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τ-SUPPLEMENTED MODULES AND τ-WEAKLY SUPPLEMENTED MODULES

Muhammet Tamer Koşan

Abstract. Given a hereditary torsion theory $\tau = (\mathbb{T}, \mathbb{F})$ in Mod-R, a module M is called *τ-supplemented* if every submodule A of M contains a direct summand C of M with A/C $\tau-$torsion. A submodule V of M is called *τ-supplement* of U in M if $U + V = M$ and $U \cap V \leq \tau(V)$ and M is *τ-weakly supplemented* if every submodule of M has a τ-supplement in M. Let M be a τ-weakly supplemented module. Then M has a decomposition $M = M_1 \oplus M_2$ where M_1 is a semisimple module and M_2 is a module with $\tau(M_2) \leq_e M_2$. Also, it is shown that; any finite sum of τ-weakly supplemented modules is a τ-weakly supplemented module.

Introduction

Throughout this paper, we assume that R is an associative ring with unity, M is a unital right R-module. The symbols, "$\leq$" will denote a submodule, "$\leq_d$" a module direct summand, "$\leq_e$" an essential submodule, "$\ll$" small submodule and "$\operatorname{Rad}(M)$" the Jacobson radical of M.

Let $\tau = (\mathbb{T}, \mathbb{F})$ be a torsion theory. Then τ is uniquely determined by its associated class $\mathbb{T}$ of τ-torsion modules $\mathbb{T} = \{M \in \text{Mod} -R \mid \tau(M) = M\}$ where for a module M, $\tau(M) = \sum\{N \mid N \leq M, N \in \mathbb{T}\}$ and $\mathbb{F}$ is referred as τ-torsion free class and $\mathbb{F} = \{M \in \text{Mod-}R \mid \tau(M) = 0\}$. A module in $\mathbb{T}$ (or $\mathbb{F}$) is called a τ-torsion module (or τ-torsionfree module). Every torsion class $\mathbb{T}$ determines in every module M a unique maximal $\mathbb{T}$-submodule $\tau(M)$, the *τ-torsion submodule* of M, and $\tau(M/\tau(M)) = 0$. In what follows τ will represent a hereditary torsion theory, that is, if $\tau = (\mathbb{T}, \mathbb{F})$ then the class $\mathbb{T}$ is closed under taking submodules, direct sums, homomorphic images and extensions by short exact sequences, equivalently the class $\mathbb{F}$ is closed under submodules, direct products, injective hulls and isomorphic copies.

Let N and K be submodules of M. N is said to be a *supplement submodule* of K in M if $M = N + K$ and $N \cap K \ll N$. M is called a *weakly supplemented module*

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if every submodule of M has a supplement in M. The module M is called a $\oplus$-*supplemented module* if every submodule of M has a supplement that is a direct summand of M. Supplemented modules and its variations have been discussed by several authors in the literature and these modules are useful in characterizing semiperfect modules and rings.

Given a hereditary torsion theory $\tau = (\mathbb{T}, \mathbb{F})$ in Mod-R, τ-complemented modules are studied in [8]. Dually, a module M is said to be a *τ-supplemented module* if every submodule A of M contains a direct summand C of M with A/C τ-torsion [4]. Some further properties of $\tau-$supplemented were studied in [4] and [5].

In this note, we define τ-supplement and τ-weakly supplemented modules. In Section 2, we will show that

Theorem. *Let M be a τ-weakly supplemented module. Then*

(1) *If M is τ-torsionfree, then M is τ-weakly supplemented if and only if M is semisimple.*
(2) *Every homomorphic image of M is again a τ-weakly supplemented module.*
(3) *$M/\tau(M)$ is semisimple*

and

Theorem. *Any finite sum of τ-weakly supplemented modules is a τ-weakly supplemented module.*

In [6], the authors defined and characterized perfect module and ring relative to a torsion theory. In this note, we define semiperfect module relative to a torsion theory and we will prove that

Theorem. *M is a τ-semiperfect module if and only if M is a τ-weakly supplemented module and each τ-supplement submodule of M is a τ-projective cover.*

We refer the reader to [3] and [9] as torsion theoretic sources sufficient for our purposes and [1] and [10] for the other notations in this paper.

1. τ-suplemented modules and τ-weakly suplemented modules

Let $\tau = (\mathbb{T}, \mathbb{F})$ be a hereditary torsion theory in Mod-R and M be a right R-module. Following [4], M is said to be a *τ-supplemented module* if every submodule A of M contains a direct summand C of M with A/C τ-torsion.

Firstly, we give some properties of τ-supplemented modules:

Theorem 1.1.

(1) *Let M be a module. Then the following are equivalent*
   (a) *M is a τ-supplemented module.*
   (b) *Every submodule A of M can be written as $A = B \oplus C$ with B a direct summand of M and $\tau(C) = C$.*
   (c) *For every submodule A of M, there exist a decomposition $M = X \oplus X'$ with $X \leq A$ and $X' \cap A \leq \tau(X')$.*
   (d) *For every submodule A of M, there is an idempotent $e \in \mathrm{End}\,(M_R)$ such that $e(M) \subseteq A$ and $(1-e)(A) \leq \tau\big((1-e)A\big)$.*

(2) *Let M be a τ-supplemented module. Then*

(a) *Every submodule of M is a τ-supplemented module.*

(b) *Every τ-torsionfree submodule of M is a direct summand of M.*

(c) *Every submodule N of M with $N \cap \tau(M) = 0$ is a direct summand of M. In particular, if M is τ-torsionfree, then M is τ-supplemented if and only if M is semisimple.*

(d) *$M/\tau(M)$ is semisimple.*

(e) *For any submodules K, N of M such that $M = N + K$, there exist a submodule X of N with $M = K + X$ and $K \cap X \subseteq \tau(X)$.*

(f) $\mathrm{Rad}\,(M) \leq \tau(M)$.

(g) *If $\tau(M) \neq \mathrm{Rad}\,(M)$, then M has a nonzero direct summand with τ-torsion.*

(h) *$\tau(M) = \mathrm{Rad}\,(M)$ or M has a nonzero τ-torsion submodule that is a direct summand of M.*

Proof. (1)(a)$\Leftrightarrow$(b) and (2)(a) are [4, Lemma 2.1].
(1)(a)$\Leftrightarrow$ (c) and (a)$\Leftrightarrow$(d) are obvious.
(2)(b) Is [4, Lemma 2.5].
(2)(c) Is [4, Corollary 2.6].
(2)(d) By [5, Theorem 4.8].
(2)(e) Let M be a τ-supplemented and K, N be submodules of M with $M = N+K$. By (2)(a), N is a τ-supplemented module. Then there exist a submodule X of N such that $N = N \cap K + X$ and $N \cap K \cap X$ is τ-torsion and so $N \cap K \cap X \leq \tau(X)$. Note that $M = X + K$. It is clear that $K \cap X = N \cap K \cap X \leq \tau(X)$.
(2)(f) By (2)(d), $M/\tau(M)$ is semisimple and so $\mathrm{Rad}\,(M) \leq \tau(M)$.
(2)(g) Assume that $\tau(M) \neq \mathrm{Rad}\,(M)$. Then there exist a maximal submodule P of M such that $\tau(M)$ is not contained in P. Since M is τ-supplemented, there exists a submodule X of K such that $M = X \oplus X'$ and $P \cap X' \leq \tau(X')$ by (1)(c). Note that $P \cap X'$ is also maximal submodule of X'. We may assume that $\tau(X') = X'$. Thus $M = X \oplus X'$, where $X' = \tau(X')$.
(2)(h) Clear from (2)(d) and (g). Also, it follows from [5, Theorem 4.9]. □

As we mentioned in introduction, a submodule V of M is called *supplement* of U in M if V is a minimal element in the set of submodules L of M with $U + L = M$. So V is a supplement of U if and only if $U + V = M$ and $U \cap V$ is small in V. An R-module M is *weakly supplemented* if every submodule of M has a supplement in M.

After considering several possible definitions for a supplement module in a torsion theory, by Theorem 2.1, we propose as; a submodule V of M is called *τ-supplement* of U in M if $U + V = M$ and $U \cap V \leq \tau(V)$ and M is said to be a *τ-weakly supplemented module* if every submodule of M has a τ-supplement in M. Clearly, every τ-supplemented is a τ-weakly supplemented.

Lemma 1.2. *Let M be a module and $V \leq M$.*

(1) *If V is a τ-torsionfree τ-supplement submodule, then V is a direct summand of M.*
(2) *If $\tau(M) = 0$, then every τ-supplement submodule of M is a direct summand.*
(3) *If V is a τ-supplement submodule of M and $V' \subseteq V$, then V/V' is also τ-supplement submodule of M/M'.*

Proof. Trivial. □

Theorem 1.3. *Let M be a τ-weakly supplemented module. Then*

(a) *If M is τ-torsionfree, then M is τ-weakly supplemented if and only if M is semisimple.*
(b) *Every homomorphic image of M is again a τ-weakly supplemented module.*
(c) *$M/\tau(M)$ is semisimple.*

Proof. They are consequences of Lemma 2.2. □

The class of τ-supplemented module is not closed under direct sums. Therefore, there are some decompositions theorems for τ- supplemented modules, for example: A τ-supplemented module M has a decomposition $M = M_1 \oplus M_2$ where M_1 is a semisimple module and M_2 is a τ-supplemented module with $\tau(M_2) \leq_e M_2$ (see [4, Lemma 2.7]).

Lemma 1.4.

(1) *Let M be a τ-weakly supplemented module. Then M has a decomposition $M = M_1 \oplus M_2$ where M_1 is a semisimple module and M_2 is a module with $\tau(M_2) \leq_e M_2$.*
(2) *For submodules N, K of M, if N is a τ-weakly supplemented module and $N + K$ has a τ-supplement in M then K has a τ-supplement in M.*

Proof. (1) For the proof, we completely follow the proof of [4, Lemma 2.7]. If $\tau(M) \leq_e M$, then proof is clear. Assume not. Let $N \leq M$ be a complement of $\tau(M)$. Therefore $N \oplus \tau(M) \leq_e M$. By Theorem 2.3, N is a semisimple module. Since M is τ-supplemented module, there exists a submodule X of M such that $M = N + X$ and $N \cap X \leq \tau(X)$. Note that $N \cap X = N \cap (N \cap X) \leq N \cap \tau(X) \leq N \cap \tau(M) = 0$. This implies $M = N \oplus X$ and $\tau(M) = \tau(N) \oplus \tau(X) = \tau(X)$ because $\tau(N) = 0$. Therefore, we have $\tau(X) \leq_e X$.
(2) Because $N + K$ has a τ-supplement in M, let A be a submodule of M with $M = (N + K) + A$ and $(N + K) \cap A \leq \tau(A)$. Since N is τ-weakly supplemented module, there exists a submodule B of N such that $[(K + A) \cap N] + B = N$ and $[(K+A)\cap N]\cap B \leq \tau(B)$. Hence $M = K+A+B$ and B is a τ-supplement of $K+A$ in M. We claim that $A+B$ is a τ-supplement of K in M. Since $B+K \leq N+K$, we have $A \cap (B + K) \leq \tau(A)$. Now, $(A + B) \cap K \leq \tau(A) + \tau(B) \leq \tau(A + B)$. □

The following theorem generalizes a part of [2, 17.13].

Theorem 1.5. *Any finite sum of τ-weakly supplemented modules is τ-weakly supplemented module.*

Proof. Let M_1 and M_2 be τ-weakly supplemented modules and $M = M_1 + M_2$. Let N be a submodule of M. Clearly, $M_1 + M_2 + N$ has a τ-supplement 0 in M. By Lemma 2.4, $M_2 + N$ has a τ-supplement in M. Again by Lemma 2.4, N has a τ-supplement in M. This implies that $M = M_1 + M_2$ is τ-weakly supplemented module. □

We recall that a module M is *τ-projective* if and only if it is projective with respect to every R-epimorphism having a τ-torsion kernel [3].

Lemma 1.6. *Let M be a module and L a direct summand of M and K a submodule of M such that M/K is $\tau-$projective and $M = L + K$ and $L \cap K$ is $\tau-$torsion. Then $L \cap K$ is direct summand of M.*

Proof. Let $M = L \oplus L'$ and $\alpha\colon M/L' \to L$ be the isomorphism and $\beta\colon L \to M/K \cong L/(L \cap K)$ the epimorphism that having $L \cap K$ as kernel. Then we have epimorphism $\beta\alpha\colon M/L' \to M/K$ having kernel $((L\cap K)\oplus L')/L' \cong L\cap K$ which is $\tau-$torsion. Since M/K is $\tau-$projective, there exists $g\colon M/K \to M/L'$ such that $1 = \beta\alpha g$. Hence $L \cap K$ is direct summand. □

An epimorphism $f\colon P \to M$ is called *a τ-projective cover* of M if P is τ-projective and $\mathrm{Ker}\,(f)$ is small τ-torsion submodule of P (see [3, Page 117]).

Lemma 1.7.

(1) *If $f\colon P \to N$ is a τ-projective cover and $g\colon N \to M$ is a τ-projective cover, then $gf\colon P \to M$ is a τ-projective cover.*

(2) *The following are equivalent for a module M and $N \leq M$.*
  (a) *If M/N has a τ-projective cover.*
  (b) *N has a τ-supplement K in M which has a τ- projective cover.*
  (c) *If N' is a submodule of M with $M = N+N'$, then N has a τ-supplement X such that $X \leq N'$ and X has a τ-projective cover.*

Proof. (1) For the proof, we claim that $\mathrm{Ker}\,(gf)$ is small τ-torsion. By [7, Lemma 4.2], $\mathrm{Ker}\,(gf)$ is small. Let $x \in \mathrm{Ker}\,(gf)$. Then $f(x) \in \mathrm{Ker}\,(g) \leq \tau(N) = f\big(\tau(P)\big)$. For any $p \in \tau(P)$, we have $f(x) = f(p)$, and so $x - p \in\ (f)\tau(P)$, that is $x \in \tau(P)$.
(2)(a)⇒(c) is [6, Lemma 3.1].
(2)(a)⇒(b) is [6, Lemma 3.3].
(2)(c)⇒(b) is clear.
(2)(b)⇒(a) assume N has a τ-supplement K in M which has a τ-projective cover, that is $f\colon P \to K$ with $\mathrm{Ker}\,(f)$ is small τ-torsion. Let $g\colon K \to K/(N \cap K)$. It is easy to see that, $\mathrm{Ker}\,(g)$ small τ-torsion. Since $N/N \cap K = M/N$, we have $gf\colon P \to M/N$ is τ-projective cover of M/N by (1). □

Following [6], a module M is said to be a *$\tau - \oplus$-supplemented* when for every submodule N of M there exists a direct summand K of M such that $M = N + K$ and $N \cap K$ is τ-torsion, and M is called a *completely $\tau - \oplus$-supplemented* if every direct summand of M is $\tau - \oplus$-supplemented and the module M is called *strongly $\tau - \oplus$-supplemented* if for any submodule N of M there exists a direct summand K of M with $M = N + K$ and $N \cap K$ is small τ-torsion in K by [6].

Theorem 1.8. *Let P be a projective $R-$module. Then the following are equivalent:*
(1) *P is $\tau-$supplemented.*
(2) *P is $\tau-\oplus-$supplemented.*

Proof. (1) $\Rightarrow$ (2) Clear from definitions.
(2) $\Rightarrow$ (1) Let N be submodule of P. By (2), there exists a direct summand K of P such that $P = N + K = K' \oplus K$ and $N \cap K$ is $\tau-$ torsion. By [7, Lemma 4.47], there exists a direct summand L of P such that $P = L \oplus K$ and $L \leq N$. Since N/L is isomorphic to $N \cap K$, N/L is $\tau-$torsion. (2) follows. □

In [6], a ring R is called a *right τ-perfect* ring if every right R-module has a τ-projective cover (compare with [11, Remark 4.5]). Every right τ-perfect ring is right perfect, and any strongly $\tau-\oplus$-supplemented module is $\tau-\oplus$-supplemented.

Theorem 1.9. *Let R be a ring. Then the following are equivalent.*
(1) *R is a right τ-perfect ring.*
(2) *Every projective R-module is a strongly $\tau-\oplus$-supplemented module.*

Proof. (1) $\Rightarrow$ (2) Let N be submodule of the projective module M. By (1), M/N has τ-projective cover. By Lemma 2.7, there exists a submodule L of M such that $M = N + L$ with $N \cap L$ is small and τ-torsion in L. Again by Lemma 2.3, N contains a submodule K such that $M = K + L$ with $K \cap L$ is small and τ-torsion in K. By [6, Lemma 3.2], $K \cap L = 0$. Hence $M = N + L = K \oplus L$ and $N \cap L$ is small and $\tau-$torsion in L. It follows that M is strongly $\tau-\oplus$-supplemented.
(2) $\Rightarrow$ (1) Let M be any R-module, P a projective module and f an epimorphism $f : P \longrightarrow M$. By (2), P has direct summands K and K' so that $P = \mathrm{Ker}\,(f) + K = K' \oplus K$ with $\mathrm{Ker}\,(f) \cap K$ small and τ- torsion in K. Hence K is the required τ-projective cover of M. □

Similar to τ-perfect module, we call a module M *τ-semiperfect* if every homomorphic image of M has a τ-projective cover.

Theorem 1.10. *The following are equivalent for a module M*

(1) *M is a τ-semiperfect module;*
(2) *M is a τ-weakly supplemented module and each τ-supplement submodule of M has τ-projective cover.*
(3) *For any submodules K, N of M such that $M = N + K$, there exist a τ-supplement submodule X of N that X has a τ-projective cover.*

Proof. Clear from Lemma 2.7 and Theorem 2.1. □

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Faculty of Science, Gebze Institute of Techonology
Çayirova Campus, 41400 Gebze- Kocaeli, Türkiye
E-mail: mtkosan@gyte.edu.tr