banach frame for E with respect to $\mathcal{A}$. But, there does not exist a Banach space $\mathcal{A}_1$ associated with E^* together with an operator $T\colon \mathcal{A}_1 \to E^*$ such that $(\{v_n^*(E^*), v_n^*\}, T)$ is a fusion Banach frame for E^* with respect to $\mathcal{A}_1$. For otherwise, $\big[\bigcup\limits_{n=1}^{\infty} G_n\big] = E$, which is not true.

Example 2.2. Let E be a Banach space defined as

$$E = c_0(X) = \big\{\{x_n\} : x_n \in X;\ \lim_{n \to \infty} \|x_n\|_X = 0\big\}$$

equipped with the norm given by

$$\|\{x_n\}\|_E = \sup_{1 \le n < \infty} \|x_n\|_X\,, \quad \text{where } (X, \|\cdot\|) \text{ is a Banach space.}$$

Define a sequence $\{G_n\}$ of subspaces of E by

$$G_{2n-1} = \{\delta_{2n-1}^x - 2^{n-1}\delta_{2n}^x : x \in X\}$$
$$G_{2n} = \{\delta_{2n}^x : x \in X\}\,.$$

Also define operators v_n on E by

$$v_{2n-1}(x) = \delta_{2n-1}^{x_{2n-1}} - 2^{n-1}\delta_{2n}^{x_{2n-1}}$$
$$v_{2n}(x) = \delta_{2n}^{2^{n-1}x_{2n-1}+x_{2n}} \quad \text{for all } x = \{x_n\} \in E \text{ and } n \in \mathbb{N}\,.$$

Then by Lemma 1.2 there exist an associated Banach space $\mathcal{A}$ and an operator $S\colon \mathcal{A} \to E$ such that $(\{G_n, v_n\}, S)$ is a fusion Banach frame for E with respect to $\mathcal{A}$.

If $\big[\bigcup_{n=1}^{\infty} G_n\big] \neq E$, then there exists $0 \neq f = \{f_i\} \in E^*$ such that $f(y) = 0$ for all $y \in G_n$, $n \in \mathbb{N}$. This would imply $f_n = 0$ for all $n \in \mathbb{N}$ and hence $f = 0$. Therefore, by Lemma 1.2 again, there exist a Banach space $\mathcal{A}_1$ associated to E^* and an operator $T\colon \mathcal{A}_1 \to E^*$ such that $(\{v_n^*(E^*), v_n^*\}, T)$ is a fusion Banach frame for E^* with respect to $\mathcal{A}_1$.

In view of the above discussion, we define the following

Definition 2.3. Let E be a Banach space. Let $\{G_n\}$ be a sequence of non-trivial subspaces of E and $\{v_n\}$ be a sequence of bounded linear projections such that $v_n(E) = G_n$, $n \in \mathbb{N}$. If there exist Banach spaces $\mathcal{A}$ and $\mathcal{A}_1$ associated with E and E^* respectively and operators $S\colon \mathcal{A} \to E$ and $T\colon \mathcal{A}_1 \to E^*$ such that $(\{G_n, v_n\}, S)$ is a fusion Banach frame for E with respect to $\mathcal{A}$ and $(\{v_n^*(E^*), v_n^*\}, T)$ is a fusion Banach frame for E^* with respect to $\mathcal{A}_1$, then we call the system $(\{G_n, v_n\}, S; \{v_n^*(E^*), v_n^*\}, T)$ a *fusion bi-Banach frame* for E

In view of Remark 3.2.1 in [9], we have

Every reflexive Banach space has a fusion bi-Banach frame.

Recall that if E is a Banach space and E_d is an associated Banach space of scalar-valued sequences, indexed by $\mathbb{N}$, $\{x_n\}$ is a sequence in E and $\{f_n\}$ is a sequence in E^*, then the pair $(\{f_n\}, \{x_n\})$ is called an *atomic decomposition* for E with respect to E_d if

(i) $\{f_n(x)\} \in E_d$, $x \in E$;

(ii) there exist constants A, B with $0 < A \leq B < \infty$ such that

$$A\|x\|_E \leq \|\{f_n(x)\}\|_{E_d} \leq B\|x\|_E\,, \quad x \in E\,;$$

(iii) $x = \sum_{n=1}^{\infty} f_n(x)x_n$, $x \in E$.

The next result is regarding the existence of fusion bi-Banach frames for a Banach space having an atomic decomposition.

Theorem 2.4. *Let E be a Banach space. If E has an atomic decomposition, then it also has a fusion bi-Banach frame.*

Proof. Let $(\{f_n\},\{x_n\})$ be an atomic decomposition for E with respect to E_d. Define $G_n = [x_n]$, $n \in \mathbb{N}$ and $v_n(x) = f_n(x)x_n$, $n \in \mathbb{N}$. Then there exist an associated Banach space $\mathcal{A} = \{\{v_n(x)\} : x \in E\}$ together with an operator $S\colon \mathcal{A} \to E$ such that $(\{G_n, v_n\}, S)$ is a fusion Banach frame for E with respect to $\mathcal{A}$. Further $\big[\bigcup_{n=1}^{\infty} G_n\big] = E$ (as $[x_n] = E$). So, $v_n^*(f) = 0$ for all $n \in \mathbb{N}$ imply $f = 0$, where $f \in E^*$. Thus, $\{v_n^*\}$ is total over E^* and so by Lemma 1.2, there exist an associated Banach space $\mathcal{A}_1$ and an operator $T\colon \mathcal{A}_1 \to E^*$ such that $(\{v_n^*(E^*), v_n^*\}, T)$ is a fusion Banach frame for E^* with respect to $\mathcal{A}_1$. Hence, $(\{G_n, v_n\}, S; \{v_n^*(E^*), v_n^*\}, T)$ is a fusion bi-Banach frame for E. □

Next, we observe that if E be a Banach space and $\{G_n\}$ be a sequence of non-trivial subspaces of E with associated sequence of projections $\{v_n\}$ with $v_n(E) = G_n$, $n \in \mathbb{N}$, then it is possible that there exist a Banach space $\mathcal{A}_1$ associated with E^* together with a bounded linear operator $T\colon \mathcal{A}_1 \to E^*$ such that $(\{v_n^*(E^*), v_n^*\}, T)$ is a fusion Banach frame for E^* with respect to $\mathcal{A}_1$ and there may not exist any Banach space $\mathcal{A}$ associated with E together with an operator $S\colon \mathcal{A} \to E$ such that $(\{G_n, v_n\}, S)$ is a fusion Banach frame for E with respect to $\mathcal{A}$. Indeed, let

$$E = \ell^2(X) = \Big\{\{x_n\} : x_n \in X; \sum_{n=1}^{\infty} \|x_n\|_X^2 < \infty\Big\},$$

where $(X, \|\cdot\|)$ is a Banach space, equipped with the norm given by

$$\|\{x_n\}\|_E = \Big(\sum_{n=1}^{\infty} \|x_n\|_X^2\Big)^{1/2}.$$

Define for $n \in \mathbb{N}$, $G_n = \{\delta_1^x + \delta_{n+1}^x : x \in X\}$ and $v_n(x) = \delta_1^{x_{n+1}} + \delta_{n+1}^{x_{n+1}}$, $x = \{x_n\} \in E$, where $\delta_n^x = (0, 0, \ldots, 0, \underset{\substack{\downarrow \\ n\text{th place}}}{x}, 0, \ldots)$, $x \in X$.

Then $\big[\bigcup_{n=1}^{\infty} G_n\big] = E$ and $v_i v_j = 0$ for all $i \neq j$.

But, since for any $0 \neq x \in X$, $\delta_1^x = (x, 0, 0, \ldots) \in E$ is such that $v_n(\delta_1^x) = 0$, for all $n \in N$, there exist no associated Banach space $\mathcal{A}$ such that $(\{G_n, v_n\}, S)$ is a fusion Banach frame for E with respect to $\mathcal{A}$. However, there exist a Banach space $\mathcal{A}_0$ and an operator $T\colon \mathcal{A}_0 \to E^*$ such that $(\{v_n^*(E^*), v_n^*\}, T)$ is a fusion Banach frame for E^* with respect to $\mathcal{A}_0$.

In view of the above discussion, we prove the following result

Theorem 2.5. *Let E be a Banach space and $\{G_n\}$ be a sequence of subspaces of E with $\big[\bigcup_{n=1}^{\infty} G_n\big] = E$. Let $\{v_n\}$ be a sequence of projections on E satisfying $v_n(E) = G_n$, $n \in \mathbb{N}$ and $v_i v_j = 0$ for all $i \neq j$. Then there exist Banach spaces $\mathcal{A}$ and $\mathcal{A}_1$ associated with E and E^*, respectively, and operators $S\colon \mathcal{A} \to E$ and $T\colon \mathcal{A}_1 \to E^*$ such that $(\{G_n, v_n\}, S; \{v_n^*(E^*), v_n^*\}, T)$ is a fusion bi-Banach frame* *for E if every sequence $\{x_n\} \subset E$ such that $x_n \in G_n$ and $x_n \neq 0$, $n \in \mathbb{N}$ satisfies* $\bigcap_{n=1}^{\infty} [x_{n+1}, x_{n+2}, \ldots] = \{0\}$.

Proof. Since $\big[\bigcup_{n=1}^{\infty} G_n\big] = E$, there exist an associated Banach space $\mathcal{A}_1$ and a bounded linear operator $T\colon \mathcal{A}_1 \to E^*$ such that $(\{v_n^*(E^*), v_n^*\}, T)$ is a fusion Banach frame for E^* with respect to $\mathcal{A}_1$. Let, if possible, there exist no Banach space $\mathcal{A}$ associated with E such that $(\{G_n, v_n\}, S)$ is a fusion Banach frame for E with respect to $\mathcal{A}$ where $S\colon \mathcal{A} \to E$ is a bounded linear operator. Now, since $\big[\bigcup_{n=1}^{\infty} G_n\big] = E$ and $v_i v_j = 0$ for all $i \neq j$, $u_n = \sum_{i=1}^{n} v_i$ is a bounded linear projection of E onto $\big[\bigcup_{i=1}^{n} G_i\big]$ along $\Big[\bigcup_{i=n+1}^{\infty} G_i\Big]$, $n \in \mathbb{N}$. Write $E = \big[\bigcup_{i=1}^{n} G_i\big] \oplus \big[\bigcup_{i=n+1}^{\infty} G_i\big]$, $n \in \mathbb{N}$. Then

$$\{x \in E : v_i(x) = 0, i = 1, 2, \ldots, n\} = \Big[\bigcup_{i=n+1}^{\infty} G_i\Big], \quad n \in \mathbb{N}.$$

Since $(\{G_n, v_n\}, S)$ is not a fusion Banach frame for E with respect to any associated Banach space, there exists $0 \neq x \in \bigcap_{n=1}^{\infty} \big[\bigcup_{i=n+1}^{\infty} G_i\big]$. So, there exists $y_1 = \sum_{i=1}^{m_1} z_i$ where $z_i \in G_i$ $(1 \leq i \leq m_1)$ such that $\operatorname{dist}(x, y_1) < 1$, that is, $\operatorname{dist}\big(x, \big[\bigcup_{i=1}^{m_1} G_i\big]\big) < 1$. Also, $x \in \big[\bigcup_{i=m_1+1}^{\infty} G_i\big]$. So, we can choose $m_2 > m_1$ and $y_2 = \sum_{i=m_1+1}^{m_2} z_i$, where $z_i \in G_i$ $(m_1 + 1 \leq i \leq m_2)$ such that $\operatorname{dist}\big(x, \big[\bigcup_{i=m_1+1}^{m_2} G_i\big]\big) < \frac{1}{2}$. Proceeding like this, for each $n \in \mathbb{N}$, we get a sequence $\{z_n\} \subset E$ and an increasing sequence $\{m_n\}$ of positive integers such that $z_n \in G_n$, $n \in \mathbb{N}$ and $\operatorname{dist}\big(x, \big[\bigcup_{i=m_{n-1}+1}^{m_n} G_i\big]\big) < \frac{1}{n}$.
Thus $x \in [z_{n+1}, z_{n+2}, \ldots]$, $n \in \mathbb{N}$. Consider a sequence $\{x_n\} \subset E$ with $0 \neq x_n \in G_n$, $n \in \mathbb{N}$ such that $x_n = z_n$ whenever $z_n \neq 0$. Then $x \in [x_{n+1}, x_{n+2}, \ldots]$, $n \in \mathbb{N}$. Hence $\bigcap_{n=1}^{\infty} [x_{n+1}, x_{n+2}, \ldots] \neq \{0\}$. □

Finally, we give a characterization of fusion bi-Banach frames in terms of a sequence in bv_0, where bv_0 is the linear space of all sequences $\{\alpha_n\}$ of scalars with $\lim_{n\to\infty} \alpha_n = 0$ and for which the norm $\|\{\alpha_n\}\| = \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n|$ is finite.

Theorem 2.6. *Let E be a Banach space and $(\{G_n, v_n\}, S)$ be a fusion Banach frame for E, where the projections $\{v_n\}$ on E are such that $v_i v_j = 0$ for all $i \neq j$. Then $(\{G_n, v_n\}, S; \{v_n^*(E^*), v_n^*\}, T)$ is a fusion bi-Banach frame for E if and only* *if for every* $x \in E$, *there exist* $\{\alpha_j\} \in bv_0$ *and* $z \in E$ *such that* $v_n(x) = \alpha_n v_n(z)$, $n \in \mathbb{N}$ *and* $\sup\limits_{1 \le n < \infty} \big\| \sum\limits_{i=1}^{n} v_i(z) \big\| < \infty$.

Proof. Let $(\{G_n, v_n\}, S; \{v_n^*(E^*), v_n^*\}, T)$ be a fusion bi-Banach frame for E. For each $k \in \mathbb{N}$, write $u_k = \sum\limits_{i=1}^{k} v_i$. Then $\lim\limits_{k\to\infty} u_k(x) = x$, $x \in E$. Therefore, there exists a sequence $\{m_n\}$ of positive integers such that

$$\|x - u_k(x)\| < \frac{1}{4^{n+1}}\,, \qquad k \ge m_n\,, \quad n \in \mathbb{N}\,.$$

Take $y_n = \sum\limits_{i=m_{n-1}+1}^{m_n} v_i(x)$, $n \in \mathbb{N}$. Then $\|y_n\| \le \dfrac{2}{4^n}$, $n \in \mathbb{N}$.
So, $\sum\limits_{n=1}^{\infty} 2^{n-1}\|y_n\| \le \sum\limits_{n=1}^{\infty} 2^{-n}$. Thus, the series $\sum\limits_{n=1}^{\infty} 2^{n-1} y_n$ converges.
Put $z = \sum\limits_{n=1}^{\infty} 2^{n-1} y_n$ and $\alpha_j = 2^{1-n}, m_{n-1} + 1 \le j \le m_n$, $n \in \mathbb{N}$.
Therefore, $\{\alpha_j\} \in bv_0$. Also, we have

$$v_j(z) = 2^{n-1} v_j(x)\,, \qquad m_{n-1} + 1 \le j \le m_n\,,\ n \in \mathbb{N}\,.$$

Hence, $v_j(x) = \alpha_j v_j(z)$, $j \in \mathbb{N}$.

Conversely, for integers $p < q$, we have

$$\Big\| \sum_{i=p}^{q} v_i(x) \Big\| = \Big\| \sum_{i=p}^{q} \alpha_i \Big(\sum_{j=1}^{i} v_j(z) - \sum_{j=1}^{i-1} v_j(z) \Big) \Big\|$$
$$\le \Big(|\alpha_p| + \sum_{i=p}^{q-1} |\alpha_i - \alpha_{i+1}| + |\alpha_q| \Big) \sup_{1 \le n < \infty} \Big\| \sum_{j=1}^{n} v_j(z) \Big\|$$

Since, $\{\alpha_j\} \in bv_0$, $\big\{ \sum\limits_{i=1}^{n} v_i(x) \big\}$ is a Cauchy sequence and hence converges.
Also, since $\{v_n\}$ is total on E and

$$v_j\Big(x - \lim_{n\to\infty} \sum_{i=1}^{n} v_i(x) \Big) = 0\,, \quad \text{for all} \quad j \in \mathbb{N}\,,$$

it follows that $x = \lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} v_i(x)$. Therefore, $\Big[\bigcup\limits_{n=1}^{\infty} G_i \Big] = E$. Thus, $\{v_n^*\}$ is total over E^* and so by Lemma 1.2, there exist a Banach space $\mathcal{A}_1$ associated with E^* and an operator $T\colon \mathcal{A}_1 \to E^*$ such that $(\{v_n^*(E^*), v_n^*\}, T)$ is a fusion Banach frame for E^* with respect to $\mathcal{A}_1$. Hence, $(\{G_n, v_n\}, S; \{v_n^*(E^*), v_n^*\}, T)$ is a fusion bi-Banach frame for E. □

Acknowledgement. The authors thank the referees for their useful suggestions towards the improvement of the paper. The research of second author is supported by the CSIR (India) (vide letter File No. 09/045(0647)/2006-EMR-I).

References

[1] Benedetto, J. J., Fickus, M., *Finite normalized tight frames*, Adv. Comput. Math. **18** (2–4) (2003), 357–385.

[2] Casazza, P. G., *Custom Building finite frames*, Wavelets, Frames and Operator Theory (Heil, C., Jorgensen, P. E. T., Larson, D. R., eds.), vol. 345, Contemp. Math., 2004, pp. 81–86.

[3] Christensen, O., *An Introduction to Frames and Reisz Bases*, Birkhäuser, 2002.

[4] Daubechies, I., Grossmann, A., Meyer, Y., *Painless non-orthogonal expansions*, J. Math. Phys. **27** (1986), 1271–1283.

[5] Duffin, R. J., Schaeffer, A. C., *A class of non-harmonic Fourier Series*, Trans. Amer. Math. Soc. **72** (1952), 341–366.

[6] Feichtinger, H. G., Gröchenig, K., *A unified approach to atomic decompositons via integrable group representations*, Proc. Conf. Function Spaces and Applications, Lecture Notes in Math. 1302, Berlin-Heidelberg-New York, Springer, 1988, pp. 52–73.

[7] Gröchenig, K., *Describing functions: Atomic decompositions versus frames*, Monatsh. Math. **112** (1991), 1–41.

[8] Jain, P. K., Kaushik, S. K., Gupta, N., *On near exact Banach frames in Banach spaces*, Bull. Austral. Math. Soc. **78** (2008), 335–342.

[9] Jain, P. K., Kaushik, S. K., Kumar, V., *Frames of subspaces for Banach spaces*, Int. J. Wavelets Multiresolut. Inf. Process. **8** (2) (2010), 243–252.

[10] Jain, P. K., Kaushik, S. K., Vashisht, L. K., *Banach frames for conjugate Banach spaces*, Z. Anal. Anwendungen **23** (4) (2004), 713–720.

[11] Jain, P. K., Kaushik, S. K., Vashisht, L. K., *On Banach frames*, Indian J. Pure Appl. Math. **37** (5) (2006), 265–272.

Department of Mathematics, Kirori Mal College
University of Delhi, Delhi 110007, India
E-mail: shikk2003@yahoo.co.in

Department of Mathematics, University of Delhi
Delhi 110007, India
E-mail: vicky.h1729@gmail.com