ARCHIVUM MATHEMATICUM (BRNO)
Tomus 44 (2008), 237–244

ON A NONCONVEX BOUNDARY VALUE PROBLEM FOR A FIRST ORDER MULTIVALUED DIFFERENTIAL SYSTEM

Aurelian Cernea

Abstract. We consider a boundary value problem for first order nonconvex differential inclusion and we obtain some existence results by using the set-valued contraction principle.

1. Introduction

This paper is concerned with the following boundary value problem for first order differential inclusions

$$x' \in A(t)x + F(t,x), \quad \text{a.e. } (I), \quad Mx(0) + Nx(1) = \eta \tag{1.1}$$

where $I=[0,1]$, $F(\cdot,\cdot)\colon I\times\mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ is a set-valued map, $A(\cdot)$ is a continuous $(n\times n)$ matrix function, M and N are $(n\times n)$ constant real matrices and $\eta\in\mathbb{R}^n$.

The present note is motivated by a recent paper of Boucherif and Chiboub ([1]), where it is considered problem (1.1) with $\eta = 0$ and several existence results are obtained under growth conditions on $F(\cdot,\cdot)$ by using topological transversality arguments, fixed point theorems and differential inequalities.

The aim of our paper is to present two additional results obtained by the application of the set-valued contraction principle due to Covitz and Nadler ([6]). The approach we propose allows to avoid the assumption that the values of $F(\cdot,\cdot)$ are convex which is an essential hypothesis in [1].

The first result follows a classical idea by applying the set-valued contraction principle in the space of solutions of the problem. The second result is a Filippov type theorem concerning the existence of solutions to problem (1.1). Recall that for a differential inclusion defined by a lipschitzian set-valued map with nonconvex values, Filippov's theorem consists in proving the existence of a solution starting from a given "quasi" solution. This time we apply the contraction principle in the space of derivatives of solutions instead of the space of solutions. In addition, as usual at a Filippov existence type theorem, our result provides an estimate between the starting "quasi" solution and the solution of the differential inclusion. The idea of applying the set-valued contraction principle in the space of derivatives of

2000 *Mathematics Subject Classification*: primary 34A60.

Key words and phrases: boundary value problem, differential inclusion, contractive set-valued map, fixed point.

Received March 31, revised May 2008. Editor O. Došlý.

the solutions belongs to Tallos ([7, 9]) and it was already used for other results concerning differential inclusions ([3, 4, 5] etc.).

For the motivation of study of problem (1.1) we refer to [1] and references therein.

The paper is organized as follows: in Section 2 we recall some preliminary facts that we need in the sequel and in Section 3 we prove our main results.

2. Preliminaries

In this short section we sum up some basic facts that we are going to use later.

Let (X, d) be a metric space and consider a set valued map T on X with nonempty values in X. T is said to be a λ-contraction if there exists $0 < \lambda < 1$ such that:

$$d_H(T(x), T(y)) \le \lambda d(x, y) \quad \forall x, y \in X\,,$$

where $d_H(\cdot,\cdot)$ denotes the Pompeiu-Hausdorff distance. Recall that the Pompeiu--Hausdorff distance of the closed subsets $A, B \subset X$ is defined by

$$d_H(A, B) = \max\left\{d^*(A, B), d^*(B, A)\right\}, \quad d^*(A, B) = \sup\left\{d(a, B); a \in A\right\},$$

where $d(x, B) = \inf_{y\in B} d(x, y)$.

The set-valued contraction principle ([6]) states that if X is complete, and $T\colon X \to \mathcal{P}(X)$ is a set valued contraction with nonempty closed values, then $T(\cdot)$ has a fixed point, i.e. a point $z \in X$ such that $z \in T(z)$.

We denote by $\mathrm{Fix}(T)$ the set of all fixed points of the set-valued map T. Obviously, $\mathrm{Fix}(T)$ is closed.

Proposition 2.1 ([8]). *Let X be a complete metric space and suppose that T_1, T_2 are λ-contractions with closed values in X. Then*

$$d_H\big(\mathrm{Fix}(T_1), \mathrm{Fix}(T_2)\big) \le \frac{1}{1-\lambda} \sup_{z\in X} d\big(T_1(z), T_2(z)\big)\,.$$

Let $I = [0, 1]$, let $|x|$ be the norm of $x \in \mathbb{R}^n$ and $\|A\|$ be the norm of any matrix A. As usual, we denote by $C(I, \mathbb{R}^n)$ the Banach space of all continuous functions from I to $\mathbb{R}^n$ with the norm $\|x(\cdot)\|_C = \sup_{t\in I} |x(t)|$, $AC(I, \mathbb{R}^n)$ is the space of absolutely continuous from I to $\mathbb{R}^n$ and $L^1(I, \mathbb{R}^n)$ is the Banach space of integrable functions $u(\cdot)\colon I \to \mathbb{R}^n$ endowed with the norm $\|u(\cdot)\|_1 = \int_0^1 \big|u(t)\big|\, dt$.

A function $x(\cdot) \in AC(I, \mathbb{R}^n)$ is called a solution of problem (1.1) if there exists a function $f(\cdot) \in L^1(I, \mathbb{R}^n)$ with $f(t) \in F\big(t, x(t)\big)$, a.e. (I) such that

$$x'(t) = A(t)x(t) + f(t)\,, \quad \text{a.e.} \quad (0, 1)\,, \quad Mx(0) + Nx(1) = \eta\,. \tag{2.1}$$

For each $x(\cdot) \in AC(I, \mathbb{R}^n)$ define

$$S_{F,x} := \left\{f(\cdot) \in L^1(I, \mathbb{R}^n);\ f(t) \in F\big(t, x(t)\big) \text{ a.e. } (I)\right\}.$$

Let $\Phi(\cdot)$ be a fundamental matrix solution of the differential equations $x' = A(t)x$ that satisfy $\Phi(0) = I$, where I is the $(n \times n)$ identity matrix.

The next result is well known (e.g. [1]).

Lemma 2.2 ([1]). *If $f(\cdot)\colon [0,1] \to \mathbb{R}^n$ is an integrable function then the problem*

$$x'(t) = A(t)x(t) + f(t)\,, \quad a.e.\ (0,1)\,, \quad Mx(0) + Nx(1) = 0 \tag{2.2}$$

has a unique solution provided $\det(M + N\Phi(1)) \neq 0$. *This solution is given by*

$$x(t) = \int_0^1 G(t,s) f(s)\, ds\,,$$

with $G(\cdot,\cdot)$ the Green function associated to the problem (2.2). *Namely,*

$$G(t,s) = \begin{cases} \Phi(t)J(s) & \text{if} \quad 0 \le t \le s\,, \\ \Phi(t)\Phi(s)^{-1} + \Phi(t)J(s) & \text{if} \quad s \le t \le 1\,, \end{cases} \tag{2.3}$$

where $J(t) = -\big(M + N\Phi(1)\big)^{-1} N\Phi(1)\Phi(t)^{-1}$.

If we consider the problem with nonhomogeneous boundary conditions, i.e. problem (2.1), then it is easy to verify that its solution is given by

$$x(t) = \Phi(t)\left(M + N\Phi(1)\right)^{-1}\eta + \int_0^1 G(t,s) f(s)\, ds\,. \tag{2.4}$$

In the sequel we assume that $A(\cdot)$ is a continuous $(n \times n)$ matrix function, M and N are $(n \times n)$ constant real matrices such that $\det\big(M + N\Phi(1)\big) \neq 0$.

In order to study problem (1.1) we introduce the following hypothesis on F.

Hypothesis 2.3. (i) $F(\cdot,\cdot)\colon I \times \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ has nonempty closed values and for every $x \in \mathbb{R}^n$ $F(\cdot,x)$ is measurable.

(ii) There exists $L(\cdot) \in L^1(I,\mathbb{R}_+)$ such that for almost all $t \in I$, $F(t,\cdot)$ is $L(t)$-Lipschitz in the sense that

$$d_H\big(F(t,x), F(t,y)\big) \le L(t)|x - y| \quad \forall\ \ x, y \in \mathbb{R}^n$$

and $d\big(0, F(t,0)\big) \le L(t)$ a.e. (I).

Denote $L_0 := \int_0^1 L(s)ds$ and $G_0 := \sup_{t,s \in I} \|G(t,s)\|$.

3. The main results

We are able now to present a first existence result for problem (1.1).

Theorem 3.1. *Assume that Hypothesis 2.3 is satisfied, $F(\cdot,\cdot)$ has compact values and $G_0 L_0 < 1$. Then the problem* (1.1) *has a solution.*

Proof. We transform the problem (1.1) in a fixed point problem. Consider the set-valued map $T\colon C(I,\mathbb{R}^n) \to \mathcal{P}\big(C(I,\mathbb{R}^n)\big)$ defined by

$$T(x) := \Big\{ v(\cdot) \in C(I,\mathbb{R}^n);\ v(t) := \Phi(t)\big(M + N\Phi(1)\big)^{-1}\eta + \int_0^1 G(t,s) f(s)\, ds,\ f \in S_{F,x} \Big\}\,.$$

Note that since the set-valued map $F\big(\cdot, x(\cdot)\big)$ is measurable with the measurable selection theorem (e.g., [2, Theorem III.6]) it admits a measurable selection $f(\cdot)\colon I \to \mathbb{R}^n$. Moreover, from Hypothesis 2.3

$$\big|f(t)\big| \le L(t) + L(t)\big|x(t)\big|\,,$$

i.e., $f(\cdot) \in L^1(I, \mathbb{R}^n)$. Therefore, $S_{F,x} \neq \emptyset$.

It is clear that the fixed points of $T(\cdot)$ are solutions of problem (1.1). We shall prove that $T(\cdot)$ fulfills the assumptions of Covitz-Nadler contraction principle.

First, we note that since $S_{F,x} \neq \emptyset$, $T(x) \neq \emptyset$ for any $x(\cdot) \in C(I, \mathbb{R}^n)$.

Secondly, we prove that $T(x)$ is closed for any $x(\cdot) \in C(I, \mathbb{R}^n)$.

Let $\{x_n\}_{n\ge 0} \in T(x)$ such that $x_n(\cdot) \to x^*(\cdot)$ in $C(I, \mathbb{R}^n)$. Then $x^*(\cdot) \in C(I, \mathbb{R}^n)$ and there exists $f_n \in S_{F,x}$ such that

$$x_n(t) = \Phi(t)\big(M + N\Phi(1)\big)^{-1}\eta + \int_0^1 G(t,s) f_n(s)\, ds\,.$$

Since $F(\cdot,\cdot)$ has compact values and Hypothesis 2.3 is satisfied we may pass to a subsequence (if necessary) to get that $f_n(.)$ converges to $f(\cdot) \in L^1(I, \mathbb{R}^n)$ in $L^1(I, \mathbb{R}^n)$.

In particular, $f \in S_{F,x}$ and for any $t \in I$ we have

$$x_n(t) \to x^*(t) = \Phi(t)\big(M + N\Phi(1)\big)^{-1}\eta + \int_0^1 G(t,s) f(s)\, ds\,,$$

i.e., $x^* \in T(x)$ and $T(x)$ is closed.

Finally, we show that $T(\cdot)$ is a contraction on $C(I, \mathbb{R}^n)$.

Let $x_1(\cdot), x_2(\cdot) \in C(I, \mathbb{R}^n)$ and $v_1 \in T(x_1)$. Then there exist $f_1 \in S_{F,x_1}$ such that

$$v_1(t) = \Phi(t)\big(M + N\Phi(1)\big)^{-1}\eta + \int_0^1 G(t,s) f_1(s)\, ds\,, \quad t \in I\,.$$

Consider the set-valued map

$$G(t) := F\big(t, x(t)\big) \cap \big\{x \in \mathbb{R}^n;\ \big|f_1(t) - x\big| \le L(t)\big|x_1(t) - x_2(t)\big|\big\}\,, \quad t \in I\,.$$

From Hypothesis 2.3 one has

$$d_H\big(F(t, x_1(t)), F(t, x_2(t))\big) \le L(t)\big|x_1(t) - x_2(t)\big|\,,$$

hence $G(\cdot)$ has nonempty closed values. Moreover, since $G(\cdot)$ is measurable, there exists $f_2(\cdot)$ a measurable selection of $G(\cdot)$. It follows that $f_2 \in S_{F,x_2}$ and for any $t \in I$

$$\big|f_1(t) - f_2(t)\big| \le L(t)\big|x_1(t) - x_2(t)\big|\,.$$

Define

$$v_2(t) = \Phi(t)\big(M + N\Phi(1)\big)^{-1}\eta + \int_0^1 G(t,s) f_2(s)\, ds\,, \quad t \in I\,,$$

and we have

$$\big|v_1(t)-v_2(t)\big| \le \int_0^1 \|G(t,s)\|\cdot\big|f_1(s)-f_2(s)\big|\,ds \le G_0\int_0^1 |f_1(s)-f_2(s)|ds$$

$$\le G_0\int_0^1 L(s)\big|x_1(s)-x_2(s)\big|\,ds \le G_0L_0\|x_1-x_2\|_C\,.$$

So, $\|v_1-v_2\|_C \le G_0L_0\|x_1-x_2\|_C$.

From an analogous reasoning by interchanging the roles of x_1 and x_2 it follows

$$d_H\big(T(x_1),T(x_2)\big) \le G_0L_0\|x_1-x_2\|_C\,.$$

Therefore, $T(\cdot)$ admits a fixed point which is a solution to problem (1.1). $\square$

The next theorem is the main result of this paper. As one can see it is, in fact, no necessary to assume that $F(\cdot,\cdot)$ has compact values as in Theorem 3.1.

Theorem 3.2. *Assume that Hypothesis 2.3 is satisfied and $G_0L_0 < 1$. Let $y(\cdot)\in AC(I,\mathbb{R}^n)$ be such that there exists $q(\cdot)\in L^1(I,\mathbb{R}_+)$ with $d\big(y'(t)-A(t)y(t), F(t,y(t))\big)\le q(t)$, a.e. (I). Denote $\mu = My(0)+Ny(1)$.*

Then for every $\varepsilon>0$ there exists $x(\cdot)$ a solution of problem (1.1) *satisfying for all $t\in I$*

$$\big|x(t)-y(t)\big| \le \frac{1}{1-G_0L_0}\sup_{t\in I}|\Phi(t)(M+N\Phi(1))^{-1}(\eta-\mu)| + \frac{G_0}{1-G_0L_0}\int_0^1 q(t)\,dt+\varepsilon\,.$$

Proof. For $u(\cdot)\in L^1(I,\mathbb{R}^n)$ define the following set valued maps

$$M_u(t) = F\big(t,\Phi(t)(M+N\Phi(1))^{-1}\eta + \int_0^1 G(t,s)u(s)\,ds\big)\,,\quad t\in I\,,$$

$$T(u) = \big\{\phi(\cdot)\in L^1(I,\mathbb{R}^n);\ \phi(t)\in M_u(t) \text{ a.e. } (I)\big\}\,.$$

It follows from the definition and (2.4) that $x(\cdot)$ is a solution of problem (1.1)–(2.2) if and only if $x'(\cdot)-A(\cdot)x(\cdot)$ is a fixed point of $T(\cdot)$.

We shall prove first that $T(u)$ is nonempty and closed for every $u\in L^1(I,\mathbb{R}^n)$. The fact that the set valued map $M_u(\cdot)$ is measurable is well known. For example the map $t\to\Phi(t)\big(M+N\Phi(1)\big)^{-1}\eta+\int_0^1 G(t,s)u(s)\,ds$ can be approximated by step functions and we can apply in [2, Theorem III.40]. Since the values of F are closed with the measurable selection theorem ([2, Theorem III.6]) we infer that $M_u(\cdot)$ admits a measurable selection ϕ. One has

$$\big|\phi(t)\big| \le d\big(0,F(t,0)\big) + d_H\Big(F(t,0),F(t,\Phi(t)(M+N\Phi(1))^{-1}\eta+\int_0^1 G(t,s)u(s)\,ds)\Big)$$

$$\le L(t)\big(1+\big|\Phi(t)\big(M+N\Phi(1)\big)^{-1}\eta\big| + G_0\int_0^1 |u(s)|\,ds\big)\,,$$

which shows that $\phi\in L^1(I,\mathbb{R}^n)$ and $T(u)$ is nonempty.

On the other hand, the set $T(u)$ is also closed. Indeed, if $\phi_n\in T(u)$ and $\|\phi_n-\phi\|_1\to 0$ then we can pass to a subsequence ϕ_{n_k} such that $\phi_{n_k}(t)\to\phi(t)$ for a.e. $t\in I$, and we find that $\phi\in T(u)$.

We show next that $T(\cdot)$ is a contraction on $L^1(I,\mathbb{R}^n)$.

Let $u, v \in L^1(I, \mathbb{R}^n)$ be given and $\phi \in T(u)$. Consider the following set-valued map:

$$H(t) = M_v(t) \cap \Big\{ x \in \mathbb{R}^n ; \big|\phi(t) - x\big| \le L(t) \Big| \int_0^1 G(t,s)\big(u(s) - v(s)\big)\, ds \Big| \Big\} .$$

From Proposition III.4 in [2], $H(\cdot)$ is measurable and from Hypothesis 2.3 ii) $H(\cdot)$ has nonempty closed values. Therefore, there exists $\psi(\cdot)$ a measurable selection of $H(\cdot)$. It follows that $\psi \in T(v)$ and according with the definition of the norm we have

$$\begin{aligned} \|\phi - \psi\|_1 &= \int_0^1 |\phi(t) - \psi(t)|\, dt \le \int_0^1 L(t) \Big(\int_0^1 \|G(t,s)\| \cdot \big|u(s) - v(s)\big|\, ds \Big)\, dt \\ &= \int_0^1 \Big(\int_0^1 L(t) \|G(t,s)\|\, dt \Big) \big|u(s) - v(s) = big|\, ds \le G_0 L_0 \|u - v\|_1 . \end{aligned}$$

We deduce that

$$d\big(\phi, T(v)\big) \le G_0 L_0 \|u - v\|_1 .$$

Replacing u by v we obtain

$$d_H\big(T(u), T(v)\big) \le G_0 L_0 \|u - v\|_1 ,$$

thus $T(\cdot)$ is a contraction on $L^1(I, \mathbb{R}^n)$.

We consider next the following set-valued maps

$$\begin{aligned} &F_1(t,x) = F(t,x) + q(t)B , \qquad (t,x) \in I \times \mathbb{R}^n , \\ &M_u^1(t) = F_1 = \Big(t, \Phi(t)(M + N\Phi(1))^{-1}\mu + \int_0^1 G(t,s)u(s)\, ds\Big) , \\ &T_1(u) = \big\{\psi(\cdot) \in L^1(I, \mathbb{R}^n);\ \psi(t) \in M_u^1(t) \text{ a.e. } (I)\big\} , \quad u(\cdot) \in L^1(I, \mathbb{R}^n) , \end{aligned}$$

where B denotes the closed unit ball in $\mathbb{R}^n$. Obviously, $F_1(\cdot,\cdot)$ satisfies Hypothesis 2.3.

Repeating the previous step of the proof we obtain that T_1 is also a G_0L_0-contraction on $L^1(I, \mathbb{R}^n)$ with closed nonempty values.

We prove next the following estimate

$$\begin{aligned} (3.1) \quad & d_H\big(T(u), T_1(u)\big) \\ & \qquad \le \sup_{t \in I} \Big|\Phi(t)\big(M + N\Phi(1)\big)^{-1}(\eta - \mu)\Big| L_0 + \int_0^1 q(t)\, dt . \end{aligned}$$

Let $\phi \in T(u)$ and define

$$H_1(t) = M_u^1(t) \cap \big\{ z \in \mathbb{R}^n ;\ \big|\phi(t) - z\big| \le L(t) \, \big|\Phi(t)\big(M + N\Phi(1)\big)^{-1}(\eta - \mu)\big| + q(t) \big\} .$$

With the same arguments used for the set valued map $H(\cdot)$, we deduce that $H_1(\cdot)$ is measurable with nonempty closed values. Hence let $\psi(\cdot)$ be a measurable selection of $H_1(\cdot)$. It follows that $\psi \in T_1(u)$ and one has

$$\|\phi-\psi\|_1 = \int_0^1 \big|\phi(t)-\psi(t)\big|\,dt \le \int_0^1 \big[L(t)\,\big|\Phi(t)\big(M+N\Phi(1)\big)^{-1}(\eta-\mu)\big| + q(t)\big]\,dt \le \int_0^1 L(t)|\Phi(t)(M+N\Phi(1))^{-1}(\eta-\mu)|dt + \int_0^1 q(t)$$
$$\le L_0 \sup_{t\in I}\big|\Phi(t)\big(M+N\Phi(1)\big)^{-1}(\eta-\mu)\big| + \int_0^1 q(t)\,dt\,.$$

As above we obtain (3.1).

We apply Proposition 2.1 and we infer that

$$d_H\big(\operatorname{Fix}(T),\operatorname{Fix}(T_1)\big) \le \frac{L_0}{1-G_0L_0}\sup_{t\in I}\big|\Phi(t)\big(M+N\Phi(1)\big)^{-1}(\eta-\mu)\big|\frac{1}{1-G_0L_0}\int_0^1 q(t)\,dt\,.$$

Since $v(\cdot) = y'(\cdot) - A(\cdot)\,y(\cdot) \in \operatorname{Fix}(T_1)$ it follows that there exists $u(\cdot) \in \operatorname{Fix}(T)$ such that for any $\varepsilon > 0$

$$\|v-u\|_1 \le \frac{L_0}{1-G_0L_0}\sup_{t\in I}\big|\Phi(t)\big(M+N\Phi(1)\big)^{-1}(\eta-\mu)\big| + \frac{1}{1-G_0L_0}\int_0^1 q(t)\,dt + \frac{\varepsilon}{G_0}\,.$$

We define $x(t) = \Phi(t)\big(M+N\Phi(1)\big)^{-1}\eta + \int_0^1 G(t,s)u(s)\,ds$, $t\in I$ and we have

$$\big|x(t)-y(t)\big| \le \big|\Phi(t)\big(M+N\Phi(1)\big)^{-1}(\eta-\mu)\big| + \int_0^1 \|G(t,s)\|\cdot\big|u(s)-v(s)\big|\,ds \le \sup_{t\in I}\big|\Phi(t)(M+N\Phi(1))^{-1}(\eta-\mu)\big|$$
$$+ \frac{G_0L_0}{1-G_0L_0}\sup_{t\in I}\big|\Phi(t)\big(M+N\Phi(1)\big)^{-1}(\eta-\mu)\big| + \frac{G_0}{1-G_0L_0}\int_0^1 q(t)\,dt + \varepsilon$$
$$\le \frac{1}{1-G_0L_0}\sup_{t\in I}\big|\Phi(t)\big(M+N\Phi(1)\big)^{-1}(\eta-\mu)\big| + \frac{G_0}{1-G_0L_0}\int_0^1 q(t)\,dt + \varepsilon\,,$$

which completes the proof. □

Remark 3.3. Taking into account Hypothesis 2.3 ii) the assumptions in Theorem 3.2 is satisfied by $y(\cdot) = 0$ and $q(\cdot) = L(\cdot)$.

References

[1] Boucherif, A., Merabet, N. Chiboub-Fellah, *Boundary value problems for first order multivalued differential systems*, Arch. Math. (Brno) **41** (2005), 187–195.

[2] Castaing, C., Valadier, M., *Convex Analysis and Measurable Multifunctions*, Springer-Verlag, Berlin, 1977.

[3] Cernea, A., *Existence for nonconvex integral inclusions via fixed points*, Arch. Math. (Brno) **39** (2003), 293–298.

[4] Cernea, A., *An existence result for nonlinear integrodifferential inclusions*, Comm. Appl. Nonlinear Anal. **14** (2007), 17–24.

[5] Cernea, A., *On the existence of solutions for a higher order differential inclusion without convexity*, Electron. J. Qual. Theory Differ. Equ. **8** (2007), 1–8.

[6] Covitz, H., Nadler jr., S. B., *Multivalued contraction mapping in generalized metric spaces*, Israel J. Math. **8** (1970), 5–11.

[7] Kannai, Z., Tallos, P., *Stability of solution sets of differential inclusions*, Acta Sci. Math. (Szeged) **61** (1995), 197–207.

[8] Lim, T. C., *On fixed point stability for set valued contractive mappings with applications to generalized differential equations*, J. Math. Anal. Appl. **110** (1985), 436–441.

[9] Tallos, P., *A Filippov-Gronwall type inequality in infinite dimensional space*, Pure Math. Appl. **5** (1994), 355–362.

Faculty of Mathematics and Informatics,
University of Bucharest
Academiei 14, 010014 Bucharest, Romania
E-mail: acernea@fmi.math.unibuc.ro