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GERSTENHABER AND BATALIN-VILKOVISKY ALGEBRAS; ALGEBRAIC, GEOMETRIC, AND PHYSICAL ASPECTS

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Abstract. We shall give a survey of classical examples, together with algebraic methods to deal with those structures: graded algebra, cohomologies, cohomology operations. The corresponding geometric structures will be described(e.g., Lie algebroids), with particular emphasis on supergeometry, odd supersymplectic structures and their classification. Finally, we shall explain how BV-structures appear in Quantum Field Theory, as a version of functional integral quantization.

1. Introduction

The present survey is an expanded version of the three lectures given by the author at Srní, in January 2009. I have tried to keep, as much as possible, the style of a graduate course; it should be accessible with only a minimal knowledge of standard differential geometry, some familiarity with graded algebra, and of course interest for mathematical physics. I do not claim here to prove any original result, but we pretend to give an approach containing the different aspects of this multiform theory; they can already be guessed from the title: Murray Gerstenhaber is well known to be the father of algebraic deformation theory, while Igor Batalin and G. Vilkovisky are famous for their approach of functional integral quantization.Let's give more details now, in order to smoothen the way:

(1) We begin with algebraic aspects, most of Chapter 1 is devoted to them; our presentation will be mainly axiomatic, and then we describe some cohomological tools, with graded algebra techniques and an approach of Hochschild cohomology and H-K-R theorem. The size of the survey doesn't allow to reach more advanced techniques, and we stop where operad theory (highly promising for the future of the subject) begins.

(2) The various geometrical aspects are extensively developed; most of examples shown in Chapter 1 belong to classical differential geometry, we give some links with Poisson geometry and the theory of Lie algebroids. The second Chapter is entirely devoted to supergeometry; we can only give a very short introduction to the famous dialectic odd-even, and then focus to

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symplectic supergeometry; its odd version, so-called "periplectic", turns out to be surprisingly analogous to (non super) symplectic geometry, although made much more rigid by the odd coordinates. A very explicit sample of supergeometric calculations is given in last section.

(3) I have tried to give some physical flavor of Batalin-Vilkovisky quantization in Chapter 3. It is a part of functional integral quantization, in which supergeometry plays a key role, through introduction of odd variables in order to treat symmetries and constraints, in an infinite dimensional context; it is linked with 'ghosts-antighosts'(Faddeev-Popov) and BRST symmetries. We give the fundamental equations, Quantum and Classical Master Equations, in their natural context of graded Lie algebras as studied in Chapter 1.

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2. BV-algebras and G-algebras. Generalities and main examples

We shall present first a purely axiomatic presentation of those algebras

2.1. A few algebraic preliminaries and notations.

We shall deal with graded vector spaces $E^* = \sum_{p\in\mathbb{Z}} E^p$ over a base field $\underline{k}$ of characteristic zero; usually, one has $E^p = 0$ for p lower than some negative bound. On those spaces will be defined algebraic structures of various kind, for which "everything" respects graduation.

The degree of an element $a \in E^*$ is denoted by $|a| = p$.

Shift of graduation: one associates to a graded space E^* another one denoted by $E[1]^*$, where $E[1]^p{=}E^{p+1}$, for $p \in \mathbb{Z}$.

Algebraic differential operators: one defines inductively the order of a differential operator of a graded commutative algebra $\mathcal{A}^*$ into itself; operators of order zero are given by multiplication $\mu_a \colon \mathcal{A}^* \to \mathcal{A}^*$, so $\mu_a(b) = ab$, for some $a \in \mathcal{A}^*$. Then, $\Delta \colon \mathcal{A}^* \to \mathcal{A}^*$ will be an operator of order n, if for any $a \in \mathcal{A}^*$, the operator $[\Delta, \mu_a] - \mu_{\Delta(a)}$ is of order $(n-1)$.Here the bracket denotes graded commutator (each operator has an **order** and a **degree**); this notion of order for algebraic differential operators is classical and due to Grothendieck.

2.2. Definition of Gerstenhaber and Batalin-Vilkovisky algebras.

2.2.1. *Gerstenhaber algebras.* A graded vector space $\mathcal{A}^*$ is a Gerstenhaber algebra if one has:

(1) An associative, graded commutative multiplication:

$$\mathcal{A}^* \times \mathcal{A}^* \xrightarrow{\;\cdot\;} \mathcal{A}^* .$$

For every a, b, one has: $a \cdot b = (-1)^{|a||b|} b \cdot a$.

(2) A graded Lie algebra bracket

$$\mathcal{A}[1]^* \times \mathcal{A}[1]^* \xrightarrow{[\,,\,]} \mathcal{A}[1]^* \,.$$

So one has:

$[b, a] = -(-1)^{(|a|-1)(|b|-1)}[a, b]$ (graded antisymmetry)

$\sum_{(a,b,c)} (-1)^{(|a|-1)(|c|-1)}[a, [b, c]] = 0$ (graded Jacobi identity).

(3) Operations $\cdot$ and $[\,,\,]$ are compatible through a Leibniz relation:

$[a, b \cdot c] = [a, b] \cdot c + (-1)^{(|a|-1)(|b|-1)} b \cdot [a, c]$

Remark. One needs five axioms to express all properties of Gerstenhaber algebras; there is an obvious analogy with the axioms of a Poisson algebra, but difficulties lie in the change of graduation. We shall omit the point when the product is obvious.

2.2.2. *Batalin-Vilkovisky algebras.* A graded vector space $\mathcal{A}^*$ is a Batalin-Vilkovisky algebra (BV-algebra) if:

(1) $\mathcal{A}^*$ is an associative graded commutative algebra.
(2) One has a differential operator $\Delta\colon \mathcal{A}^* \to \mathcal{A}^*$ of order 2 and degree (-1).
(3) Δ^2=0.

Four axioms are needed to define BV-algebras. More explicitly, Δ of order 2 means that for every a, b, c in $\mathcal{A}^*$, one has:

$$\begin{aligned}\Delta(abc) &= \Delta(ab)c + (-1)^a a\Delta(bc) + (-1)^{|b|(|a|+1)} b\Delta(ac) \\ &\quad - \Delta(a)bc - (-1)^{|a|} a\Delta(b)c - (-1)^{|a|+|b|} ab\Delta(c)\,.\end{aligned}$$

2.2.3. *A Batalin-Vilkovisky algebra is a Gerstenhaber algebra.* More precisely, one can associate canonically to any Batalin-Vilkovisky algebra, a structure of Gerstenhaber algebra; the associative multiplication remains the same, and the graded Lie algebra bracket is the obstruction of Δ being a derivation:

(1) $[a, b] = (-1)^{|a|}\big(\Delta(ab) - \Delta(a)b - (-1)^{|a|} a\Delta(b)\big)$. Then, the couple of operations $(\,\cdot\,, [\,,\,])$ define a Gerstenhaber algebra structure on $\mathcal{A}^*$. Moreover Δ is then a graded derivation of $[\,,\,]$:
(2) $\Delta([a, b]) = [\Delta(a), b] + (-1)^{|a|-1}[a, \Delta(b)]$.

Remark. For a Gerstenhaber algebra, equation $[a, b] = (-1)^{|a|}\big(\Delta(ab) - \Delta(a)b - (-1)^{|a|} a\Delta(b)\big)$ can be valid for some Δ, then called **generator** of bracket $[\,,\,]$, which doesn't necessarily satisfy $\Delta^2 = 0$. We shall see later that if Δ^2 derives associative product, then Δ derives the graded Lie bracket.

Exercise. Let $\mathcal{A}^*$ be a Gerstenhaber algebra, such that $\mathcal{A}^p = 0$ for $p < 0$. Show that the axioms imply:

(1) $\mathcal{A}^0$ is an associative commutative algebra.
(2) $\mathcal{A}^1$ is a Lie algebra.
(3) There exists a Lie algebra morphism $\mathcal{A}^1 \to \mathrm{Der}(\mathcal{A}^0)$, the Lie algebra of derivations of $\mathcal{A}^0$. See [18] for more details.

2.3. **Basic examples of Gerstenhaber structures.**

2.3.1. *Schouten bracket.* Let X be a differentiable manifold, let $\tau_X \to X$ be its tangent bundle, and $\Lambda_*\tau_X \to X$ the associated exterior algebra bundle. Let $\Omega_*(X) = \Gamma(X, \Lambda_*\tau_X)$ be its space of sections, in other words the space of antisymmetric contravariant smooth tensor fields. Then: $(\Omega_*(X), \wedge, [\,,\,])$ is a Gerstenhaber algebra for exterior product $\wedge$ of tensor fields, $[\,,\,]$ being the Schouten bracket. It can be defined as the unique graded extension of Lie bracket of vector fields; for precise definition and more details, cf. [26].

2.3.2. *Exterior algebra of a Lie algebra.* Let $\mathfrak{g}$ be a Lie algebra, then $\Lambda_*(\mathfrak{g})$ is naturally a Gerstenhaber algebra, for exterior product and natural prolongation of the bracket of $\mathfrak{g}$. One sees easily that this example can be deduced from the previous one, since $\Lambda_*(\mathfrak{g}) = \mathrm{Inv}_G\, \Omega_*(G)$, invariance being with respect to natural action of G on the space of contravariant tensor fields.

2.3.3. *Algebraization of the previous cases.* Let A be a commutative associative unital algebra, and M an A-module, Set $\mathcal{P}^n(A, M)$ the space of antisymmetric mappings which are multiderivations, i.e. derivations w.r.t. each entry. Let $\mathcal{P}^n(A) = \mathcal{P}^n(A, M)$, and $\mathcal{P}^*(A) = \sum_{p=0}^{+\infty} \mathcal{P}^n(A)$. Then: $\big(\mathcal{P}^*(A), \cdot, [\,,\,]_S\big)$ is a Gerstenhaber algebra for:

(1) The **cup-product** of cochains, defined as follows:

$$(c_1 \cdot c_2)(x_1, x_2, \ldots, x_{m+n}) = (-1)^{mn} \sum_{\sigma\in\Sigma_{m+n}} \epsilon(\sigma)\, c_1(x_{\sigma(1)}, \ldots, x_{\sigma(m)}) c_2(x_{\sigma(m+1)}, \ldots, x_{\sigma(m+n)})\,.$$

(2) The generalized Schouten bracket $[\,,\,]_S$, being defined as the unique graded Lie bracket which prolongates: $[a, b]_S = 0$ if $|a|{=}|b| = 0$;
$[a, b]_S = 0$ if $|a| = 1$, and $|b| = 0$.

For the case $A = C^\infty(X)$, one recovers geometric Schouten bracket as above.

2.3.4. *Geometric generalization: Lie algebroids.* (In fact all previous examples are particular cases of this one).

Definition 1. A **Lie algebroid** on X is a vector bundle $A \to X$ together with a bundle map $a\colon A \to \tau_X$ such that:

(1) $\Gamma(A)$ is equipped with a Lie bracket.
(2) $a : \Gamma(A) \to \Gamma(X, \tau_X) = \mathrm{Vect}(X)$ is a Lie morphism.
(3) one has the following relation:
$$[\xi, f\eta] = f[\xi, \eta] + (L_{a(\xi)}f)\,\eta$$
for every ξ, $\eta \in \Gamma(A)$ and every $f \in C^\infty(X)$.

Besides, for any vector bundle $A \to X$, then $\mathcal{A} = \oplus_{k=0}^{n}\Gamma(\Lambda_k A)$ is an associative graded-commutative algebra for exterior product; one has the following:

Theorem. *A is a Lie algebroid $\iff$ $\mathcal{A}$ is a Gerstenhaber algebra.*

For a proof cf. [37, 22].

Exercise. Let A be a Lie algebroid, let A' be its dualvector bundle, prove the existence on the space of sections $\Gamma^*(A')$ of a differential d, such that $\Gamma^*(A')$ becomes a differential graded algebra (DGA for short). In some sense, the notion of Gerstenhaber algebra is dual to the notion of DGA).

Examples

(1) $A = \tau_X$, with $a = \mathrm{Id}$, one gets the first example above.

(2) $A = \mathfrak{g}$ and $X = \mathit{point}$, one gets the second example above.

(3) A is a tangent bundle to some regular foliation on X, a being the natural inclusion.

(4) Let (P, Λ) be a Poisson manifold. Set $A = \tau^* P$, the cotangent bundle of P, and let $a =: \tau^* P \to \tau P$ be the "musical" [1] morphism associated to Λ. It means that for 1-forms on P, α and β, one has $a(\alpha)(\beta) = \Lambda(\alpha, \beta)$; one could also define this operator as an inner product: $a(\alpha) = i_\alpha \Lambda$. Then following Koszul, one can define the algebroid bracket of forms α and β:

$$[\alpha, \beta] = -d\big(\Lambda(\alpha, \beta)\big) + L_{a(\alpha)} - L_{a(\beta)} \,.$$

If Λ turns out to be of maximal rank, it defines a symplectic structure and a is an isomorphism: one recovers the first case above. See [37] for details. For a good textbook in Poisson geometry, cf. [36].

2.4. **Examples of Batalin-Vilkovisky structures.**
We shall see that most examples from the above section are in fact Batalin-Vilkovisky algebras

(1) We shall begin with a particular case of case 2.3.3 above. Let $A_n = \underline{k}[x_1, x_2, \ldots, x_n]$. One easily determines the derivations:
$\mathrm{Der}(A_n) = \big\{ \sum_{i=1}^n p_i \theta_i | p_i \in A_n,\ \theta_i = \frac{\partial}{\partial x_i} \big\}$. Then,

$$\mathcal{P}^*(A_n) = \Lambda^*_{A_n}(\mathrm{Der}(A_n) = \Lambda^*(\theta_1, \ldots, \theta_n) \otimes \underline{k}[x_1, x_2, \ldots, x_n] \,.$$

Let's now settle some notations, in order to simplify the formulas. For $\Phi \in \mathcal{P}^*(A_n)$, set $\Phi = \Phi^I \theta_I$, where $\theta_I = \theta_{i_1, \ldots, i_m}$ for $I = \{i_1, \ldots, i_m\}$,and $\Phi^I = \Phi^{i_1, \ldots, i_m} = (-1)^{\frac{m(m-1)}{2}} \frac{\Phi(x_{i_1}, \ldots, x_{i_m})}{m!}$, where Einstein convention is used. One can now write down explicit formula for Schouten bracket of $\Phi \in \mathcal{P}^m(A_n)$ and $\Psi \in \mathcal{P}^l(A_n)$:

$$[\Phi, \Psi] = \sum_{k=1}^m \Phi^{i_1, \ldots, i_m} \frac{\partial}{\partial x_{i_k}} (\Psi^{j_1, \ldots, j_l}) \theta_{I_k} \theta_J - (-1)^{(m-1)(l-1)} \times \sum_{k=1}^l \Psi^{j_1, \ldots, j_l} \frac{\partial}{\partial x_{j_k}} (\Phi^{i_1, \ldots, i_m}) \theta_{J_k} \theta_I \,.$$

[1] terminology of musical morphism, because # and ♭ upper and lower indices respectively.

For $I = \{i_1, \dots, i_m\} \in \{1, \dots, n\}$, then $I_k = \{i_1, \dots, i_{k-1}, i_{k+1}, \dots, i_m\}$ and similarly for $J = \{j_1, \dots, j_l\} \in \{1, \dots, n\}$, then $J_k = \{j_1, \dots, j_{k-1}, j_{k+1}, \dots, j_m\}$.

Remark. One has the inner product $i(x)\colon \mathcal{P}^k(A_n) \to \mathcal{P}^{k-1}(A_n)$, defined as $i(x)\Phi = [\Phi, x]$, for $x \in A_n = \mathcal{P}^0(A_n)$; then for x^k, $k = 1, \dots, n$, one has $i(x_k) = \frac{\partial}{\partial \theta_k}$.

Theorem. *Let* $\Delta = -\frac{\partial}{\partial x_i} \frac{\partial}{\partial \theta_i}$ *(using Einstein convention once more). Then* $\Delta^2 = 0$*,* Δ *is a differential operator of order* 2 *and degree* (-1) *which generates Schouten bracket on* $\mathcal{P}^*(A_n)$*.*

Proof. Exercise. $\square$

Remark. There is a curious analogy, and not only because of notations, between Δ and the Laplacian. Things will become clear in the next chapter, in the context of supergeometry.

$\big(\Omega_*(X), \wedge, [\,,\,]\big)$ is a BV-algebra with de Rham codifferential on contravariant tensor fields as BV-operator, provided X is orientable. Take $\omega \in \Omega_*(X)$ a volume form, it defines "musical" isomorphisms (i.e. moving indices up and down), which transfers De Rham differential d to codifferential δ:

$$\begin{tikzcd} \Omega^p \arrow[r, "d"] \arrow[d, "\#"'] & \Omega^{p+1} \arrow[d, "\#"] \\ \Omega_{n-p} \arrow[r, "\delta"] & \Omega_{n-p-1} \end{tikzcd}$$

Operator δ is of order 2 and degree (-1), and $\delta^2 = 0$ is obvious. One checks easily that $(\Omega_*(X), \wedge, \delta)$ is a BV-algebra.

Remark. The operator is non unique since it depends on the choice of volume form. Operator δ can be changed into $\delta^{'} = \delta + i(d\varphi)$, for some function φ.

(3) The same construction works for $\Lambda_*(\mathfrak{g})$, and one gets that $(\Lambda_*(\mathfrak{g}), \wedge, \delta)$ is a BV-algebra, where δ is the differential of the homological Chevalley-Eilenberg complex of the Lie algebra $\mathfrak{g}$ with scalar coefficients:

$$\delta(x_1 \wedge \cdots \wedge x_n) = \sum_{1 \le i < j \le p} (-1)^{i+j} [\hat{x}_i, \hat{x}_j] \wedge x_1 \wedge \cdots \wedge x_i \wedge \cdots \wedge x_i \wedge \cdots \wedge x_p\,.$$

(4) For a Poisson manifold (P, Λ), let $\big(\Omega_*(P), \wedge, [\,,\,]\big)$ the Gerstenhaber structure naturally associated to it. Then operator $d_\Lambda = [i(\Lambda),\ d]$ generates the Gerstenhaber bracket, as it can be checked easily. So one obtains a BV-algebra $(\Omega_*(P), \wedge, d_\Lambda)$.

Moreover, complex $\big(\Omega_*(P), d_\Lambda\big)$ is the Poisson homology of (P, Λ) as defined by J.-L. Brylinski [6]. If the Poisson structure is symplectic, this BV-algebra is isomorphic to (2.4) above; if the Poisson structure is the linear Poisson structure on the dual of some finite dimensional Lie algebra, then one recovers the BV-structure of case (2.4) above.

(5) For the general case of Lie algebroids, the problem has been fully geometrized by Ping Xu [37]. For any algebroid A, he defines the notion of A-connection, straightforwardly generalizing linear connections. So covariant derivative acts:

$$\nabla\colon \Gamma(A)\times\Gamma(E)\to\Gamma(E)$$

satisfying standard axioms of covariant derivative (here Γ denotes the space of sections as usual). Then, he associates to any A-connection on the determinant bundle $\Lambda^n A$, a covariant derivative $D_\Lambda\colon \Gamma(\Lambda^k A)\to\Gamma(\Lambda^{k-1}A)$, which satisfies the following properties:

(a) D_Λ generates the Gerstenhaber bracket on $\mathcal{A}=\oplus_{k=0}^n\Gamma(\Lambda_k A)$

(b) $D_\Lambda^2=-i(R)$, where R denotes the curvature of ∇; it belongs to $\Gamma(\Lambda^2A^*\otimes\mathrm{End}(\Lambda^n A))$.

So, one obtains finally:

(a) An isomorphism between the set of coboundaries for Gerstenhaber structure on $\mathcal{A}=\oplus_{k=0}^n\Gamma(\Lambda_k A)$, and the set of A-connections on the determinant bundle $\Lambda^n A$.

(b) An isomorphism between the set of BV-structures associated to the Gerstenhaber structure on $\mathcal{A}=\oplus_{k=0}^n\Gamma(\Lambda_k A)$ and the set of **flat** A-connections on the determinant bundle $\Lambda^n A$.

In the C^∞ context, there is no obstruction to the existence of connections[2], so any Gerstenhaber algebra associated to an algebroid admits a coboundary; as long as determinant bundle $\Lambda^n A$ is trivial (orientability!), it admits flat connections as well, so the corresponding algebroid $(\mathcal{A}=\oplus_{k=0}^n\Gamma(\Lambda_k A),\wedge,D_\Lambda)$ is a BV-algebra.

2.5. **Algebraic computations through Graded Lie Algebras (GLA). Relations with Hochschild cohomology and Chevalley-Eilenberg cohomology.**

2.5.1. We shall make use of algebraic deformation methods initiated by Gerstenhaber in the early sixties [14, 15], and extensively developed by Nijenhuis and Richardson (cf. for example [26]). Our approach of vanishing square formalism and associated cohomology has been borrowed from Lecomte and De Wilde (cf. [12]).

Let L^* be a GLA, with $L^k=0$ when $k\le k_0$. An element with vanishing square is a $c\in L^1$, such that $[c,c]=0$. If one sets $\partial_c(x)=[c,x]$, then $(L^*,\ \partial_c)$ is a cohomological complex.

Exercise. Check that the cohomology space of this complex is also a GLA, for the induced bracket. This GLA will be denoted as $H_c(L^*)$.

[2] unlike the complex analytic case, cf. the famous Atiyah class.

2.5.2. *Deformation theory through* GLA. Let c be a square-vanishing object, a deformation of c will be an element $c+\gamma$, also square-vanishing. One easily deduces from equation $[c+\gamma, c+\gamma]=0$, the Maurer-Cartan equation:

$$\partial_c(\gamma) + \frac{[\gamma,\gamma]}{2} = 0\,. \tag{1}$$

[*Historical remark*: the analogy with vanishing curvature equation for connexion is obvious, this terminology was first used in algebraic deformation theory by Kontsevich [21]].

One then obtains inductive classification of deformations (infinitesimal, order 2, order k, formal...) through $H^1_c(L^*)$, and obstructions, using square map $\mathrm{Sq}\colon H^1_c(L^*) \to H^2_c(L^*)$.

2.5.3. *Fundamental examples.*

(1) Let E be a vector space, define a graded vector space by $\mathbb{M}^*(E) = \oplus_{p=-1}^{p=\infty}\mathbb{M}^p(E)$, where $\mathbb{M}^p(E) = C^{p+1}(E,E)$, the space of $(p+1)$-linear mappings from E into E. For $c_a \in \mathbb{M}^a(E)$, $c_b \in \mathbb{M}^b(E)$ define $i(c_a)c_b \in \mathbb{M}^{a+b}(E)$ as:

$$\begin{aligned} &i(c_a)c_b(x_0,\dots,x_{a+b}) \\ &\qquad = \sum_{k=0}^{k=b}(-1)^{ak}c_b(x_0,\dots,x_{k-1},c_a(x_k,\dots,x_{k+a}),x_{k+a+1},\dots,x_{a+b})\,. \end{aligned}$$

Then $[c_a,c_b] = i(c_a)c_b - (-1)^{ab}i(c_b)c_a$ defines a Graded Lie Algebra (GLA) bracket on $\mathbb{M}^*(E)$ (it was already implicit in the work of Gerstenhaber [14]); then one has for $c \in \mathbb{M}^1(E)$:

$$[c,c] = 0 \iff c \quad \text{is associative}\,.$$

Cohomology space $H_c(\mathbb{M}^*(E))$ is then Hochschild cohomology of associative algebra structure defined by c on E.

(2) Let E be a vector space, define a graded vector space by $\mathbb{A}^*(E) = \oplus_{p=-1}^{p=\infty}\mathbb{A}^p(E)$, where $\mathbb{A}^p(E) = Alt^{p+1}(E,E)$, the space of completely antisymmetric $(p+1)$ mappings from E into E. If E is finite dimensional, one has an identification between $\mathbb{A}^p(E)$ and $\Lambda^{p+1}(E')\otimes E$. The GLA bracket on $\mathbb{A}^*(E)$ is obtained from the previous one by antisymmetrization; n terms of elements in $\Lambda^{p+1}(E')\otimes E$, one has:

$$[\alpha\otimes X, \beta\otimes Y] = \alpha\wedge i(X)\beta\otimes Y - (-1)^{ab}\beta\wedge\ i(Y)\alpha\otimes X$$

where $|\alpha| = a$ and $|\beta| = b$. Then one has for $c \in \mathbb{A}^1(E)$:

$$[c,c] = 0 \quad \iff c \quad \text{satisfies Jacobi identity}\,.$$

Cohomology space $H_c(\mathbb{A}^*(E))$ is then Lie algebra cohomology (Chevalley--Eilenberg) for the adjoint representation of the Lie algebra structure on E defined by c.

Remark. We shall see in next chapter a supergeometric interpretation of this GLA. One has $\mathbb{A}^*(E) = \mathrm{Der}(\Lambda^* E)$, so $\mathbb{A}^*(E) = \mathrm{Vect}(0|n)$ if $n = \mathrm{Dim}\, E$. If E is a graded vector space for some intrinsic graduation, then $\mathbb{M}^*(E)$ and $\mathbb{A}^*(E)$ become bigraded Lie algebras, and indices with respect to the intrinsic graduation will be written below.

2.5.4. *Applications to Gerstenhaber and BV-structures.* We give in this subsection some applications of GLA computations to classification or generalization of Gerstenhaber and BV structures, under the form of small exercises; computations are sometimes lengthy, but straightforward.

1. Let E be the graded vector space underlying a Gerstenhaber structure, let μ and c its associative multiplication and graded Lie bracket respectively; then $\mu \in \mathbb{M}^1(E)_0$, and $c \in \mathbb{A}^1(E)_{-1} \subset \mathbb{M}^1(E)_{-1}$. Then if operator $\Delta\colon E \in E$ defines a BV-structure associated to the Gerstenhaber structure given by (μ, c), one has $\Delta \in \mathbb{M}^0(E)_{-1}$, satisfying $[\Delta, \mu] = c$ (check it!). So, in the graded Hochschild cohomology for the associative algebra structure on E defined by μ, c is the coboundary of Δ.

2. Deduce from Leibniz property of bracket c, that $[c, \mu] = 0$ in $\mathbb{M}^2(E)_{-1}$, so c is a 2-cocycle in graded Hochschild cohomology (what about the converse?).

3. Deduce from above a cohomological interpretation of existence and classification of BV-structures associated to a given Gerstenhaber structure.

4. Let Δ be a coboundary for c, so $[\Delta, \mu] = c$. Prove that $[\Delta^2, \mu] = [\Delta, c]$ (up to sign). So Δ is a derivation of c if and only if Δ^2 is a derivation of μ, and in particular if $\Delta^2 = 0$!

5. Suppose now that $c = [\Delta, \mu]$ without assuming that c is a Lie algebra structure; compute $Sq(c) = [c, c] \in \mathbb{A}^2(E)_{-2}$ and prove: $Sq(c) = 0 \iff \Delta^2$ of order 2. For more details about this kind of computations, cf. the work of Penkava and Schwarz [28], or F. Akman [1].

2.5.5. *More results about Hochschild cochains.*
1. A new Gerstenhaber algebra. Let E be a vector space (not necessarily graded), and $\mu \in \mathbb{M}^1(E)$ an associative multiplication. Denote by A the associative algebra defined by multiplication μ on E, and let $C^{p+1}(A, A) = \mathbb{M}^p(E)$ be the space of Hochschild cochains. Then Gerstenhaber bracket defines a GLA bracket:

(1)
$$C^*(A, A)[1] \times C^*(A, A)[1] \xrightarrow{[\ ,\ ]} C^*(A, A)[1]\,.$$

One has moreover the naturally defined cup-product[3]

(2)
$$C^*(A, A) \times C^*(A, A) \xrightarrow{\cup} C^*(A, A)\,.$$

[3]the name cup-product, standard in algebraic topology is naturally extended to this context

For $c \in C^k(A,A)$ and $c' \in C^l(A,A)$, one has $c \cup c' \in C^{k+l}(A,A)$ defined as follows:

$$(c \cup c')(x_1, \dots, x_{k+l}) = (-1)^{kl} \mu\big(c(x_1, \dots, x_k), c'(x_{k+1}, \dots, x_{k+l})\big) .$$

The two operations defined above doesn't give a structure of Gerstenhaber algebra on $C^*(A,A)$, since cup-product is not graded-commutative, for example, but everything works well on cohomological level. One has:

Theorem (Gerstenhaber 1963 [14])**.** *For any associative algebra, Hochschild cohomology space $HH^*(A,A)$ admits a Gerstenhaber algebra structure.*

2. H-K-R Theorem. This example can be considered as a generalization of case $(\Omega_*(X), \wedge, [\,,\,])$ above; let's now mention the:

Hochschild-Kostant-Rosenberg Theorem [17]**:**

If A is a smooth commutative $\underline{k}$-algebra, then one has an isomorphism:

$$\Lambda^*_A\big(\operatorname{Der}_{\underline{k}}(A)\big) \longrightarrow HH^*(A,A) .$$

So, as a particular example of smooth algebra, if $A = C^\infty(X)$, then $\operatorname{Der}_{\underline{k}}(A) = \operatorname{Vect}(X)$, and $\Lambda^*_A(\operatorname{Der}_{\underline{k}}(A)) = \Omega_*(X) = HH^*\big(C^\infty(X), C^\infty(X)\big)$, and we obtain $\big(\Omega_*(X), \wedge, [\,,\,]\big)$ Explicitly, one associates to each antisymmetric contravariant tensor $T \in \Omega_k(X)$ the Hochschild cochain $c_T \in C^p\big(C^\infty(X), C^\infty(X)\big)$, given by:

$$c_T(f_1, \dots, f_p) = \langle T, df_1 \wedge \dots \wedge df_p \rangle ,$$

where $\langle\ \rangle$ denotes evaluation of contravariant tensors on differential forms. One can then check that cup-product of Hochschild cochains give exterior product of tensors in cohomology. There exists also a homological version of this theorem, one can construct explicitly a map:

$$HH_*(A,A) \longrightarrow \Omega^*_{\underline{k}}(A)$$

which is an isomorphism, the latter space being the space of Kähler differentials on A which is exactly the space of differential forms on X when $A = C^\infty(X)$. So Hochschild cohomology (resp. homology) can be considered as a natural generalization of the space of contravariant tensors (resp. differential forms); noncommutative geometry broadly generalizes that point of view (cf. [11]).

3. Note about the proof of H-K-R Theorem: cf. also [12, 7], or [36, p. 417 sqq.]. This paragraph presents a sketch of a proof of H-K-R Theorem for C^∞ manifolds, using standard though apparently sophisticated tools of cohomology; it is intended for devotees of homological algebra, others can skip it without inconvenients.

(1) Consider first the case when $A = \underline{k}[x_1, \dots, x_n]$. From the very definition of Hochschild cohomology, one has:

$$HH^*(A,A) = \operatorname{Ext}_{A \otimes A^{\mathrm{op}}}(A,A)$$

(cf. [24, p.283]), where A^{op} denotes the opposite algebra to A; since A is commutative, one has $A^{\mathrm{op}} = A$, whence $A \otimes A = A[y_1, \dots, y_n]$, and finally one has to compute $HH^*(A,A) = \operatorname{Ext}_{A[y_1,\dots,y_n]}(A,A)$. This space can now be determined using Koszul resolution of a polynomial algebra (cf.

again [24, p.204]); one has to take an exterior algebra over n generators $(\partial_1, \ldots, \partial_n)$ and so one can easily conclude that:

$$HH^*(A, A) = A \otimes_{\underline{k}} \Lambda^*(\partial_1, \ldots, \partial_n) = \Lambda^*_A\big(\operatorname{Der}_{\underline{k}}(A)\big)\,,$$

as required.

(2) For the case of an arbitrary C^∞ manifold X, we shall use basic techniques of sheaf theory. Let $\underline{\mathcal{O}}$ be the structural sheaf of X, its sections on an open subset U are simply the smooth functions on U; let $\underline{\mathcal{C}}^*(\underline{\mathcal{O}}, \underline{\mathcal{O}})$ be the sheaf of local Hochschild cochains on $\underline{\mathcal{O}}$, Hochschild differential gives a complex of sheaves, denoted by $(\underline{\mathcal{C}^*}, d)$ for short. We shall consider hypercohomology of X with coefficients in this complex, denoted by $\mathbb{H}^*\big(X, (\underline{\mathcal{C}^*}, d)\big)$. This hypercohomology is obtained through a bicomplex, which induces two spectral sequences (see [9] for definitions and appropriate techniques for hypercohomology); the first one gives:

$$E_1^{p,q} = H^q\big(X,\, (\underline{\mathcal{C}}^p, d)\big)\,.$$

It is now easy to get convinced that $(\underline{\mathcal{C}}^*, d)$ is a complex of fine sheaves: Peetre's theorem (cf. [27]) shows that any local multilinear map from $\underline{\mathcal{O}}$ into itself is locally given by a multidifferential operator, so sheaf $\underline{\mathcal{C}}^*$ is locally free over $\underline{\mathcal{O}}$, hence fine; so cohomology vanishes, except in degree zero and one has $E_1^{p,q} = 0$, for $q \neq 0$, and $E_1^{p,0} = \mathcal{C}^p(X)$, the space of global sections of $\underline{\mathcal{C}}^p$, in other words the space of Hochschild p-cochains on $C^\infty(X)$. So this first spectral sequence degenerates from E_2 and one has $E_2^{p,0} = HH^p\big(C^\infty(X), C^\infty(X)\big)$. So hypercohomology is exactly the expected Hochschild cohomology.

Now, degeneracy of the second spectral sequence will give the result; we must compute cohomology beginning with the differential d of the complex of sheafified Hochschild cochains, it gives the cohomology sheaves $\underline{\mathcal{H}}^*$, and the second spectral sequence satisfies:

$$E_2^{p,q} = H^p(X, \underline{\mathcal{H}}^q)\,.$$

The delicate part is now to identify those cohomology sheaves $\underline{\mathcal{H}}^*$; going to inductive limits on charts around some point $x \in X$ allows identification between fiber $\underline{\mathcal{H}}^*_x$ and the space of Hochschild cohomology $HH^*(\mathcal{O}_x, \mathcal{O}_x)$, where $\mathcal{O}_x$ is the fiber of $\underline{\mathcal{O}}$ in x, i.e. the ring of germs of smooth functions at x.

(3) We shall compute $HH^*(\mathcal{O}_x, \mathcal{O}_x)$ using change of rings; the choice of a local chart in x induces a morphism of rings $i_x\colon A \to \mathcal{O}_x$, and we use now the theorem of change of rings, following Cartan and Eilenberg [9, p. 172, Prop. 5.1]. It gives:

$$\begin{aligned} &HH^*(\mathcal{O}_x, \mathcal{O}_x) \\ &\qquad = \operatorname{Ext}_{\mathcal{O}_x \otimes \mathcal{O}_x^{\mathrm{op}}}(\mathcal{O}_x, \mathcal{O}_x) \xrightarrow{\;i_x^*\;} \operatorname{Ext}_{A \otimes A^{\mathrm{op}}}(A, \mathcal{O}_x) = HH^*(A, \mathcal{O}_x) \end{aligned}$$

is an isomorphism. In order to determine $HH^*(A, \mathcal{O}_x)$, we shall use flatness; it follows from a theorem of Tougeron [34, chap. VI, p. 118, Cor. 1.3] that $\mathcal{O}_x$ is flat on the ring of germs of analytic functions in x, since the latter is flat on the polynomial ring A (classical result of commutative algebra), one has $\mathcal{O}_x$ flat on A; then tensor product commutes with cohomology, so one gets: $HH^*(A, \mathcal{O}_x) = HH^*(A, A) \otimes_A \mathcal{O}_x$. One obtains finally:

$$\begin{aligned} HH^*(\mathcal{O}_x, \mathcal{O}_x) &= HH^*(A, A) \otimes_A \mathcal{O}_x = \big(A \otimes_{\underline{k}} \Lambda^*(\partial_1, \ldots, \partial_n)\big) \otimes_A \mathcal{O}_x \\ &= \mathcal{O}_x \otimes_{\underline{k}} \Lambda^*(\partial_1, \ldots, \partial_n) \end{aligned}$$

It is now easy to identify the latter space with $\Omega_{*\,x}$, the space of germs in x of contravariant antisymmetric tensor fields. Summarizing, we have identified the cohomology sheaves $\underline{\mathcal{H}}^*$ with the sheaves $\underline{\Omega}_*$ of contravariant antisymmetric tensor fields.The latter being a fine sheaf, its cohomology vanishes except in degree zero, so one gets for the second spectral sequence: $E_2^{p,q} = 0$ for $p \neq 0$, and $E_2^{0,q} = \Omega_q(X)$ the space of contravariant antisymmetric tensor fields on X. Finally, this spectral sequence and one gets H-K-R Theorem for rings of smooth functions.

Final remarks. In fact, the right tool to deal with the above defined algebraic structures on Hochschild cochains $C^*(A, A)$ are the "structures up to homotopy". It turns out that the space of Hochschild cochains $C^*(A, A)$ admits a structure of **Gerstenhaber algebra up to homotopy**, obtained through various constructions of operators, called "braces" (cf. [1] and [35]). The combinatorics of those braces can be very complicated, the appropriate formalism being the theory of **operads** (cf. [25]). As a recent result, let's mention the work of B. Vallette and collaborators [8], in which the right operad for BV-structures is constructed, and so allows to handle with BV-structures up to homotopy.

3. BV-structures and supergeometry

3.1. **A short sketch of supergeometry.**

We shall only give here the few definitions really unavoidable in order to make this chapter reasonably self-contained; the reader is referred to [13] for a nice and rigorous introduction to supergeometry.

3.1.1. *Superspace.* Basically a superspace will be a vector space equipped with a $\mathbb{Z}/2\mathbb{Z}$-graded commutative algebra of functions, called superfunctions; this point of view might seem strange at first glance, but it is nothing but a (very)particular case of basic principle of considering geometry as given by a ring of functions on the space! So, we shall consider superspace $\mathbb{R}^{p|q}$, as a space with a ring of superfunctions $C^\infty(\mathbb{R}^{p|q}) = C^\infty(\mathbb{R}^p) \otimes \Lambda^*(\mathbb{R}^q)$; set the generators of exterior algebra as odd, it will uniquely determine the parity.

3.1.2. *Superdomain.* Analogously, we shall consider superdomain $\mathcal{U} \subset \mathbb{R}^{p|q}$, defined by $C^\infty(\mathcal{U}) = C^\infty(U) \otimes \Lambda^*(\mathbb{R}^q)$, where $U \subset \mathbb{R}^p$ is an open set. Dimension of a superdomain will be a couple of integers, here Dim $\mathcal{U} = p|q$. One has an algebra of superfunctions on a superdomain, which is associative and graded commutative.

3.1.3. *Supermanifold.* A supermanifold will be defined as a ringed space, in Grothendieck's sense [16], locally isomorphic to some superdomain; more precisely, it will be a differentiable manifold equipped with a sheaf of superfunctions, as follows: one has $\mathcal{X} = (X, \underline{\mathcal{O}}_{\mathcal{X}})$ a ringed space whose underlying space is a differentiable manifold X of dimension n, and for each open set $U \subset X$, one has $\underline{\mathcal{O}}_{\mathcal{X}}(U) = C^\infty(U) \otimes \Lambda^*(\mathbb{R}^m)$. We set Dim $\mathcal{X} = n|m$. A typical example of a supermanifold is the following: consider a vector bundle of rank m on a manifold X, say $E \to X$; then take the bundle in exterior algebras,and its sheaf of sections, then with $\underline{\mathcal{O}}_{\mathcal{X}} = \underline{\Gamma}(\Lambda^* E)$. Up to some minor details, one can prove that all supermanifolds in the C^∞ category are of this type (Batchelor Theorem, cf. [13]).

3.1.4. *Change of parity.* Functions on a supermanifold form naturally a $\mathbb{Z}/2\mathbb{Z}$-graded commutative algebra; explicitly $\underline{\mathcal{O}}_{\mathcal{X}} = \underline{\mathcal{O}}_{\mathcal{X}}{}^{\text{even}} \oplus \underline{\mathcal{O}}_{\mathcal{X}}{}^{\text{odd}}$, where $\underline{\mathcal{O}}_{\mathcal{X}}(U)^{\text{even}} = C^\infty(U) \otimes \Lambda^{\text{even}}(\mathbb{R}^m)$ (resp. for odd). One defines then the functor of change of parity, denoted by Π: for a superspace E, one has $(\Pi E)^{\text{odd}} = E^{\text{even}}$ and $(\Pi E)^{\text{even}} = E^{\text{odd}}$; one checks immediately that Π is functorial. One can construct very useful and interesting supermanifolds using this functor: let X be any differentiable manifold, then ΠTX (resp. ΠT^*X) is the supermanifold obtained by making the fibers of tangent (resp. cotangent) bundle odd. In terms of the above definition of supermanifolds, one has $\Pi TX = (X, \underline{\Omega}^*)$ (sheaf of differential forms), and $\Pi T^*X = (X, \underline{\Omega}_*)$ (sheaf of antisymmetric contravariant tensor fields).

3.1.5. *About supergroups and Lie superalgebras.* Basic notions of differential geometry can be more or less extended to the super case, with some specific difficulties with volume form and integration, some of them will be discussed below. In particular, one has frame bundles, and G-structures, for various supergroups $G \subset GL(n|m)$. The latter is simply the group of even graded linear automorphisms of superspace $\mathbb{R}^{n|m}$, which can be described through block matrices and graded commutator(some examples will be given below); the corresponding Lie superalgebra $\mathfrak{gl}(n|m)$ is described similarly. The general problem of constructing a supergroup associated with a general Lie superalgebra is rather delicate and we shall not use it here (it uses the notion of Harish-Chandra pair, or one can consider the more sophisticated notion of "functor of points", like in algebraic group theory, cf. once more [16]).

3.2. **Supersymplectic geometry.**

The notion of supersymplectic form on a supermanifold $\mathcal{X}$ can be naturally defined; a 2 form $\omega \in \Omega^2(\mathcal{X})$ will be called supersymplectic, if it is closed and non degenerate. For any $x \in X$, the underlying manifold, one has a superantisymmetric mapping

$$\omega(x) \colon T_x\mathcal{X} \times T_x\mathcal{X} \to \mathbb{R}^{1|1}$$

Superantisymmetry reads: $\omega(x)(a,b) = -(-1)^{|a||b|}\omega(x)(b,a)$, and in terms of parity, one has: $|\omega(x)(a,b)| = |a| + |b| + |\omega|$ (mod. 2), where $|\omega|$ denotes the parity of the form ω itself. So one can distinguish to geometrically very different cases, according to the parity of ω, keeping in mind the splitting $T_x\mathcal{X} = T_x\mathcal{X}^{\text{even}} \oplus T_x\mathcal{X}^{\text{odd}}$, and $T_x\mathcal{X}^{\text{even}} = T_xX$:

(1) ω is even: one has an orthosymplectic form, which restricts to a symplectic form on $T_x\mathcal{X}^{\text{even}}$, and to a symmetric non degenerate form on $T_x\mathcal{X}^{\text{odd}}$. So the underlying manifold carries a symplectic structure; we shall not consider this case here (cf. [29]).

(2) ω is odd: it defines an isomorphism between $T_x\mathcal{X}^{\text{even}} = T_x\mathcal{X}^{\text{odd}}$, so this case can occur only if $n = m$. We shall call this form a **periplectic** form, following Leites and Poletaeva [29].

One has a canonical periplectic form on $\mathbb{R}^{n|n}$, we shall denote by $P(n) \subset GL(n|n)$ its group of invariance (this group enters the famous classification of finite dimensional supergroups, due to V. Kac).

Super Darboux theorem.
Let $\mathcal{X}$ be a supermanifold with a periplectic form $\omega \in \Omega^2{}_{\text{odd}}(\mathcal{X})$, then there exists at every point a chart $\mathcal{U} \subset \mathcal{X}$ with coordinates $(x_1, \ldots, x_n, \theta_1, \ldots, \theta_n)$, such that

$$\omega_{|\mathcal{U}} = \sum_{i=1}^{n} dx_i \wedge d\theta_i \,.$$

Then the usual formalism of symplectic geometry extends straightforwardly to the periplectic case, one has the construction of Hamiltonian (called Leitesian here) and odd Poisson bracket, known as Buttin bracket[4]. Explicitly, one has for f, $g \in \underline{\mathcal{O}}_{\mathcal{X}}$:

$$\{f, g\} = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial \theta_i} + (-1)^{|f|}\frac{\partial f}{\partial \theta_i}\frac{\partial g}{\partial x_i}\right).$$

Example. Consider ΠT^*X with canonical Liouville form made odd is a periplectic manifold. Then, through identification $C^\infty(\Pi T^*X) = \Omega_*(X)$, Buttin bracket on superfunctions and Schouten bracket on contravariant antisymmetric tensor fields, as an immediate calculation shows. So if Dim $X = n$, then Dim $\Pi T^*X = (n|n)$. One can easily generalize this construction: let $\mathcal{X}$ be a supermanifold, one easily constructs a super-Liouville form on cotangent after change of parity on the fiber, and $\Pi T^*\mathcal{X}$ is a periplectic manifold; if Dim $\mathcal{X} = (n|m)$ then Dim $\Pi T^*\mathcal{X} = (n+m|n+m)$ (cf. [31, 20]) for details. In fact this example will turn out to be the only one, up to isomorphism!

Theorem (Albert Schwarz [31]). *Let $\mathcal{X} = (X, \underline{\mathcal{O}}_{\mathcal{X}})$ be a $(n|n)$-dimensional supermanifold with a periplectic form. Then $\mathcal{X}$ is equivalent, through a diffeomorphism exchanging periplectic forms*[5]*, to ΠT^*X with periplectic form defined as above.*

[4]Claudette Buttin (1936-1972)
[5]one should call it "periplectomorphism" !

3.3. **The Berezinian.** We shall now deal with determinants and volume forms; the standard definition of determinant cannot be directly extended to the super-geometric case, since the exterior algebra over odd space is infinite dimensional. The **supertrace** is naturally defined; any linear endomorphism of a superspace $E = E^{\text{even}} \oplus E^{\text{odd}}$ can be decomposed as a 4 blocks matrix as follows:

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} .$$

The **supertrace** is then naturally defined

$$\mathrm{sTr}(M) = \mathrm{Tr}(A) - \mathrm{Tr}(D) .$$

It has the good property one should expect from a trace: it is a supersymmetric invariant. One want to define a determinant, in order to keep the well known relation between trace and determinant :$\mathrm{Det}(\exp(M)) = \exp(\mathrm{Tr}(M))$, for M a (classical) matrix. We shall call the superdeterminant **Berezinian**, from its inventor, and denote it as Ber. So we require

$$\mathrm{Ber}\left(\exp(M)\right) = \exp\left(\mathrm{sTr}(M)\right)$$

for every endomorphism of a superspace. The solution is given by the following explicit formula:

$$\mathrm{Ber}(M) = \mathrm{Det}(A - BD^{-1}C)\,\mathrm{Det}(D)^{-1}$$

(cf. [13] for details) for a matrix $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ as above. One has the expected property, that mapping $\mathrm{Ber}\colon \mathrm{GL}(n|m) \to \mathrm{GL}(1|0)$ is a group homomorphism, whose kernel is $\mathrm{SL}(n|m)$. One can now define a subgroup of periplectic group $P(n)$ as $\mathrm{SP}(n) = P(n) \cap \mathrm{SL}(n|n)$, or equivalently

$$\mathrm{SP}(n) = \{M \in P(n) |\, \mathrm{Ber}(M) = 1\} .$$

Then a direct computation shows that if

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in P(n) .$$

Then

$$\mathrm{Ber}(M) = \mathrm{Det}(A)^2 .$$

This formula will play a crucial role in the next sections.

3.4. **The Berezin integral.** We need an integration of functions on supermanifolds, or 'superfunctions', which preserves the fundamental principle that integration of a differential on a closed cycle gives zero, as well as the integral of a Lie derivative: $\int_X (L_\xi f) d\mu = 0$.

In the purely odd case, one has: $\int_{\mathbb{R}^{0|1}} (a + b\theta)\mathcal{D}\theta = b$, since $a = \frac{\partial}{\partial\theta}(a\theta)$; and for the same reason

$$\int_{\mathbb{R}^{0|n}} \Sigma\ c_I \theta_I \mathcal{D}\theta_1, \dots, \theta_n = c_{1,\dots,n}$$

(summation on $I \in \{1, \dots, n\}$)

On supermanifolds $\Pi T^* X$, one has a **canonical Berezin measure** $\mathcal{D}(x,\theta)$. On a coordinate chart $U \subset X$, one has $\mathcal{D}(x,\theta)_{|\Pi T^* U} = \prod_{i=1}^n dx_i \prod_{i=1}^n \mathcal{D}\theta_i$.

Warning: This is not a supervolume form (it doesn't exist on periplectic manifolds, for the reason mentioned at the beginning of previous section). If one considers homothetic transformation on coordinates $x_i \to \lambda x_j$, and $\theta_i \to \lambda\theta_j$, then $dx_i \to \lambda dx_j$ but $\mathcal{D}\theta_i \to \lambda^{-1}\mathcal{D}\theta_j$ (and this is the reason why this Berezin measure is canonical).

Now, on any supermanifold $\mathcal{X}$, one has the **sheaf of densities** $\mathrm{Ber}(\Omega^1_{\mathcal{X}})$, defined as linear forms on the space of (super)functions. The sheaf of half-densities is then a tensor square root of the sheaf of densities. One then deduces the notion of **integral forms**, which can be integrated on submanifolds; it differs from differential forms (cf. [13, Vol1, p 84] for details). One has moreover a formula of change of variables for integrals on superdomains ...:

$$\int_{\Phi(\mathcal{U})} f(y,\psi)\mathcal{D}(y,\psi) = \int_{\mathcal{U}} f(\Phi(x,\theta))\,|\mathrm{Ber}(T\Phi_{x,\theta})|\,\mathcal{D}(x,\theta)\,.$$

So, this formula is formally the same as the classical one, the Berezinian replacing usual determinant. One characterizes integral forms through the following:

Theorem (Khudaverdian, cf. [19])**.** *Integral forms on a supermanifold $\mathcal{X}$ can be identified with half densities on the periplectic supermanifold $\Pi T^*\mathcal{X}$.*

Sketch of proof: Let Φ be an odd symplectomorphism and set:

$$T\Phi = \begin{pmatrix} T\Phi_{1,1} & * \\ * & * \end{pmatrix}.$$

So, using 3.3, one sees that $\mathrm{Det}(T\Phi_{1,1}) = \sqrt{\mathrm{Ber}}(T\Phi)$; if one now considers integral forms on $\mathcal{X}$, such as $\sigma = s(x,\theta)[dx_1,\dots,dx_n]$, then one deduces:

$$\Phi^*(\sigma) = \sigma\,\mathrm{Det}(T\Phi_{1,1}) = \sigma\sqrt{\mathrm{Ber}}(T\Phi)\,.$$

So, according to formula 3.4 above, σ transforms as a half-density, as if it were a square root of integral forms on $\Pi T^*\mathcal{X}$; so, it can be written as

$$\sigma = s(x,\theta)\sqrt{\mathcal{D}}(x,\theta)$$

(for full details, cf. [20]).

3.4.1. *BV operator in supergeometry.* One can now find a naturally defined Gerstenhaber algebra in supergeometrical context. Following the scheme: $\mathcal{X}$ supermanifold $\to \Pi T^*\mathcal{X}$ periplectic supermanifold $\to C^\infty(\Pi T^*\mathcal{X}) = \Omega_*\mathcal{X}$. The latter is naturally a Gerstenhaber algebra, just the superization of example 2.3.1 above: associative product is simply the supercommutative product of functions, and graded Lie algebra bracket being the Buttin bracket 3.2.

We shall now construct a BV operator which is a generator of this bracket, but non canonically .If : $\omega_{|U} = \sum_{i=1}^n dx_i \wedge d\theta_i$ is the periplectic form, then: $\Delta_{|U} = \sum_{i=1}^n \frac{\partial}{\partial x_i}\frac{\partial}{\partial \theta_i}$ is a BV operator which generates the Gerstenhaber algebra (Warning: here, x_i denotes coordinates on supermanifold $\mathcal{X}$, were they even or odd). But this BV operator, acting on superfunctions, also called for obvious reasons **Superlaplacian**, depends on the choice of a system of coordinates. It is associated to De Rham differential, using so-called "odd Fourier transform" (cf. once more [31]), and has the following nice properties:

(1) Δ acts canonically on half densities: $\sigma = s(x,\theta)\sqrt{\mathcal{D}(x,\theta)}$ gives $\Delta(\sigma) = \sum_{i=1}^{n} \frac{\partial^2\sigma}{\partial x_i \partial\theta_i}\sqrt{\mathcal{D}(x,\theta)}$, this transformation being covariant under odd symplectomorphism.

(2) If the integral form is changed by multiplication by a factor ρ, then Δ is changed into $\Delta_\rho = \Delta + \frac{1}{2}\{\mathrm{Log}(\rho), \cdot\}$ (find a cohomological interpretation).

3.5. **An integration formula in supergeometry.**
This section follows an example given in the article of A. Losev [32]. For physical applications, it happens rather often that really interesting relations, such as the presence of a group of symmetries, are valid only on the space of solutions of the equation; or, in theoretical physicist's language, valid only "on shell". In most case this space of solutions is not simple to handle, many technicalities were introduced to circumvent those difficulties; we shall show here how supergeometry can be used for this purpose, through some extension of the classical notion of Lagrange multipliers. Let X be the space of fields and $f\colon X \to \mathbb{R}$ the structure equation, and $\{f^{-1}(0)\}$ the space of solutions. As an example of computations "on shell" one needs to understand $\int_{\{f^{-1}(0)\}} \omega$, for some differential form ω. Using Poincaré duality, one can write

$$\int_{\{f^{-1}(0)\}} \omega = \int_X \delta_f \wedge \omega\,,$$

where δ_f is a current in De Rham's sense, whose cohomology class $[\delta_f] \in H^1(X)$ is Poincaré dual of $[\{f^{-1}(0)\}] \in H_{n-1}(X)$. Recall that De Rham's currents are continuous linear forms on spaces of differential forms, just as distributions are linear forms on spaces of differentiable functions; they can also be regularized as limits of differential forms. For our case, if one sets $\delta_f^{(m)} = \frac{1}{\sqrt{\pi}}\exp(-m^2 f^2) m\, df$, one has $\delta_f \to \delta_f^{(m)}$ for the weak topology when $m \to \infty$. So, as a consequence:

$$\int_{\{f^{-1}(0)\}} \omega = \lim_{m\to\infty} \int_X \delta_f^{(m)} \wedge \omega\,.$$

Now, supergeometry can enter the scenario: let's first extend manifold X to a supermanifold by adding one odd coordinate; so we consider $\mathcal{X} = X \times \mathbb{R}^{0|1}$, we shall denote by η the odd coordinate. Let's then consider the tangent space on $\mathcal{X}$ with inverse parity on the fibers $\Pi T\mathcal{X}$, which is isomorphic to $\Pi TX \times \mathbb{R}^{1|1}$, with $\mathbb{R}^{1|1}$ identified with $\Pi T\mathbb{R}^{0|1}$; we shall denote by t the even tangent coordinate in $\Pi T\mathbb{R}^{0|1}$. Since differential forms on X are functions on ΠTX, one can consider the function $\exp(-m^2 f^2 + \eta\, m\, df)$ on $\Pi T\mathcal{X}$; it satisfies, from the properties of odd variables:

$$\exp(-m^2 f^2 + \eta\, m\, df) = \exp(-m^2 f^2) + \eta\, \exp(-m^2 f^2) m\, df\,.$$

One then deduces from the properties of Berezin integral:

$$\delta_f^{(m)} = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}^{0|1}} \exp(-m^2 f^2 + \eta\, m\, df)\, \mathcal{D}\eta\,.$$

One further uses properties of Fourier transforms for gaussian functions,and the even tangent variable t as a Lagrange multiplier to get:

$$\delta_f^{(m)} = \frac{1}{2\pi}\int_{\mathbb{R}^{1|1}} \exp(itmf - \frac{t^2}{4} + \eta\, m\, df)\, \mathcal{D}(\eta, t)\,.$$

One can now use the following change of variables $l = mt2\pi$, $\theta = m\eta$, and one obtains:

$$\delta_f^{(m)} = \int_{\mathbb{R}^{1|1}} \exp(2i\pi lf - (\frac{l\pi}{m})^2 + \theta df)\, \mathcal{D}(\theta, l)\,.$$

So finally when $m \to \infty$, one has:

$$\delta_f = \int_{\mathbb{R}^{1|1}} \exp(2i\pi lf + \theta df)\, \mathcal{D}(\theta, l)\,.$$

In a system of coordinates (x_i, ψ_i) on ΠTX, one can write: $2i\pi lf + \theta\, df = 2i\pi lf + \theta(\sum_{i=1}^n \frac{\partial f}{\partial x_i}\psi_i) = [2i\pi l\frac{\partial}{\partial\theta} + \sum_{i=1}^n \psi_i\frac{\partial}{\partial x_i}](\theta f) = D(\theta f)$. Here D represents exterior derivative of functions on supermanifold $\mathcal{X}$ and the 1-form $D(\theta f)$ is then seen as a function on $\Pi T\mathcal{X}$(up to homothety $l \to 2i\pi l$). One obtains finally:

$$\int_{\{f^{-1}(0)\}} \omega = \int_{\Pi T\mathcal{X}} \omega\, \exp(D(\theta f))\, \mathcal{D}(x_i, \psi_i, \theta, l)\,.$$

So the integral of ω on a complicated space, is replaced by integration of a function $\omega\, \exp\big(D(\theta f)\big)$ on a bigger space, but regular; this exactly the old idea of Lagrange multipliers where constraints are integrated in the phase space, but adapted to the supergeometric context; the supplementary odd variables are known as “twisted fermions”.

3.6. About symplectomorphisms of ΠT^*X.

The theorem of A. Schwarz, mentioned above 3.2, says that any periplectic manifold is equivalent to some ΠT^*X, but non canonically, i.e. up to some odd symplectomorphism. We shall say now some words about them, following results from Schwarz [31] and Khudaverdian [19] to periplectic form ω.

3.6.1. *Homotopy type.* Let’s consider the cotangent vector bundle $T^*X \to X$, and its group of vector bundle automorphisms $\mathcal{AUT}(T^*X)$. The structure of the latter group is well known from classical Gauge Theory; it enters an exact sequence:

$$C^\infty(X; \mathrm{GL}(n)) \longrightarrow \mathcal{AUT}(T^*X) \longrightarrow \mathrm{Diff}(X)$$

Here $C^\infty(X; \mathrm{GL}(n))$ is simply the gauge group associated to the corresponding frame bundle. Now, one has an obvious inclusion $\mathcal{AUT}(T^*X) \subset \mathrm{Sympl}(\Pi T^*X, \omega)$, the latter group being the group of odd symplectomorphisms of manifold ΠT^*X, with respect to periplectic form ω. The image of this inclusion consists of automorphisms which are linear in the odd variables. A result of Schwarz [31] shows that this inclusion is a homotopy equivalence.

3.6.2. *Various kind of odd symplectomorphisms.* In [19], Khudaverdian distinguishes 3 kinds of odd symplectomorphisms:

(1) **ponctual:** They form the image of $\mathrm{Diff}(X) \subset \mathrm{Sympl}(\Pi T^*X, \omega)$;
(2) **special:** They form the image of an embedding $\Omega^1(X) \subset \mathrm{Sympl}(\Pi T^*X, \omega)$, defined as follows: a 1-form $\alpha = \sum_{i=1}^n \alpha_i dx_i$ gives$(x, \theta_i) \to (x, \theta_i + \alpha_i)$
(3) **adjusted:** They are much more mysterious, they mix odd and even variables.

Ponctual and special automorphisms generate together a copy of semi-direct product of diffeomorphism group with one-forms. The deep result is now that those three kind of symplectomorphisms together generate the full group $\mathrm{Sympl}(\Pi T^*X, \omega)$ (see [19] for proof).

4. BV-structures in field theory

Those structures appeared for the first time as a version of quantum field theory (QFT), using functional integration for quantizing classical action $\gamma \to \mathcal{S}(\gamma)$ into a partition function $\mathcal{Z}\colon \int \exp(i\frac{\mathcal{S}(\gamma)}{h})\mathcal{D}(\gamma)$, using some kind of "measure" on a space of paths γ (for Quantum Mechanics) or fields ϕ (for Quantum Field Theory). One then computes mean values of functionals over the space of fields, or some correlation functions:

$$\langle \mathcal{F}(\phi) \rangle = \frac{\int \mathcal{F}(\phi) \exp(i\frac{\mathcal{S}(\phi)}{h})\, \mathcal{D}(\phi)}{\mathcal{Z}}\,.$$

Recall that when $h \to 0$, the value of the integral tends to be concentrated on critical values of the functional, so one obtains as classical limit the Euler-Lagrange equation $\frac{\delta \mathcal{S}}{\delta \gamma} = 0$ as usual.

4.1. General framework of BV quantization.

We shall start from some space of fields, some manifold or supermanifold $\mathcal{X}$, usually infinite dimensional. The fields will be defined on $\prod T^*\mathcal{X}$, equipped with its periplectic and Gerstenhaber structure, constructed in Chapter 2, and with some BV operator associated to it; The choice of some density enables to integrate functions $f\colon \prod T^*\mathcal{X} \to \mathbb{C}$, on a Lagrangian submanifold $L \subset \prod T^*\mathcal{X}$. For classical actions $\Sigma\colon \prod T^*\mathcal{X} \to \mathbb{C}$, one wants to define quantum action as the integral

$$\int_L \exp\left(\frac{i\Sigma}{h}\right)$$

as a quantum action. One then has

Theorem (Main theorem of BV quantization (Albert Schwarz))**.**

(1) $\Delta f = 0$ *and* L_1 *homologous to* L_2 *imply* $\int_{L_1} f = \int_{L_2} f$
(2) *If* $f = \Delta g$*, then* $\int_L f = 0$ *for any Lagrangian* L*.*

Before entering the arguments of proof, one needs some precision about geometry of Lagrangian submanifolds.

4.2. **About Lagrangian submanifolds.**
The theory is completely parallel to the case of symplectic geometry, but much more rigid. A submanifold $L \subset \prod T^*\mathcal{X}$ is Lagrangian if it is maximal isotropic (recall that 'isotropic' means $\omega_{|L} = O$). A standard example of isotropic submanifold is the following: let $\mathcal{N}$ be a k-dimensional submanifold of $\mathcal{X}$, and set $T'\mathcal{N} = \{\alpha \in T^*\mathcal{X} | \alpha(\xi) = O,\ \text{if}\ \xi \in T\mathcal{N}\}$, then $\prod T'\mathcal{N} \subset \prod T^*\mathcal{X}$ is a Lagrangian submanifold, called standard Lagrangian.

In terms of local coordinates: let $x_1, \dots, x_n, \theta_1, \dots, \theta_n$ be a system of coordinates in $\prod T^*\mathcal{X}$, let $\mathcal{N}$ be a k-dimensional submanifold defined locally by $x_{k+1} = \cdots = x_n = 0$ (so here $\mathcal{X}$ is an ordinary manifold); then $\prod T^\perp\mathcal{N}$ admits local equations $x_{k+1} = \cdots = x_n = 0, \theta_1 = \cdots = \theta_k = 0$. So $\operatorname{Dim}(\prod T^\perp\mathcal{N}) = (k|n-k)$.

Remark. This construction can be generalized for the case when $\mathcal{X}$ a supermanifold.

Theorem. *Let $L \subset \prod T^*\mathcal{X}$ be a Lagrangian submanifold. Then there exists a smooth family of Lagrangian submanifolds (L_t), $t \in [0,1]$, with $L_0 = L$ and L_1 of the form $\prod T^\perp\mathcal{N}$ for some $\mathcal{N} \subset \mathcal{X}$. A detailed proof can be found in* [20]*, it makes an extensive use of classification of odd symplectomorphisms.*

4.3. **Classical and quantum BV master equation.**
Let $\Sigma\colon \prod T^*\mathcal{X} \to \mathbb{C}$ be a classical action. One wants to compute $\Delta\big(\exp\big(\frac{i\Sigma}{h}\big)\big)$, where Δ is the BV-operator.

Proposition 1. *One has:* $\Delta\big(\exp\big(\frac{i\Sigma}{h}\big)\big) = \exp\big(\frac{i\Sigma}{h}\big)\big(\Delta(\Sigma) - \frac{[\Sigma,\Sigma]}{2ih}\big)$.

Proof. Develop $\exp\big(\frac{i\Sigma}{h}\big)$ into power series, use the fact that Δ is of order 2, and compute! □

Corollary. $\Delta\big(\exp\big(\frac{i\Sigma}{h}\big)\big) = 0 \iff 2ih\Delta\Sigma = [\Sigma, \Sigma]$.

Remark. The first equation is called **BV Equation**, and the second one **Quantum Master Equation** (QME). One must point out the analogy with deformation theory and Maurer-Cartan equation (cf. 2.5.2).

Let now $\Psi\colon \prod T^*\mathcal{X} \to \mathbb{C}$ be another function. If Σ satisfies (QME), then one has

$$\Delta\Big(\Psi \exp\Big(\frac{i\Sigma}{h}\Big)\Big) = \Big(\Delta\Psi + (-1)^{|\Psi|}\frac{i}{h}[\Psi, \Sigma]\Big)\exp\Big(\frac{i\Sigma}{h}\Big)\Big).$$

So one has:

Corollary. *If* Σ *satisfies* (QME)*, then one has*

$$\Delta\Big(\Psi \exp\Big(\frac{i\Sigma}{h}\Big)\Big) = 0 \iff ih\Delta\Psi = (-1)^{|\Psi|}[\Psi, \Sigma]\,.$$

Such a function Ψ will be called a **quantum observable**.

Remark. Set $\Delta_\Sigma(\Psi) = ih\Delta\Psi - (-1)^{|\Psi|}[\Psi, \Sigma]$. Then if Σ satisfies (QME), one has ${\Delta_\Sigma}^2 = 0$. So there exists a cohomology naturally associated to it! If $h \to 0$, the QME tends to the Classical Master Equation

$$[\Sigma, \Sigma] = 0\,.$$

So when $\mathcal{X}$ is a classical finite dimensional manifold, $[\ ,\ ]$ turns out to be Schouten bracket (cf. 2.3.1), and CME is the equation of Poisson tensors. Moreover, the equation for classical observables (cf. above) yields:

$$[\Psi, \Sigma] = 0 .$$

It means that Ψ is a cocycle for Poisson cohomology defined by tensor Σ. This cohomology is a typical example of cohomologies obtained through vanishing square objects (cf. 1); it has been introduced by A. Lichnerowicz [23], for maximal rank (symplectic) Poisson structures, it yields De Rham cohomology of $\mathcal{X}$, but in general it is very difficult to understand.

Remark (Exercise, or cf. Cattaneo [10])**.** If Σ satisfies QME, then one obtains two homological differentials Δ and Δ_Σ, since $\Delta^2 = 0$ and ${\Delta_\Sigma}^2 = 0$. Prove they define the same cohomology, using the above computation for $\Delta\big(\Psi \exp\big(\frac{i\Sigma}{h}\big)\big)$.

4.4. **Integration on Lagrangian submanifolds and quantization.**
The global scheme works as follows:

(1) Let $v = \underline{vol}$ a volume form on $\mathcal{X}$;
(2) One deduces from v, the density $\mathcal{D}_v$ on $\prod T^*\mathcal{X}$ (Berezinian);
(3) One deduces a generator Δ_v for the BV structure;
(4) Finally one gets a density on any Lagrangian submanifold $L \subset \prod T^*\mathcal{X}$, denoted by $\sqrt{\mathcal{D}_v}$.

Those successive constructions have been described in previous sections; one first gets the partition function

$$\mathcal{Z} = \int_L \exp\left(\frac{i\Sigma}{h}\right)\sqrt{\mathcal{D}_v}\,,$$

for Σ satisfying QME. For any quantum observable Ψ, one computes the mean value:

$$\langle \Psi \rangle = \frac{\int_L \Psi \exp\left(\frac{i\Sigma}{h}\right)\sqrt{\mathcal{D}_v}}{\mathcal{Z}} .$$

The main theorem above imply that $\mathcal{Z}$, $\langle\Psi\rangle$, and correlation functions $\langle\Psi_1\Psi_2\ldots\Psi_p\rangle$ do not depend on the choice of a particular Lagrangian submanifold, inside a same homology class; this kind of independence looks very much like 'gauge invariance'.

Remark. If one changes densities, and thus the operator Δ, then one modifies action and the corresponding quantum observables according to previous formulas.

4.5. **Miscellaneous.**

4.5.1. *Algebraic quantization.* Starting from a particular solution of CME, one can try to construct inductively a solution of QME, in the spirit of perturbative theory or deformation quantization; more explicitly, let S be a solution of CME, so $[S, S] = 0$, and look for solutions of QME following the Ansatz:

$$\Sigma = S + \sum_{p=1}^{+\infty} h^p \Sigma_p .$$

So, from $2ih\Delta\Sigma = [\Sigma, \Sigma]$, one deduces successive equations

$$\begin{aligned} i\Delta S &= [S, \Sigma_1] \\ &\vdots \\ i\Delta\Sigma_p &= [S, \Sigma_p] + \frac{1}{2}\sum_{a+b=p} [\Sigma_a, \Sigma_b] \,. \end{aligned}$$

So Quantum Master Equation leads to successive and recurrent cohomological equations, for the cohomology defined by bracket with S (cf. 1).

4.5.2. *Ghosts.* Following Losev [32], if one has a classical action $f\colon X \to \mathbb{R}$, and a group G acting on X such that f is invariant, then one can make G enter the configuration space, but with inversed parity. Let $\mathfrak{g}$ the Lie algebra of G and let $\{c^a\}$ be a basis of $\mathfrak{g}$; consider the supermanifold $X \times \Pi\mathfrak{g}$ and then its associated cotangent space with inverse parity on the fibers:

$$\mathcal{X} = \Pi T^*(X \times \Pi\mathfrak{g}) = \Pi T^*(X) \times \Pi\mathfrak{g} \times \mathfrak{g}^* \,.$$

This supermanifold admits a natural periplectic structure, and its space of (super) functions form a BV algebra. Choosing coordinates x_i on X, one deduces coordinates on $\mathcal{X}$: $(x_i, c^a, \theta_i, c^*_a)$. One can now define the BV action,which is the quantization of the previous f:

$$S(x, \theta, c, c^*) = f(x) + c^a v^i_a(x)\theta_i + \frac{1}{2} C^c_{a\,b} c^a c^b c^*_c \,.$$

[NB: here we use Einstein convention, i.e. summations on indices are understood, and $C^c_{a\,b}$ indicates structure constants].

Then, for the above mentioned BV structure, the BV equation $\Delta S = 0$ is equivalent to the fact that vector fields $v^i_a(x)\frac{\partial}{\partial x_i}$ generate a Lie algebra isomorphic to $\mathfrak{g}$, and they leave action f invariant. In the BV terminology, the x_i are the fields, θ_i the antifields, and the c^a the c-ghosts (following Faddeev-Popov). Here we briefly checked how to introduce symmetries in the configuration space; in the previous chapter (see 3.5) we showed in a very particular case, how to introduce constraints in the configuration space, generalizing Lagrange multipliers with odd coordinates; in the general case one gets this way b-ghosts. Both approach are strongly linked, in some sense dual to each other, ie symmetries gives cohomology, while constraints give homology. The cohomology one gets from the full quantized action is a mixture of Lie algebra homology and cohomology. Well known examples of this kind of cohomology are BRST[6], and semi-infinite cohomology [5, 33].

4.5.3. *Further applications.* For more details about the physical roots of those constructions, see the original articles [3, 2], or the beautiful survey [30]. Recent applications of the BV-quantization scheme to the study of Poisson sigma-models has been given in [4].

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