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A NEW CHARACTERIZATION OF MATHIEU GROUPS

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Abstract. Let G be a finite group and $\mathrm{nse}(G)$ the set of numbers of elements with the same order in G. In this paper, we prove that a finite group G is isomorphic to M, where M is one of the Mathieu groups, if and only if the following hold:

(1) $|G| = |M|$,
(2) $\mathrm{nse}(G) = \mathrm{nse}(M)$.

1. Introduction

It is well known that the conjugacy class sizes play an important role in determining the structure of a finite group. The connection between the conjugacy class sizes and the structures of finite groups has been studied extensively (see [3], [5], [8], for example).

Analogically, let $m_i(G) := |\{g \in G \mid \text{ the order of } g \text{ is } i\}|$ (m_i for short), be the number of elements of order i, and $\mathrm{nse}(G) := \{m_i(G) \mid i \in \pi_e(G)\}$, the set of sizes of elements with the same order. We now consider the influence of the set $\mathrm{nse}(G)$ and $|G|$ on G.

For the set $\mathrm{nse}(G)$, the most important problem is related to the Thompson's problem.

Let G be a finite group and $M_t(G) = \{g \in G \mid g^t = 1\}$. Two finite groups G_1 and G_2 are of the same order type if and only if $|M_t(G_1)| = |M_t(G_2)|$, where $t = 1, 2, \dots$. In 1987, J. G. Thompson put forward the following problem:

Thompson's problem. Suppose G_1 and G_2 are of the same order type. If G_1 is solvable, is G_2 necessarily solvable?

Professor W. J. Shi made the above problem public in 1989 (see [10]). Unfortunately, no one can solve it or even give a counterexample till now.

We found that the set $\mathrm{nse}(G)$ plays an important role in determining structure of a finite group, too. Surely, the set $\{|M_t(G)| \mid t = 1, 2, \dots\}$ can determine the set $\mathrm{nse}(G)$. However, if the set $\mathrm{nse}(G)$ is known, what can we say about $|M_t(G)|$?

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Main theorem. A group G is isomorphic to M, where M is a Mathieu group, if and only if the following hold:

(1) $|G| = |M|$,

(2) $\mathrm{nse}(G) = \mathrm{nse}(M)$.

In this paper, $n_p(G)$ always denotes the number of Sylow p-subgroups of G, that is, $n_p(G) = |\operatorname{Syl}_p(G)|$, $\pi(G)$ the set of all prime divisors of $|G|$. And $\varphi(x)$ denotes the Euler function of x. We always use $|G|$ for order of finite group G, $|$ to denote division relationship and $||$ denote that the prime upon the left is in its highest possible power divides the argument upon the right. All further unexplained notation is standard (see [10]).

2. Lemmas

Lemma 2.1 ([6])**.** *Suppose G is a finite solvable group with $|G| = mn$, where $m = p_1^{\alpha_1} \dots p_r^{\alpha_r}, (m,n) = 1$, $p_1, \dots, p_r$ are distinct primes. Let $\pi = \{p_1, \dots, p_r\}$ and let h_m be the number of π-Hall subgroups of G. Suppose that $h_m = q_1^{\beta_1} \dots q_s^{\beta_s}$, where $q_1, \dots, q_s$ are distinct primes. Then following conditions are true for all i:*

(1) $q_i^{\beta_i} \equiv 1 \pmod{p_j}$ *for some p_j.*

(2) *The order of some chief factor of G is divided by $q_i^{\beta_i}$.*

A finite group G is called a K_n-group, if $|\pi(G)| = n$.

Lemma 2.2 ([12])**.** *Let G be a simple K_4-group, then G is isomorphic to one of the following groups:*

1) A_7, A_8, A_9, A_{10};

2) M_{11}, M_{12}, J_2;

3) (a) $L_2(r)$, *where r is a prime and satisfies*

$$r^2 - 1 = 2^a \cdot 3^b \cdot u^c$$

with $a \geq 1, b \geq 1, c \geq 1, u > 3, u$ is prime;

(b) $L_2(2^m)$, *where m satisfies:*

$$\begin{cases} 2^m - 1 = u\,; \\ 2^m + 1 = 3t^b\,. \end{cases}$$

with $m \geq 1, u, t$ primes, $t > 3$, $b \geq 1$.

(c) $L_2(3^m)$, *where m satisfies:*

$$\begin{cases} 3^m + 1 = 4t\,; \\ 3^m - 1 = 2u^c\,. \end{cases}$$

or

$$\begin{cases} 3^m + 1 = 4t^b\,; \\ 3^m - 1 = 2u\,. \end{cases}$$

with $m \geq 1, u, t$ odd primes, $b \geq 1$, $c \geq 1$.

(d) $L_2(16)$, $L_2(25)$, $L_2(49)$, $L_2(81)$, $L_3(4)$, $L_3(5)$, $L_3(7)$, $L_3(8)$, $L_3(17)$, $L_4(3)$, $S_4(4)$, $S_4(5)$, $S_4(7)$, $S_4(9)$, $S_6(2)$, $O_8{}^+(2)$, $G_2(3)$, $U_3(4)$, $U_3(5)$, $U_3(7)$, $U_3(8)$, $U_3(9)$, $U_4(3)$, $U_5(2)$, $Sz(8)$, $Sz(32)$, ${}^3D_4(2)$, ${}^2F_4(2)'$.

Corollary 2.3 ([12])**.** *Let G be a simple group of order $2^a \cdot 3^b \cdot 5 \cdot p^c$, where p is a prime, $p \neq 2, 3, 5$ and $abc \neq 0$. Then G is isomorphic to one of the following groups: A_7, A_8, A_9; M_{11}, M_{12}; $L_2(q)$, $q = 11, 16, 19, 31, 81$; $L_3(4)$, $L_4(3)$, $S_6(2)$, $U_4(3)$ or $U_5(2)$. In particular, if $p = 11$, then $G \cong M_{11}$, M_{12} or $L_2(11)$. If $p = 7$, then $G \cong A_7$, A_8, A_9, $L_3(4)$, $S_6(2)$ or $U_4(3)$.*

Lemma 2.4. *Let G be a simple K_4-group and $\pi(G) \subseteq \{2, 3, 5, 7, 11, 23\}$. Then G is isomorphic to one of the following simple groups:*
(1) A_7, A_8, A_9, A_{10};
(2) M_{11}, M_{12}, J_2;
(3) $L_2(11)$, $L_2(23)$;
(4) $L_2(49)$, $L_3(4)$, $S_4(7)$, $S_6(2)$, $O_8^+(2)$, $U_3(5)$, $U_4(3)$, $U_5(2)$.

Proof. If G is isomorphic to one of the groups of 1), 2), or 3)(d) in Lemma 2.2, we can easily get (1), (2) and (4) by [4].

Suppose now that G is isomorphic to one of the groups of (a), (b) or (c) in 3) of Lemma 2.2.

(I) If G is isomorphic to $L_2(r)$ in Lemma 2.2, then $r \in \{5, 7, 11, 23\}$.
If $r = 5$ or 7, then $|\pi(r^2 - 1)| = 2$, a contradiction.
If $r = 11$, then $r^2 - 1 = 2^3 \cdot 3 \cdot 5$. Thus $G \cong L_2(11)$.
If $r = 23$, then $r^2 - 1 = 2^4 \cdot 3 \cdot 11$. Thus $G \cong L_2(23)$.

(II) If G is isomorphic to $L_2(2^m)$ in Lemma 2.2, then $u \in \{3, 5, 7, 11, 23\}$.
If $u = 3$, then $m = 2$ and $3t^b = 5$, a contradiction.
If $u = 5$, then $2^m - 1 = 5$, a contradiction.
If $u = 7$, then $m = 3$ and $3t^b = 9$, thus $t = 3, b = 1$, this contradicts $t > 3$.
If $u = 11$, then $2^m - 1 = 11$, a contradiction.
If $u = 23$, then $2^m - 1 = 23$, a contradiction.

Similarly, we can prove that G is not isomorphic to $L_2(3^m)$ in Lemma 2.2. □

Lemma 2.5 ([2])**.** *Let α_i be a positive integer ($i = 1, \ldots, 5$), p a prime and $p \notin \{2, 3, 5, 7\}$. If G is a simple group and $|G| = 2^{\alpha_1} \cdot 3^{\alpha_2} \cdot 5^{\alpha_3} \cdot 7^{\alpha_4} \cdot p^{\alpha_5}$, then G is isomorphic to one of the following simple groups: A_{11}, A_{12}, M_{22}, HS, McL, He, $L_2(q)$ ($q = 2^6, 5^3, 7^4, 29, 41, 71, 251, 449, 4801$), $L_3(3^2)$, $L_4(2^2)$, $L_4(7)$, $L_5(2)$, $L_6(2)$, $O_5(7^2)$, $O_7(3)$, $O_9(2)$, $S_6(3)$, $O_8^+(3)$, $G_2(2^2)$, $G_2(5)$, $U_3(19)$, $U_4(5)$, $U_4(7)$, $U_5(3)$, $U_6(2)$, ${}^2D_4(2)$. In particular, if $p = 11$, then G is isomorphic to one of the following simple groups: A_{11}, A_{12}, M_{22}, HS, McL, $U_6(2)$.*

Lemma 2.6 ([1])**.** *Let α_i be a positive integer ($i = 1, \ldots, 6$), $p > 11$ a prime. If G is a simple group and $|G| = 2^{\alpha_1} \cdot 3^{\alpha_2} \cdot 5^{\alpha_3} \cdot 7^{\alpha_4} \cdot 11^{\alpha_5} \cdot p^{\alpha_6}$, then G is isomorphic to one of the following simple groups: A_{13}, A_{14}, A_{15}, A_{16}, M_{23}, M_{24}, J_1, Suz, Co_2,Co_3, $M(22)$, F_3, $L_2(769)$, $L_2(881)$, $L_3(11)$, $L_6(3)$, $U_7(2)$, ${}^2D_5(2)$. In particular, if $p = 23$, then G is isomorphic to one of the following simple groups: M_{23}, M_{24}, Co_2, Co_3.*

Lemma 2.7 ([7])**.** *If G is a simple K_3-group, then G is isomorphic to one of the following groups: A_5, A_6, $L_2(7)$, $L_2(8)$, $L_2(17)$, $U_3(3)$, $L_3(3)$, $U_4(2)$.*

Lemma 2.8. *Let G be a finite group, $P \in \mathrm{Syl}_p(G)$, where $p \in \pi(G)$. Suppose that G has a normal series $K \trianglelefteq L \trianglelefteq G$. If $P \leq L$ and $p \nmid |K|$, then the following hold:*

(1) $N_{G/K}(PK/K) = N_G(P)K/K$;

(2) $|G\colon N_G(P)| = |L\colon N_L(P)|$, *that is,* $n_p(G) = n_p(L)$;

(3) $|L/K\colon N_{L/K}(PK/K)|t = |G\colon N_G(P)| = |L\colon N_L(P)|$, *that is,* $n_p(L/K)$ $t = n_p(G) = n_p(L)$, *for some positive integer t. And $|N_K(P)|t = |K|$.*

Proof. (1) follows from [7].

(2) By Frattini argument, $G = N_G(P)L$. Hence $|G\colon N_G(P)| = |N_G(P)L\colon N_G(P)|$ $= |L\colon N_L(P)|$.

(3) By (1), we have $|L/K\colon N_{L/K}(PK/K)| = |L/K\colon N_L(P)K/K| = |L\colon N_L(P)K|\big||L\colon N_L(P)|$, then $|L/K\colon N_{L/K}(PK/K)|t = |G\colon N_G(P)| = |L\colon N_L(P)|$ for some positive integer t.

As $|L/K\colon N_{L/K}(PK/K)|t = |L\colon N_L(P)|$, then $|N_L(P)|t = |N_L(P)K|$. Hence $|N_K(P)|t = |K|$. □

Lemma 2.9 ([9])**.** *Each simple K_5-group is isomorphic to one of the following simple groups:*

(a) $L_2(q)$ *where q satisfies* $|\pi(q^2-1)| = 4$;

(b) $L_3(q)$ *where q satisfies* $|\pi(q^2-1)(q^3-1)| = 4$;

(c) $U_3(q)$ *where* $|\pi(q^2-1)(q^3+1)| = 4$;

(d) $O_5(q)$ *where* $|\pi(q^4-1)| = 4$;

(e) $S_z(2^{2^m+1})$ *where* $|\pi((2^{2^m+1}-1)(2^{2^{4m+2}}+1))| = 4$;

(f) $R(q)$ *where q is an odd power of 3 and* $|\pi(q^2-1)| = 3$;

(h) *one of the 30 other simple groups:* A_{11}, A_{12}, M_{22}, J_3, HS, He, McL, $L_4(4)$, $L_4(5)$, $L_4(7)$, $L_5(2)$, $L_5(3)$, $L_6(2)$, $O_7(3)$, $O_9(2)$, $PSp_6(3)$, $PSp_8(2)$, $U_4(4)$, $U_4(5)$, $U_4(7)$, $U_4(9)$, $U_5(3)$, $U_6(2)$, ${O_8}^+(3)$, ${O_8}^-(2)$, ${}^3D_4(3)$, $G_2(4)$, $G_2(5)$, $G_2(7)$, $G_2(9)$.

Lemma 2.10. *Let G be a simple K_5-group and $|G|\big|2^{10}\cdot 3^3\cdot 5\cdot 7\cdot 11\cdot 23$. Then $G \cong M_{22}$.*

Proof. Assume G is isomorphic to $L_2(q)$ in Lemma 2.9. Then 11 or $23\big|\ |L_2(q)|$.

(1) If $11\big||L_2(q)|$, we claim $q \neq 11$, otherwise if $q = 11$, then $|\pi(q^2-1)| = 3$, which contradicts $|\pi(q^2-1)| = 4$.

If $q = 2^m$, then $11\big|2^{2m}-1$. So we have $5\big|m$ and $31\big||L_2(q)|$, a contradiction.

If $q = 3, 3^2, 5, 7$ or 23, then $|\pi(q^2-1)| < 4$, a contradiction.

If $q = 3^3$, then $13\big|q^2-1\big||G|$, a contradiction.

(2) If $23\big||L_2(q)|$, we also get a contradiction similar to (1).

Hence G is not isomorphic to $L_2(q)$.

Similarly, G is not isomorphic to $L_3(q)$, $U_3(q)$, $O_5(q)$, $Sz(2^{2m+1})$ and $R(q)$.

In (h) of Lemma 2.9, we see that $M_{22} \cong G$. □

3. Proof of the main theorem

We shall present a separate proof for each of the Mathieu groups.

Theorem 3.1. *Let G be a group. Then $G \cong M_{11}$ if and only if the following hold:*
(1) $|G| = |M|$,
(2) $\mathrm{nse}(G) = \mathrm{nse}(M_{11}) = \{1, 165, 440, 990, 1584, 1320, 1980, 1440\}$.

Proof. The necessity is obvious. We only need to prove the sufficiency.

First, we prove that G is unsolvable.

If G is solvable, let H be a $\{2,5,11\}$-Hall subgroup of G. All $\{2,5,11\}$-Hall subgroups in G are conjugate, so the number of $\{2,5,11\}$-Hall subgroups of G is: $|G\colon N_G(H)|\big|3^2$.

Now we calculate the number of elements of order 11 in G. By Sylow theorem we have that $n_{11}(H) = 1$ in H. So the number m of elements with order 11 in G is: $10 \le m \le 90$ and $10\big|m$, but $m \notin \mathrm{nse}(G)$.

Thus G is unsolvable. Since $p\big||G|$, where $p \in \{5, 11\}$, G has a normal series:

$$1 \trianglelefteq K \trianglelefteq L \trianglelefteq G\,,$$

such that L/K is a simple K_3-group or a simple K_4-group.

If L/K is a simple K_3-group. Since 5 or $11\big||L/K|\big|2^4 \cdot 3^2 \cdot 5 \cdot 11$, then $L/K \cong A_5$ or A_6.

(1) Assume $L/K \cong A_5$. If $P_5 \in \mathrm{Syl}_5(G)$, then $P_5K/K \in \mathrm{Syl}_5(L/K)$. Also $n_5(L/K)t = n_5(G)$ for some positive integer t by Lemma 2.8.

By [4], $n_5(L/K) = n_5(A_5) = 6$. Hence $n_5(G) = 6t$ and $5 \nmid t$.

So the number m of elements of order 5 in G is: $m = n_5(G) \cdot 4 = 24t$. Since $m \in \mathrm{nse}(G)$, $m = 1584$ and $t = 66$. By Lemma 2.8, $66|N_K(P_5)| = |K|$. As $|K|\big|2^2 \cdot 3 \cdot 11$, thus $n_{11}(K) = 1$ or 12. So we get that the number of elements of order 11 in G is 10 or 120. But 10, $120 \notin \mathrm{nse}(G)$, a contradiction.

(2) Assume $L/K \cong A_6$. If $P_5 \in \mathrm{Syl}_5(G)$, then $P_5K/K \in \mathrm{Syl}_5(L/K)$. Also $n_5(L/K)t = n_5(G)$ for some positive integer t by Lemma 2.8.

By [4], $n_5(L/K) = n_5(A_6) = 36$. Hence $n_5(G) = 36t$ and $5 \nmid t$.

So the number m of elements of order 5 in G is: $m = n_5(G) \cdot 4 = 36t$. Since $m \in nse(G)$, $m = 1584$ and $t = 44$. As $|G| = |G\colon L|\,|L\colon K|\,|K|$ and $44|N_K(P_5)| = |K|$ by Lemma 2.8. Thus $44\big||K|$, then $44 \cdot |A_6|\big|G|$, that is, $2^5 \cdot 3^2 \cdot 5 \cdot 11\big||G|$, which is a contradiction.

So L/K is a simple K_4-group and $\pi(L/K) = \pi(G) = \{2,3,5,11\}$. By Corollary 2.3 $L/K \cong M_{11}, M_{12}$ or $L_2(11)$. But $|L/K|\big|2^4 \cdot 3^2 \cdot 5 \cdot 11$, thus $L/K \cong M_{11}$ or $L_2(11)$.

Assume $L/K \cong L_2(11)$. Let $P_{11} \in \mathrm{Syl}_{11}(G)$, then $P_{11}K/K \in \mathrm{Syl}_{11}(L/K)$. By Lemma 2.8, $n_{11}(L/K)t = n_{11}(G)$ for some positive integer t and $11 \nmid t$.

By [4], $n_{11}(L/K) = n_{11}(L_2(11)) = 12$. Hence $n_{11}(G) = 12t$.

Thus the number m of elements of order 11 in G is: $m = n_{11}(G) \cdot 10 = 120t$. Since $m \in \{1, 165, 440, 990, 1584, 1320, 1980, 1440\}$, $m = 1440$ and $t = 12$. Therefor $12|N_K(P_{11})| = |K|$ by Lemma 2.8. As $|K|\big|12$, so $N_K(P_{11}) = 1$ and $|K| = 12$. And then $K \cap N_G(P_{11}) = K \cap C_G(P_{11}) = 1$. So $K \rtimes P_{11}$ is a Frobenius group, which means that $|P_{11}|\big|\,|\mathrm{Aut}(K)|$, a contradiction.

So $L/K \cong M_{11}$, and hence $|L/K| = |M_{11}|$. Thus $K = 1$ and $G = L \cong M_{11}$. $\square$

Theorem 3.2. *Let G be a group. Then $G \cong M_{12}$ if and only if the following hold:*
(1) $|G| = |M|$,
(2) $\mathrm{nse}(G) = \mathrm{nse}(M_{12}) = \{1, 891, 4400, 5940, 9504, 23760, 9504, 17280\}$.

Proof. The necessity is obvious. We only need to prove the sufficiency.

First, we prove that G is unsolvable.

If G is solvable, let H be a $\{2,5,11\}$-Hall subgroup of G. G is solvable, and therefore all the $\{2,5,11\}$-Hall subgroups of G are conjugate. Hence the number of $\{2,5,11\}$-Hall subgroups of G is:

$$|G \colon N_G(H)| \big| 3^3 .$$

We have $n_{11}(H) = 1$ or 320 by Sylow theorem. Let m be the number of elements of order 11 in G. If $n_{11}(H) = 1$, then $10 \le m \le 270$. But $m \notin \mathrm{nse}(G)$, a contradiction.

If $n_{11}(H) = 320$, then $3200 \le m \le 86400$ and $10 \big| m$. Since $m \in \{1, 891, 4400, 5940, 9504, 23760, 9504, 17280\}$, $m = 4400, 5940, 23760$ or 17280. And we have $n_{11}(G) \cdot 10 = m$ in G, that is, $n_{11}(G) = 11k+1 = 440, 594, 2376$ or 1728 for some positive integer k. If $n_{11}(G) = 11k+1 = 440, 594$ or 2376, then this equation has no solution in N. If $n_{11}(G) = 11k+1 = 1728 = 2^6 \cdot 3^3$, then we have $2^6 \equiv 1 \pmod{11}$ and $3^3 \equiv 1 \pmod{11}$ by Lemma 2.1, a contradiction.

Hence G is unsolvable. Since $p \big| \, |G|$, where $p \in \{5, 11\}$, G has a normal series as follows:

$$1 \trianglelefteq K \trianglelefteq L \trianglelefteq G ,$$

such that L/K is a non-Abelian simple group. Since $|\pi(G)| = 4$, then $|\pi(L/K)| = 3$ or 4.

If $|\pi(L/K)| = 3$, then L/K is a simple K_3-group and $\pi(L/K) \subset \pi(G) = \{2,3,5,11\}$. Hence G is isomorphic to one of the group: A_5, A_6 or $U_4(2)$ by Lemma 2.7.

(1) Assume $L/K \cong A_5$. If $P_5 \in \mathrm{Syl}_5(G)$, then $P_5K/K \in \mathrm{Syl}_5(L/K)$. By Lemma 2.8, $n_5(L/K)t = n_5(G)$ for some positive integer t and $5 \nmid t$.

By [4], $n_5(L/K) = n_5(A_5) = 6$. Hence $n_5(G) = 6t$.

So the number m of elements of order 5 in G is: $m = n_5(G) \cdot 4 = 24t$. Since $m \in \mathrm{nse}(G)$, then $m = 9504$ and $t = 396$. Therefore $396|N_K(P_5)| = |K|$ by Lemma 2.8. As $|K| \big| 2^4 \cdot 3 \cdot 11$, and hence $n_{11}(K) = 1$, 12 or 144. So the number of elements of order 11 in G is: 10 or 120. But $10, 120, 1440 \notin \mathrm{nse}(G)$, a contradiction.

(2) Assume $L/K \cong A_6$. If $P_5 \in \mathrm{Syl}_5(G)$, then $P_5K/K \in \mathrm{Syl}_5(L/K)$. By Lemma 2.8, $n_5(L/K)t = n_5(G)$ for some positive integer t and $5 \nmid t$.

By [4], $n_5(L/K) = n_5(A_6) = 36$. Hence $n_5(G) = 36t$.

So the number m of elements of order 5 in G is: $m = n_5(G) \cdot 4 = 144t$. Since $m \in \mathrm{nse}(G)$, and hence $m = 9504$ and $t = 66$. By Lemma 2.8, $66|N_K(P_5)| = |K|$. As $|G| = |G \colon L|\,|L \colon K|\,|K|$, then $|K| \big| 2^3 \cdot 3 \cdot 11$. So we have $n_{11}(K) = 1$ or 12. And then the number m of elements of order 11 in G is: $m = 10$ or 120. But 10, $120 \notin \mathrm{nse}(G)$, a contradiction.

(3) Assume $L/K \cong U_4(2)$, then $|U_4(2)| \,\big|\, |G|$, that is, $2^6 \cdot 3^4 \cdot 5 \big| \, |G| = 2^6 \cdot 3^3 \cdot 5 \cdot 11$, a contradiction.

Hence L/K is a simple K_4- group and $\pi(L/K) = \{2, 3, 5, 11\}$, therefore $L/K \cong M_{11}, M_{12}$ or $L_2(11)$ by Corollary 2.3.

(1) Assume $L/K \cong M_{11}$, Let $P_{11} \in \mathrm{Syl}_{11}(G)$, then $P_{11}K/K \in \mathrm{Syl}_{11}(L/K)$. Also $n_{11}(L/K)t = n_{11}(G)$ for some positive integer t and $11 \nmid t$.

By [4], $n_{11}(L/K) = n_{11}(M_{11}) = 144$. Hence $n_{11}(G) = 144t$.

So the number m of elements of order 5 in G is: $m = n_{11}(G) \cdot 10 = 1440t$. Since $m \in \mathrm{nse}(G)$, $m = 17280$ and $t = 12$. By Lemma 2.8, $12|N_K(P_5)| = |K|$. As $|K| \big| 2^2 \cdot 3$, then $|K| = 2^2 \cdot 3$ and $N_K(P_{11}) = 1$. And $1 = N_K(P_{11}) \geq C_K(P_{11})$. So $K \rtimes P_{11}$ is a Frobenius group, and hence $|P_{11}| \,\big|\, |\mathrm{Aut}(K)|$, a contradiction.

(2) Assume $L/K \cong L_2(11)$. If $P_{11} \in \mathrm{Syl}_{11}(G)$, then $P_{11}K/K \in \mathrm{Syl}_{11}(L/K)$. Also $n_{11}(L/K)t = n_{11}(G)$ for some positive integer t and $5 \nmid t$ by Lemma 2.8.

By [4], $n_{11}(L/K) = n_{11}(L_2(11)) = 12$. Hence $n_{11}(G) = 12t$ and $11 \nmid t$.

So the number m of elements of order 11 in G is: $m = n_{11}(G) \cdot 10 = 120t$. Since $m \in \mathrm{nse}(G)$, $m = 17280$ and $t = 144$. By Lemma 2.8, $144|N_K(P_{11})| = |K|$. As $|K| \,\big| 2^4 \cdot 3^2$, then $|K| = 2^4 \cdot 3^2$ and $N_K(P_{11}) = 1$. And $1 = N_K(P_{11}) \geq C_K(P_{11})$. So $K \rtimes P_{11}$ is a Frobenius group, and hence $|P_{11}| \,\big|\, |\mathrm{Aut}(K)|$, a contradiction.

So we get $L/K \cong M_{12}$, and hence $|L/K| = |M_{12}| = |G|$. Thus $K = 1$ and $G = L \cong M_{12}$. □

Theorem 3.3. *Let G be a group. Then $G \cong M_{22}$ if and only if the following hold:*

(1) $|G| = |M|$,

(2) $\mathrm{nse}(G) = \mathrm{nse}(M_{22}) = \{1, 1155, 12320, 41580, 88704, 36960, 126720, 55440, 80640\}$.

Proof. The necessity is obvious. We only need to prove the sufficiency.

First, we prove that G is unsolvable.

If G is solvable, let H be a $\{3, 5, 7, 11\}$-Hall subgroup of G. G is solvable, and therefore all the $\{3, 5, 7, 11\}$-Hall subgroups of G are conjugate. Hence the number of $\{3, 5, 7, 11\}$-Hall subgroup of G is:

$$|G\colon N_G(H)| \,\big| 2^7 .$$

We have $n_{11}(H) = 1$ or 45 by Sylow theorem. Let m be the number of elements of order 11 in G.

If $n_{11}(H) = 1$, then $10 \leq m \leq 1280$ and $10 \big| \, m$. But $m \notin \mathrm{nse}(G)$, a contradiction.

If $n_{11}(H) = 45$, then $450 \leq m \leq 57600$ and $10 \big| \, m$. Since $m \in \mathrm{nse}(G)$, $m = 12320, 41580, 36960$ or 55440. And we have $n_{11}(G) \cdot 10 = m$, that is, $11k + 1 = 1232, 4158, 3696$ or 5544 in G for some positive integer k. But this equation has no solution in N.

Hence, G is unsolvable. Since $p \big| \,\big|\, |G|$, where $p \in \{5, 7, 11\}$, G has a normal series:

$$1 \trianglelefteq K \trianglelefteq L \trianglelefteq G \,,$$

such that L/K is a non-Abelian simple group.

(1) L/K is not a simple K_3-group. Otherwise, $L/K \cong A_5, A_6, L_2(7)$ or $L_2(8)$ by Lemma 2.7 and [4].

Assume $L/K \cong A_5$. If $P_5 \in \mathrm{Syl}_5(G)$, then $P_5K/K \in \mathrm{Syl}_5(L/K)$. Also by Lemma 2.8, $n_5(L/K)t = n_5(G)$ for some positive integer t and $5 \mid t$.

By [4], $n_5(L/K) = n_5(A_5) = 6$. Hence $n_5(G) = 6t$.

So the number m of elements of order 5 in G is: $m = n_5(G) \cdot 4 = 24t$. Since $m \in \mathrm{nse}(G)$, then $m = 88704$ and $t = 3696$. By Lemma 2.8, $3696|N_K(P_5)| = |K|$.

As $|K| \,\big|\, 2^5 \cdot 3 \cdot 7 \cdot 11$, there must be $n_{11}(K) = 1, 56$ or 672. So the number m of elements of order 11 in G is: 10, 560 or 6720. But $m \notin \mathrm{nse}(G)$, a contradiction.

Similarly, L/K is not isomorphic to $A_6, L_2(7)$ or $L_2(8)$.

(2) L/K is not a simple K_4-group. Otherwise, by Corollary 2.3, we have $L/K \cong A_7$, A_8, M_{11}, $L_2(11)$ or $L_3(4)$.

Assume $L/K \cong A_7$. If $P_5 \in \mathrm{Syl}_5(G)$, then $P_5K/K \in \mathrm{Syl}_5(L/K)$. Also $n_5(L/K)$ $t = n_5(G)$ for some positive integer t and $5 \nmid t$.

By [4], $n_5(L/K) = n_5(A_7) = 126$. Hence $n_5(G) = 126t$.

So the number m of elements of order 5 in G is: $m = n_5(G) \cdot 4 = 504t$. Since $m \in nse(G)$, then $m = 88704$ and $t = 176$. By Lemma 2.8, $176|N_K(P_5)| = |K|$. As $|G| = |G\colon L|\,|L\colon K|\,|K|$, then $|K| \,\big|\, 2^4 \cdot 11$, that is, $|K| = 2^4 \cdot 11$. And then $n_{11}(K) = 1$. So the number m of elements of order 11 in G is: $m = 10$. But $10 \notin \mathrm{nse}(G)$, a contradiction.

Similarly, we can get that L/K is not isomorphic to A_8, M_{11}, $L_2(11)$ and $L_3(4)$.

Hence L/K is a simple K_5-group. By Lemma 2.10, $L/K \cong M_{22}$. So we have $|L/K| = |M_{22}| = |G|$. Thus $K = 1$ and $G = L \cong M_{22}$. □

Theorem 3.4. *Let G be a group. Then $G \cong M_{23}$ if and only if the following hold:*

(1) $|G| = |M|$,

(2) $\mathrm{nse}(G) = \mathrm{nse}(M_{23}) = \{1, 3795, 56672, 318780, 680064, 850080, 1457280, 1275120, 1854720, 1360128, 887040\}$.

Proof. The necessity is obvious. We only need to prove the sufficiency.

First, we prove that G is unsolvable.

If G is solvable, then G contains a $\{3, 5, 7, 11, 23\}$-Hall subgroup. By Sylow theorem, $n_{23}(H) = 1$ or 231.

Moreover,

$$|G\colon N_G(H)| \,\big|\, 2^7\,.$$

If $n_{23}(H) = 1$, then $22 \le m \le 2816$ and $22 \mid m$, but $m \notin \mathrm{nse}(G)$.

If $n_{23}(H) = 231$, then $5082 \le m \le 650496$ and $22 \mid m$, but $m \in \mathrm{nse}(G)$.

Hence $m = 56672$ or 318780. And we have $n_{23}(G) \cdot 22 = m$ in G, that is, $23k + 1 = 2576$ or 1440 for some positive integer k, but the equation has no solution in N.

Thus, G is unsolvable. Since $p\big|\,\big||G|$, where $p \in \{5, 7, 11, 23\}$, G has a normal series:

$$1 \trianglelefteq K \trianglelefteq L \trianglelefteq G\,,$$

such that L/K is a non-Abelian simple group.

(1) L/K is not a simple K_3-group. Otherwise, L/K is isomorphic to A_5, A_6, $L_2(7)$, $L_2(8)$ or $U_3(3)$ by Lemma 2.7 and [4].

Assume $L/K \cong A_5$. If $P_5 \in \mathrm{Syl}_5(G)$, then $P_5K/K \in \mathrm{Syl}_5(L/K)$. Also by Lemma 2.8, $n_5(L/K)t = n_5(G)$ for some positive integer t and $5 \nmid t$.

By [4], $n_5(L/K) = n_5(A_5) = 6$. Hence $n_5(G) = 6t$.

So the number m of elements of order 5 in G is: $m = n_5(G) \cdot 4 = 24t$. Since $m \in \mathrm{nse}(G)$, there must be $m = 680064$ or 1360128 and $t = 28336$ or 56672, respectively.

If $m = 680064$ and $t = 28336$, then $28336|N_K(P_5)| = |K|$ by Lemma 2.8. As $|K| \bigm| 2^5 \cdot 3 \cdot 7 \cdot 11 \cdot 23$, we obtain $n_{23}(K) = 1$. So the number m of elements of order 23 in G is: $m = 22$. But $22 \in \mathrm{nse}(G)$, a contradiction.

If $m = 1360128$ and $t = 56672$. Similarly as above, we also get a contradiction.

Similarly, $L/K \ncong A_6, L_2(7), L_2(8)$ or $U_3(3)$.

(2) L/K is not a simple K_4-group. Otherwise, by Lemma 2.4 and [4], $L/K \cong A_7$, A_8, M_{11}, $L_3(4)$, $L_2(11)$ or $L_2(23)$.

If $L/K \cong A_7$, then $n_7(L/K) = 120$, $n_7(G) = 120t$ and $7 \nmid t$ for some positive integer t and $7 \nmid t$ by Lemma 2.8.

So the number m of elements of order 7 in G is: $m = n_7(G) \cdot 6 = 720t$. Since $m \in \mathrm{nse}(G)$, then $m = 1457280$ and $t = 2024$. So we have $2024|N_K(P_7)| = |K|$ by Lemma 2.8. As $|K| \mid 2^4 \cdot 11 \cdot 23$, there must be $n_{23}(K) = 1$. And then the number m of elements of order 23 in G is: $m = 22$. but $22 \notin nse(G)$, a contradiction.

Similar to the case in (1), we can get that $L/K \ncong A_7, A_8$ or $L_3(4)$.

(3) L/K is not a simple K_5-group. Otherwise, $L/K \cong M_{22}$ by Lemma 2.10. So we have $n_{11}(G) = n_{11}(L/K)t = 8064t$, where $11 \nmid t$, and the number m of elements of order 11 in G is: $m = n_{11}(G)10 = 80640t$. Since $m \in \mathrm{nse}(G)$, there must be $m = 1854720$ or 887040 and $t = 23$ or 11, respectively.

Assume $m = 1854720$ and $t = 23$. If $P_{11} \in \mathrm{Syl}_{11}(G)$, there is $23|N_K(P_{11})| = |K|$. As $|K| \mid 23$, then $|K| = 23$. So we have that the number of elements of order 23 in G is 22, but $22 \notin \mathrm{nse}(G)$, a contradiction.

Assume $m = 887040$ and $t = 11$. If $P_{11} \in \mathrm{Syl}_{11}(G)$, then $11|N_K(P_{11})| = |K|$. We have $|K| \mid 23$, which is a contradiction.

So $\pi(L/K) = \{2, 3, 5, 7, 11, 23\}$ and $|L/K| \mid 2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23$. By Lemma 2.6 and [4], $L/K \cong M_{23}$. Thus $|L/K| = |M_{23}| = |G|$. This implies $K = 1$ and $G = L \cong M_{23}$. □

Theorem 3.5. *Let G be a group. Then $G \cong M_{24}$ if and only if the following hold:*

(1) $|G| = |M|$,

(2) $\mathrm{nse}(G) = \mathrm{nse}(M_{24}) = \{1, 43263, 712448, 5100480, 4080384, 20401920, 11658240, 15301440, 12241152, 22256640, 40803840, 34974720, 32643072, 23316480, 21288960\}$.

Proof. The necessity is obvious. We only need to prove the sufficiency.

First, we will prove that G is unsolvable.

If G is solvable, then G contains a $\{3, 5, 7, 11, 23\}$-Hall subgroup. Moreover, the number of $\{3, 5, 7, 11\}$-Hall subgroups in G is $|G \colon N_G(H)| \mid 2^{10}$. By Sylow theorem, $n_{23}(H) = 1$ or 231.

If $n_{23}(H)$=1, then $10 \le m \le 40960$ and $22 \mid m$, but $m \notin \mathrm{nse}(G)$.

If $n_{23}(H)$=231, then $5082 \leq m \leq 5203968$ and $22 \mid m$. Since $m \in \mathrm{nse}(G)$, there must be $m = 712448, 5100480$ or 4080384.

So we have $n_{23}(G) \cdot 22 = m$, that is, $23k+1 = 32384, 231840$ or 185472, but the equation has no solution in N.

Hence, G is unsolvable. Since $p\big|\,|G|$, where $p \in \{5,7,11\}$, G has a normal series:

$$1 \trianglelefteq K \trianglelefteq L \trianglelefteq G\,,$$

such that L/K is a non-Abelian simple group.

(1) L/K is not a simple K_3-group. Otherwise, L/K is isomorphic to: A_5, A_6, $L_2(7)$, $L_2(8)$ or $U_4(2)$ by Lemma 2.7 and [4].

Assume $L/K \cong A_5$. If $P_5 \in \mathrm{Syl}_5(G)$, then $P_5K/K \in \mathrm{Syl}_5(L/K)$. Also $n_5(L/K)t = n_5(G)$ for some positive integer t and $5 \nmid t$ by Lemma 2.8.

By [4], $n_5(L/K) = n_5(A_5) = 6$. Hence $n_5(G) = 6t$.

So the number m of elements of order 5 in G is: $m = n_5(G) \cdot 4 = 24t$. Since $m \in \mathrm{nse}(G)$, there must be $m = 4080384$, 12241152 or 32643072 and $t = 170016$, 510048 or 1360128, respectively.

Whenever $t = 170016$, 510048 or 1360128, we can get that $p\big|\,|K|$, where $p \in \{7,11,23\}$.

If K is solvable, we can get that the number m of elements of order 23 in G is: $m \notin \mathrm{nse}(G)$.

If K is unsolvable, similar to theorem 3.1, we also can get a contradiction.

Similarly, $L/K \ncong A_6, L_2(7), L_2(8)$ or $U_3(3)$.

(2) L/K is not a simple K_4-group. Otherwise, by Lemma 2.4 and [4], $L/K \cong A_7, A_8, M_{11}, M_{12}, L_2(11), L_2(23)$, or $L_3(4)$.

If $L/K \cong A_7$, then $n_7(L/K) = 120$, $n_7(G) = 120t$ and $7 \nmid t$ by Lemma 2.8.

So the number m of elements of order 7 in G is: $m = n_7(G) \cdot 6 = 720t$. Since $m \in \mathrm{nse}(G)$, there must be $m = 11658240$, 34974720 or 23316480 and $t = 16192$, 48676 or 32384, respectively.

If $m = 11658240$ and $t = 16192$, then $16192|N_K(P_7)| = |K|$. As $|K| \mid 2^6 \cdot 3 \cdot 11 \cdot 23$, we obtain $n_{23}(K) = 1$ or 24. And then the number m of elements of order 23 in G is: $m = 22$ or 528. But 22, $528 \notin \mathrm{nse}(G)$, a contradiction.

If $m = 34974720$ and $t = 48676$, then $48676|N_K(P_7)| = |K|$. Now $|K| \mid 2^6{\cdot}3{\cdot}11{\cdot}23$, a contradiction.

If $m = 23316480$ and $t = 32384$, then $32384|N_K(P_7)| = |K|$. Now $|K| \mid 2^6{\cdot}3{\cdot}11{\cdot}23$, a contradiction.

Similarly as above, we also get that $L/K \ncong A_8$, A_9, M_{11}, M_{12}, $L_2(11)$, $L_2(23)$, or $L_3(4)$.

(3) L/K is not a simple K_5-group. Otherwise, $L/K \cong M_{22}$ by Lemma 2.10. So we have $n_{11}(G) = n_{11}(L/K)t = 8064t$, where $11 \nmid t$, and the number m of elements of order 11 in G is: $m = n_{11}(G)10 = 80640t$. Since $m \in \mathrm{nse}(G)$, there must be $m = 22256640$ and $t = 276$. If $P_{11} \in \mathrm{Syl}_{11}(G)$, then $276|N_K(P_{11})| = |K|$. Now $|K| \mid 2^3 \cdot 3 \cdot 23$. Therefore $n_{23}(K) = 1$ or 24. So we have that the number of elements of order 23 in G is 22 or 528. But 22, $528 \notin \mathrm{nse}(G)$, a contradiction.

So $\pi(L/K) = \{2,3,5,7,11,23\}$ and $|L/K| \mid 2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23$. By Lemma 2.6 and [4], $L/K \cong M_{23}$ or M_{24}.

If $L/K \cong M_{23}$, we have $n_{23}(L/K)t = n_{23}(G)$ and $23 \nmid t$ by Lemma 2.8. Now, $n_{23}(L/K) = 40320$. Choose $P_{23} \in \mathrm{Syl}_{23}(G)$. Then $n_{23}(G) = 40320t$, where $23 \nmid t$. Hence the number m of elements of order 23 in G is: $m = n_{23}(G)22 = 887040t$. Since $m \in \mathrm{nse}(G)$, there must be $m = 21288960$ and $t = 24$. As $24|N_K(P_{23})| = |K|$ and $|K| \mid 2^3 \cdot 3$, we obtain $N_K(P_{23}) = 1$ and $|K| = 24$. And $1 = K \cap N_G(P_{23}) \geq K \cap C_G(P_{23})$. So $K \rtimes P_{23}$ is a Frobenius group, therefore $|P_{23}| \mid |\mathrm{Aut}(K)|$, a contradiction.

Hence $|L/K| = |M_{24}| = |G|$. We get $K = 1,\ G = L \cong M_{24}$.

□

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