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CONDITIONS UNDER WHICH $R(x)$ AND $R\langle x\rangle$ ARE ALMOST Q-RINGS

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Abstract. All rings considered in this paper are assumed to be commutative with identities. A ring R is a Q-ring if every ideal of R is a finite product of primary ideals. An almost Q-ring is a ring whose localization at every prime ideal is a Q-ring. In this paper, we first prove that the statements, R is an almost ZPI-ring and $R[x]$ is an almost Q-ring are equivalent for any ring R. Then we prove that under the condition that every prime ideal of $R(x)$ is an extension of a prime ideal of R, the ring R is a (an almost) Q-ring if and only if $R(x)$ is so. Finally, we justify a condition under which $R(x)$ is an almost Q-ring if and only if $R\langle x\rangle$ is an almost Q-ring.

1. Introduction

Let R be a ring and let $f \in R[x]$. Then $C(f)$ denotes the ideal of R generated by the coefficients of f. If $S = \{f \in R[x] : C(f) = R\}$ and $W = \{f \in R[x] : f \text{ is monic}\}$, then S and W are regular multiplicatively closed subsets of $R[x]$ and the rings $S^{-1}R[x]$ and $W^{-1}R[x]$ are denoted by $R(x)$ and $R\langle x\rangle$ respectively. Some basic properties and related Theorems of $R(x)$ and $R\langle x\rangle$ can be found in [2].

Recall that a ring R is called a Laskerian ring if every ideal of R is a finite intersection of primary ideals. A ring R is a Q-ring if every ideal of R is a finite product of primary ideals. This class of rings has come as a generalization of an important class of rings called the ZPI-rings that are defined as rings in which every ideal is a product of prime ideals. Equivalently, a ring R is a Q-ring if and only if R is Laskerian and every non maximal prime ideal of R is finitely generated and locally principal, see [1]. If the localization R_P of a ring R is a Q-ring for every prime ideal P of R, then R is called an almost Q-ring. The classes of Q-rings and almost Q-rings were studied in detail in [1] and [5].

One of the main results appeared in [1] is that a ring R is a ZPI-ring if and only if $R[x]$ is a Q-ring. In this paper, we first generalized this result to almost Q-rings and then we have tried to find a condition under which a ring R is a (an

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almost) Q-ring if and only if $R(x)$ is a (an almost) Q-ring. We have investigated that this is true if every prime ideal of $R(x)$ is an extension of a prime ideal of R. Those rings that satisfy this property are said to satisfy the property $(*)$, see [2]. We gave some examples of such rings and in order to achieve our result, we proved that the localization of a ring that satisfies the property $(*)$ at every prime ideal satisfies the property $(*)$ as well.

Finally, we proved that under the condition that a ring R is one dimensional reduced ring, $R(x)$ is an almost Q-ring if and only if $R\langle x\rangle$ is so.

The following Lemma will be needed in the proof of the next main Theorem. It can be proved by using [7, Theorem 3.16].

Lemma 1.1. *Let R be any ring and let Q be a prime ideal of $R[x]$, then $R[x]_Q \cong R_P[x]_{QR_P[x]}$ where $P = Q \cap R$.*

By [4, Theorem 14.1], each maximal ideal of $R(x)$ is of the form $MR(x)$ where M is a maximal ideal of R and $R(x)_{MR(x)} \cong R_M(x) \cong R[x]_{M[x]}$. Hence, $R(x)$ is an almost ZPI-ring if and only if $R_M(x)$ is a ZPI- ring for each maximal ideal M of R.

Theorem 1.2. *Let R be a ring. The following are equivalent*
(1) *R is an almost ZPI-ring.*
(2) *$R(x)$ is an almost ZPI-ring.*
(3) *$R[x]$ is an almost Q-ring.*

Proof. (1) ⇒ (3): Suppose that R is an almost ZPI-ring. Let $\widehat{P}$ be a prime ideal of $R[x]$. Then $P = \widehat{P} \cap R$ is a prime ideal of R and so R_P is a ZPI-ring. By Lemma 1.1, $R[x]_{\widehat{P}} \cong R_P[x]_{\widehat{P}R_P[x]}$ and since R_P is a ZPI-ring, $R_P[x]$ is a Q-ring by [1, Theorem 14]. Hence, $R[x]_{\widehat{P}}$ is a ring of quotients of a Q-ring and so it is a Q-ring. Therefore, $R[x]$ is an almost Q-ring.

(3) ⇒ (2): Suppose that $R[x]$ is an almost Q-ring. Let M be a maximal ideal of R and let $\widehat{M}$ be a maximal ideal of $R[x]$ such that $M[x] \subset \widehat{M}$. Then $R[x]_{\widehat{M}}$ is a Q-ring and hence any non maximal prime ideal of $R[x]_{\widehat{M}}$ is principal by [1, Lemma 5]. Since $M[x] \subset \widehat{M}$, $M[x]$ is a principal ideal of $R[x]_{\widehat{M}}$ and so $M[x]_{M[x]}$ is principal in $R[x]_{M[x]}$. Thus, all prime ideals of $R_M(x) \cong R[x]_{M[x]}$ are principal and so $R_M(x)$ is a PIR. Hence, $R_M(x)$ is a ZPI- ring by [4, Theorem 18.8]. Since M was arbitrary, $R(x)$ is an almost ZPI-ring.

(2) ⇒ (1): Suppose $R(x)$ is an almost ZPI-ring. Let P be a prime ideal of R. Then $PR(x)$ is a prime ideal of $R(x)$. Hence, $R_P(x) \cong R(x)_{PR(x)}$ is a ZPI-ring. Again by [4, Theorem 18.8], R_P is a ZPI-ring and so R is an almost ZPI-ring. □

2. Rings that satisfy the property $(*)$

The definition of rings that satisfy the property $(*)$ was appeared in [2] as follows: A ring R is said to satisfy the property $(*)$ if for each prime ideal P of $R[x]$ with $P \subseteq MR[x]$ for some maximal ideal M of R, we have $P = QR[x]$ for some prime ideal Q of R.

In the following proposition, we can see one characterization of rings that satisfy the property $(*)$.

Proposition 2.1. *A ring R satisfies the property $(*)$ if and only if every prime ideal of $R(x)$ is an extension of a prime ideal of R.*

Proof. $\Rightarrow$): Suppose that R satisfies the property $(*)$. Let $\widehat{P}$ be a prime ideal of $R(x) = S^{-1}R[x]$. Then $\widehat{P} = S^{-1}P$ where P is a prime ideal of $R[x]$ with $P \cap S = \phi$. Let $\{M_\alpha \colon \alpha \in \Lambda\}$ be the set of all maximal ideals of R. Then $S = R[x] \setminus \bigcup_{\alpha\in\Lambda} M_\alpha[x]$ by [4, Theorem 14.1]. Hence, $P \subseteq \bigcup_{\alpha\in\Lambda} M_\alpha[x]$ and then $P \subseteq M_\alpha[x]$ for some $\alpha \in \Lambda$. By assumption, there exists a prime ideal Q of R such that $P = Q[x]$. Hence, $\widehat{P} = S^{-1}P = S^{-1}Q[x] = QR(x)$.

$\Leftarrow$): Conversely, suppose that any prime ideal of $R(x)$ is an extension of a prime ideal of R. Let P be a prime ideal of $R[x]$ with $P \subseteq M[x]$ for some maximal ideal M of R. Then $P \subseteq \bigcup_{\alpha\in\Lambda} M_\alpha[x]$ and so $P \cap \big(R[x] \setminus \bigcup_{\alpha\in\Lambda} M_\alpha[x]\big) = \emptyset$. Hence, $P \cap S = \emptyset$ and then $S^{-1}P$ is a prime ideal of $R(x)$. Thus, by assumption there exists a prime ideal Q of R such that $S^{-1}P = QR(x) = Q(S^{-1}R[x]) = S^{-1}Q[x]$. Hence, $P = S^{-1}P \cap R[x] = S^{-1}Q[x] \cap R[x] = Q[x]$ as required. □

Two examples of rings satisfying the property $(*)$ can be seen in the following proposition

Proposition 2.2. *A zero dimensional ring and a one dimensional Noetherian domain are satisfying the property $(*)$.*

Proof. Suppose that R is a zero dimensional ring. Let $\widehat{P}$ be a non zero prime ideal of $R(x)$. Since R is zero dimensional, $R(x)$ is also zero dimensional by [4, Theorem 17.3] and [7, Theorem 7.13]. Hence, $\widehat{P}$ is a maximal ideal of $R(x)$ and so by [4, Theorem 14.1], $\widehat{P} = MR(x)$ for some maximal ideal M of R. Therefore, R satisfies the property $(*)$ by Proposition 2.1. For one dimensional Noetherian domain, one can use [4, Corollary 17.5] to get a similar proof. □

Recall that a ring R is called an arithmetical ring if each finitely generated ideal of R is locally principal. Equivalently, a ring R is arithmetical if and only if every ideal of $R(x)$ is of the form $IR(x)$ for some ideal I of R. It follows that any arithmetical ring satisfies the property $(*)$.

Proposition 2.3. *Let R be a ring that satisfies the property $(*)$. Then R_P satisfies the property $(*)$ for each prime ideal P of R.*

Proof. Let P be a prime ideal of R and let $\widehat{M}$ be any prime ideal of $R_P(x) \simeq R(x)_{PR(x)}$. Then $\widehat{M} = M_{PR(x)}$ for some prime ideal M of $R(x)$ such that $M \subseteq PR(x)$. Since R satisfies the property $(*)$, $M = QR(x)$ for some prime ideal Q of R. Hence, $\widehat{M} = QR(x)_{PR(x)} = Q_P R_P(x)$ and Q_P is a prime ideal of R_P since $Q \subseteq P$. So, R_P satisfies the property $(*)$ by Proposition 2.1.

Let R be a ring and let $X = \operatorname{spec}(R)$ denotes the set of all prime ideals of R. For each subset $L \subseteq R$, we let $V(L) = \{P \in \operatorname{spec}(R) \colon L \subseteq P\}$. Then the collection $\tau = \{V(L) \colon L \subseteq R\}$ satisfies the axioms for closed sets in some topology on X which is called the prime spectral topology on X. Now, if $X = \operatorname{spec}(R)$ with the above topology is Noetherian (the closed subsets of X satisfy the DCC), we say that

R has a Noetherian spectrum. Equivalently, a ring R has a Noetherian spectrum if and only if it satisfies the ACC for the radical ideals. If R has a Noetherian spectrum, then there are only finitely many prime ideals that are minimal over any ideal of R, see [8]. In [1], we can see that any Q-ring has a Noetherian spectrum. □

Proposition 2.4. *Let R be a ring that satisfies the property $(*)$. Then R has a Noetherian spectrum if and only if $R(x)$ has a Noetherian spectrum.*

Proof. $\Rightarrow$): Suppose that R has a Noetherian spectrum. Then by [8, Theorem 2.5], $R[x]$ has a Noetherian spectrum and so the ring of quotients $R(x)$ of $R[x]$ has a Noetherian spectrum.

$\Leftarrow$): Conversely, suppose that $R(x)$ has a Noetherian spectrum. Let $I_1 \subseteq I_2 \subseteq I_3 \subseteq \dots$ be an ascending chain of radical ideals of R. The $I_1R(x) \subseteq I_2R(x) \subseteq I_3R(x) \subseteq \dots$ is an ascending chain of radical ideals of $R(x)$. Indeed, let I be an ideal of R such that $I = \text{rad}I$ and let $P_1R(x)$, $P_2R(x), \dots, P_nR(x)$ be the set of all minimal prime ideals of $R(x)$ over $IR(x)$. Then clearly, $P_1, P_2, \dots, P_n$ are the set of all minimal prime ideals of R over I. Hence, by [4, Theorem 14.1], we have $\text{rad}\big(IR(x)\big) = \bigcap\limits_{i=1}^{n} P_iR(x) = (\bigcap\limits_{i=1}^{n} P_i)R(x) = (\text{rad}I)R(x) = IR(x)$. Since $R(x)$ has a Noetherian spectrum, there exists $m \in N$ such that $I_mR(x) = I_{m+1}R(x) = \dots$ Hence, $I_m = I_{m+1} = \dots$ and so R has a Noetherian spectrum. □

By using the above proposition, we can prove the following main theorem

Theorem 2.5. *Let R be a ring that satisfies the property $(*)$. Then R is a Q-ring if and only if $R(x)$ is a Q-ring.*

Proof. $\Rightarrow$): Suppose that R is a Q-ring. Let $\widehat{P}$ be any non maximal prime ideal of $R(x)$. Since R satisfies the property $(*)$, then $\widehat{P} = PR(x)$ where P is a non maximal prime ideal of R by Proposition 2.1. Since R is a Q-ring, then P is finitely generated and locally principal and hence $PR(x)$ is finitely generated and locally principal by [2, Theorem 2.2]. Since R has a Noetherian spectrum, then $R[x]$ and its ring of quotients $R(x)$ have a Noetherian spectrum. Since also any non maximal prime ideal of $R(x)$ is finitely generated, then $R(x)$ is Laskerian by [3, Corollary 2.3]. Therefore, $R(x)$ is a Q-ring.

$\Leftarrow$): Suppose that $R(x)$ is a Q-ring. Then $R(x)$ has a Noetherian spectrum and so by Proposition 2.4, R has a Noetherian spectrum. If P is a non maximal prime ideal of R, then $PR(x)$ is a non maximal prime ideal of $R(x)$. So, $PR(x)$ is finitely generated and locally principal and then P is finitely generated and locally principal again by [2, Theorem 2.2]. Thus, R is Laskerian again by [3, Corollary 2.3] and each non maximal prime ideal of R is finitely generated and locally principal. Therefore, R is a Q-ring. □

By using Proposition 2.3 and Theorem 2.5, we have

Theorem 2.6. *Let R be a ring that satisfies the property $(*)$. Then R is an almost Q-ring if and only if $R(x)$ is so.*

Proof. $\Rightarrow$): Suppose that R is an almost Q-ring. Let $PR(x)$ be a prime ideal of $R(x)$. Then $R(x)_{PR(x)} \simeq R_P(x)$. Since R_P satisfies the property $(*)$ by Proposition 2.3 and R_P is a Q-ring, Then by Theorem 2.5, $R_P(x)$ is a Q-ring. Hence, $R(x)$ is an almost Q-ring.

$\Leftarrow$): Suppose that $R(x)$ is an almost Q-ring. Let P be a prime ideal of R. Then $PR(x)$ is a prime ideal of $R(x)$ and so $R(x)_{PR(x)}$ is a Q-ring. Therefore, $R_P(x)$ is a Q-ring. Again, since R_P satisfies the the property $(*)$ and by using Theorem (2.5), we see that R_P is a Q-ring and so R is an almost Q-ring. □

Remark 2.7. If a ring R is a zero dimensional ring, then $R(x)$ and $R\langle x\rangle$ are coincide, see (i.e. [4, Theorem 17.11]). Hence, in this case, the following are equivalent

(1) R is a (an almost) Q-ring.
(2) $R(x)$ is a (an almost) Q-ring.
(3) $R\langle x\rangle$ is a (an almost) Q-ring.

Finally, we show that if a ring R satisfies a certain condition, then $R(x)$ is an almost Q-ring if and only if $R\langle x\rangle$ is so. Recall that a ring R is said to be reduced if its nilradical is 0, the zero ideal of R.

Theorem 2.8. *Let R be a reduced one dimensional ring. Then $R(x)$ is an almost Q-ring if and only if $R\langle x\rangle$ is an almost Q-ring.*

Proof. $\Leftarrow$): Suppose that $R\langle x\rangle$ is an almost Q-ring. Since $R(x)$ is a ring of quotients of $R\langle x\rangle$and clearly the ring of quotients of an almost Q-ring is again an almost Q-ring, then the result follows.

$\Rightarrow$): Suppose that $R(x)$ is an almost Q-ring. Let $\widehat{P}$ be a prime ideal of $R\langle x\rangle$. Then $\widehat{P} = W^{-1}Q$ where Q is a prime ideal of $R[x]$ such that $Q \cap W = \phi$. Now, $R\langle x\rangle_{\widehat{P}} = (W^{-1}R[x])_{W^{-1}Q} \simeq R[x]_Q$. Hence, it is enough to show that $R[x]_Q$ is a Q-ring for each prime ideal Q of $R[x]$ with $Q \cap W = \emptyset$. Take an arbitrary chain $P_0 \subsetneq P_1$ of prime ideals of R. Then P_0 is minimal and P_1 is a maximal ideal of R since $\dim R = 1$. We look for the prime ideals in $R[x]$ that contract to P_0 or P_1. First, we have the prime ideals $P_0[x]$ and $P_1[x]$ for which we see that $R[x]_{P_i[x]} \simeq R_{P_i}(x)$ is a Q-ring for $i = 1, 2$.

If Q_1 is any other prime ideal of $R[x]$ such that $Q_1 \cap R = P_1$, then Q_1 is a maximal ideal of $R[x]$ since P_1 is a maximal ideal of R, $P_1[x] \subsetneq Q_1$ and there is no chain of three distinct prime ideals of $R[x]$ with the same contraction in R, see [7, Corollary 7.12]. By Theorem 28 in [6], Q_1 contains a monic polynomial and so need not be considered. It remains to consider the prime ideals of $R[x]$ that contract to P_0. Let Q_0 be a prime ideal of $R[x]$ such that $Q_0 \cap R = P_0$. Then $Q_0 \cap (R \setminus P_0) = \phi$ in $R[x]$ and so $(R \setminus P_0)^{-1}Q_0$ is a prime ideal in $(R \setminus P_0)^{-1}R[x] = R_{P_0}[x]$. Hence, we have, $R[x]_{Q_0} \simeq ((R \setminus P_0)^{-1}R[x])_{(R\setminus P_0)^{-1}Q_0} \simeq (R_{P_0}[x])_{(R\setminus P_0)^{-1}Q_0}$. Since P_0 is minimal and R is reduced, then R_{P_0} is a field, see [6]. Hence, $R_{P_0}[x]$ is a PID and so it is a Q-ring. Thus, $R[x]_{Q_0}$ is a ring of quotients of a Q-ring and then it is a Q-ring. Hence, for each prime ideal Q of $R[x]$ such that $Q \cap W = \phi$, $R[x]_Q$ is a Q-ring and it follows that $R\langle x\rangle$ is an almost Q-ring. □

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