ARCHIVUM MATHEMATICUM (BRNO)
Tomus 42 (2006), Supplement, 169 – 183

SPECTRUM GENERATING ON TWISTOR BUNDLE

THOMAS BRANSON AND DOOJIN HONG

Abstract. Spectrum generating technique introduced by Ólafsson, Ørsted, and one of the authors in the paper [5] provides an efficient way to construct certain intertwinors when K-types are of multiplicity at most one. Intertwinors on the twistor bundle over $S^1 \times S^{n-1}$ have some K-types of multiplicity 2. With some additional calculation along with the spectrum generating technique, we give explicit formulas for these intertwinors of all orders.

1. Introduction

It was shown in [5] that one can construct intertwining operators of principal series representations induced from maximal parabolic subgroups without too much effort when K-types occur with multiplicity at most one. On the differential form bundle over $S^1 \times S^{n-1}$, a double cover of the compactified Minkowski space, some K-types occur with multiplicity two. One of the authors showed that the spectrum generating technique can also handle this multiplicity 2 case provided that some extra computation is performed.

It is thus natural to do the same thing on general tensor-spinor bundle. Intertwinors on spinors like the Dirac operator have eigenspaces with multiplicity one over $S^1 \times S^{n-1}$and explicit spectral function was given in [7]. On twistors, however, the eigenspaces of the intertwinors including Rarita Schwinger operator have multiplicity two on some K-types. In this paper, we present the spectral function for these operators.

We briefly review conformal covariance and intertwining relation (for more details, see [2], [5]).

Let M be an n-dimensional spin manifold. We enlarge the structure group $\mathrm{Spin}(n)$ to $\mathrm{Spin}(n) \times \mathbb{R}_+$ in conformal geometry. $(V(\lambda), \lambda^r)$ are finite dimensional $\mathrm{Spin}(n) \times \mathbb{R}_+$ representations, where $(V(\lambda), \lambda)$ are finite dimensional representations of $\mathrm{Spin}(n)$ and $\lambda^r(h, \alpha) = \alpha^r \lambda(h)$ for $h \in \mathrm{Spin}(n)$ and $\alpha \in \mathbb{R}_+$. The corresponding associated vector bundles are $\mathbb{V}(\lambda) = P_{\mathrm{Spin}(n)} \times_\lambda V(\lambda)$ and

Research supported by the Eduard Cech Center for Algebra and Geometry Grant Nr. LC505 and by the KRF 2005-070-C00007. The author gratefully acknowledges the services of the Eduard Cech Center and the Seoul National University.

This paper is in final form and no version of it will be submitted elsewhere.

$\mathbb{V}^r(\lambda) = P_{\mathrm{Spin}(n)\times\mathbb{R}_+} \times_{\lambda^r} V(\lambda)$ with structure groups $\mathrm{Spin}(n)$ and $\mathrm{Spin}(n) \times \mathbb{R}_+$. r is called the conformal weight of $\mathbb{V}^r$. Tangent bundle TM carries conformal weight -1 and cotangent bundle T^*M carries conformal weight $+1$. In general, if V is a subbundle of $(TM)^{\otimes p} \otimes (T^*M)^{\otimes q} \otimes (\Sigma M)^{\otimes r} \otimes (\Sigma^* M)^{\otimes s}$, then V carries conformal weight $q - p$, where ΣM is the contravariant spinor bundle.

A conformal covariant of bidegree (a, b) is a $\mathrm{Spin}(n)\times\mathbb{R}_+$-equivariant differential operator $D : \mathbb{V}^r(\lambda) \to \mathbb{V}^s(\sigma)$ which is a polynomial in the metric g, its inverse g^{-1}, the volume element E, and the fundamental tensor-spinor γ with a conformal covariance law

$$\omega \in C^\infty(M), \quad \overline{g} = e^{2\omega} g, \quad \overline{E} = e^{n\omega} E, \quad \overline{\gamma} = e^{-\omega}\gamma \Rightarrow \overline{D} = e^{-b\omega} D\mu(e^{a\omega}),$$

where $\mu(e^{a\omega})$ is multiplication of $e^{a\omega}$.

Given a conformal covariant of bidegree (a, b), $D : \mathbb{V}^r(\lambda) \to \mathbb{V}^s(\sigma)$, we can assign new conformal weights to get $D : \mathbb{V}^{r'}(\lambda) \to \mathbb{V}^{s'}(\sigma)$ whose bidegree is then $(a - r' + r, b - s' + s)$. Calling this D again is an abuse of notation. If $r' = r + a$ and $s' = s + b$, then $D : \mathbb{V}^{r+a}(\lambda) \to \mathbb{V}^{s+b}(\sigma)$ becomes conformally invariant and we call $(a + r, b + s)$ the reduced conformal bidegree of D. To see how conformal covariants behave under a conformal transformation and a conformal vector field, we recall followings.

A diffeomorphism $h : M \to M$ is called a conformal transformation if $h \cdot g = e^{2\omega_h} g$, where "$\cdot$" is the natural action of h on tensor fields; in particular, $h\cdot = (h^{-1})^*$ on purely covariant tensors like g. A conformal vector field is a vector field X with $\mathcal{L}_X g = 2\omega_X g$ for some $\omega_X \in C^\infty(M)$. A conformal covariant $D : \mathbb{V}^0(\lambda) \to \mathbb{V}^0(\sigma)$ of reduced bidegree (a, b) satisfies

$$D(e^{a\omega_h} h \cdot \varphi) = e^{b\omega_h} h \cdot (D(\varphi)) \quad \text{and} \quad D(\mathcal{L}_X + a\omega_X)\varphi = (\mathcal{L}_X + b\omega_X)D\varphi .$$

for all $\varphi \in \Gamma(\mathbb{V}^0(\lambda))$. Thus if $D : \mathbb{V}^r(\lambda) \to \mathbb{V}^s(\sigma)$ of reduced bidegree (a, b), then

$$D(\mathcal{L}_X + (a - r)\omega_X)\varphi = (\mathcal{L}_X + (b - s)\omega_X)D\varphi \tag{1.1}$$

for $\varphi \in \Gamma(\mathbb{V}^r(\lambda))$ and $D\varphi \in \Gamma(\mathbb{V}^s(\sigma))$.

Note that conformal vector fields form a Lie algebra $\mathfrak{c}(M, g)$ and give rise to the principal series representation

$$U^\lambda_a : \mathfrak{c}(M, g) \to \mathrm{End}\Gamma(\mathbb{V}^0(\lambda)) \quad \text{by} \quad X \mapsto \mathcal{L}_X + a\omega_X .$$

So a conformal covariant $D : \mathbb{V}^r(\lambda) \to \mathbb{V}^s(\sigma)$ of reduced bidegree (a, b) intertwines the principal series representation

$$DU^\lambda_{a-r}\varphi = U^\sigma_{b-s} D\varphi$$

for $\varphi \in \Gamma(\mathbb{V}^r(\lambda))$ and $D\varphi \in \Gamma(\mathbb{V}^s(\sigma))$.

2. Spinors and twistors

Let $M = S^1 \times S^{n-1}$, n even, be a manifold endowed with the Lorentz metric $-dt^2 + g_{S^{n-1}}$.

To get a fundamental tensor-spinor α for M from the corresponding object γ on S^{n-1}, let

$$\alpha^j = \begin{pmatrix} \gamma^j & 0 \\ 0 & -\gamma^j \end{pmatrix}, \qquad j = 1, \dots, n-1,$$

and

$$\alpha^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Since M is even-dimensional, there is a *chirality* operator χ_M, equal to some complex unit times $\alpha^0 \tilde{\chi}_S$, where

$$\tilde{\chi}_S = \begin{pmatrix} \chi_S & 0 \\ 0 & -\chi_S \end{pmatrix},$$

χ_S being the chirality operator on S. The chirality operator is always normalized to have square 1; thus $(\chi_S)^2$ and $(\tilde{\chi}_S)^2$ are identity operators, and since $\alpha^0\alpha^0 = 1$, we have $(\alpha^0\tilde{\chi}_S)^2 = -1$. As a result, we may take

$$\chi_M = \pm\sqrt{-1}\alpha^0\tilde{\chi}_S.$$

A *spinor* on M can be viewed as a pair of time-dependent spinors on S^{n-1}, i.e., $\begin{pmatrix} \varphi \\ \psi \end{pmatrix}$, where φ and ψ are t-dependent spinors on S^{n-1}. But by chirality consideration ([6]), we get $\Xi = \pm 1$ spinors:

$$\begin{pmatrix} \varphi \\ \psi \end{pmatrix} = \begin{pmatrix} \Xi\psi/\sqrt{-1} \\ \psi \end{pmatrix}.$$

Recall that *twistors* are spinor-one-forms Φ_λ with $\alpha^\lambda\Phi_\lambda = 0$. Given a chirality Ξ, a twistor Ψ is determined by a t-dependent spinor-one-form ψ_j on S^{n-1} via

$$\Psi = dt \wedge \begin{pmatrix} \varphi_0 \\ \psi_0 \end{pmatrix} + \begin{pmatrix} \varphi_j \\ \psi_j \end{pmatrix},$$

where

$$\begin{aligned} \varphi_j &= -\Xi\sqrt{-1}\psi_j, \\ \psi_0 &= \Xi\sqrt{-1}\gamma^k\psi_k, \\ \varphi_0 &= \gamma^k\psi_k. \end{aligned}$$

Let θ_j be a spinor-one-form on S^{n-1}. Then, it can be written as

$$\theta_j = \gamma_j(-\frac{1}{n-1}\gamma^i\theta_i) + (\theta_j + \frac{1}{n-1}\gamma_j\gamma^i\theta_i) =: \gamma_j\theta + \pi_j, \tag{2.2}$$

where θ is a spinor and π_j is a twistor on S^{n-1} since $\gamma^j(\theta_j + \frac{1}{n-1}\gamma_j\gamma^i\theta_i) = 0$. It turned out ([6]) that we can Hodge decompose the twistor bundle over the sphere so that a twistor π_j can be written as

$$\pi_j = \mathcal{T}_j\tau + (-\nabla^i\eta_{ij}),$$

where $\mathcal{T}_j\tau := \nabla_j\tau + \gamma_j D\tau$ (here D is the Dirac operator on the sphere) is the j-th component of the twistor operator applied to a spinor τ and η_{ij} is a spinor-two form with $\gamma^i\eta_{ij} = 0$.

Therefore, a twistor on M can be decomposed as follows:

$$\begin{pmatrix} -(n-1)\theta & -\Xi\sqrt{-1}\gamma_i\theta \\ -(n-1)\Xi\sqrt{-1}\theta & \gamma_i\theta \end{pmatrix} + \begin{pmatrix} 0 & -\Xi\sqrt{-1}\mathcal{T}_i\tau \\ 0 & \mathcal{T}_i\tau \end{pmatrix}$$

$$+ \begin{pmatrix} 0 & -\Xi\sqrt{-1}\nabla^j\eta_{ji} \\ 0 & \nabla^j\eta_{ji} \end{pmatrix} =: \langle\theta\rangle + \{\tau\} + [\eta]\,, \tag{2.3}$$

for some spinors θ, τ and some spinor-two form η.

3. Intertwining relation on twistors

Let us briefly review some standard materials on the conformal structure on the manifold $S^1 \times S^{n-1}$. Let $G = \mathrm{Spin}_0(2,n)$ and P the maximal parabolic subgroup for which G/P is the 4-fold cover of the compactified Minkowski space $(S^1 \times S^{n-1})/\mathbb{Z}_2$, where the $\mathbb{Z}_2$ action comes from the product of antipodal maps on S^1 and on S^{n-1}. $G^{'}/P^{'}$, where $G^{'} = \mathrm{SO}_0(2,n)$ and P' its maximal parabolic subgroup, is the double cover $S^1 \times S^{n-1}$ of $(S^1 \times S^{n-1})/\mathbb{Z}_2$. Then G/P is the double cover of $S^1 \times S^{n-1}$ obtained from the standard covering of S^1 factor. The Lie algebra $\mathfrak{g}$ can be realized in homogeneous coordinates $(\xi_{-1}, \ldots, \xi_n)$ ([1, 9]):

$$L_{\alpha\beta} = \varepsilon_\alpha\xi_\alpha\partial_\beta - \varepsilon_\beta\xi_\beta\partial_\alpha \quad \alpha, \beta = -1, \ldots, n\,,$$

where $\partial_\alpha = \partial/\partial\xi_\alpha$, and $-\varepsilon_{-1} = -\varepsilon_0 = \varepsilon_1 = \cdots = \varepsilon_n = 1$. The $L_{-1,0}$ generates $\mathrm{SO}(2)$ group of isometries and the $L_{\alpha\beta}$ for $\alpha, \beta = 1, \ldots, n$ generate $\mathrm{SO}(n)$ group of isometries. If $\mathfrak{g} = \mathfrak{k} + \mathfrak{s}$ is a Cartan decomposition of $\mathfrak{g}$, then $\mathfrak{k}$ corresponds to the $\mathfrak{so}(2) \times \mathfrak{so}(n)$ and $\mathfrak{s}$ corresponds to the *proper* conformal vector fields:

$$\mathcal{L}_{L_{\alpha\beta}} g = 2\omega_{\alpha\beta} g, \quad \text{with } \omega_{\alpha\beta} \neq 0\,,$$

where $\mathcal{L}$ denotes Lie derivative. So they are just the ones with mixed indices: $L_{\alpha\beta}$ for $-1 \leq \alpha \leq 0 < \beta \leq n$. Let t be the angular parameter on S^1 so that $\xi_{-1} = \cos t$ and $\xi_0 = \sin t$. And set $\xi_n = \cos\rho$ and complete a set of spherical angular coordinates $(\rho, \theta_1, \ldots, \theta_{n-2})$ on S^{n-1} so that ∂_ρ is $g_{S^{n-1}}$-orthogonal to the ∂_{θ_i}. Then we get a typical conformal vector field T and its conformal factor ϖ:

$$L_{-1,n} = \cos\rho\sin t\partial_t + \cos t\sin\rho\partial_\rho := T$$
$$\omega_{-1,n} = \cos t\cos\rho := \varpi\,.$$

Let $A = A_{2r}$ be an intertwinor of order $2r$. That is, an operator satisfying the intertwining relation ((1.1), [2, 3, 5])

$$A\left(\tilde{\mathcal{L}}_T + \left(\frac{n}{2} - r\right)\varpi\right) = \left(\tilde{\mathcal{L}}_T + \left(\frac{n}{2} + r\right)\varpi\right)A\,, \tag{3.4}$$

where $\tilde{\mathcal{L}}_T$ is the *reduced Lie derivative.* On a tensor-spinor with $\begin{pmatrix} p \\ q \end{pmatrix}$ tensor content, this is

$$\tilde{\mathcal{L}}_T = \mathcal{L}_T + (p-q)\varpi\,.$$

So here (with only 1-form content), it is $\mathcal{L}_T - \varpi$. Note that we are using the convention where spinors do not have an internal weight; otherwise the spinor

content would influence the reduction.
Since intertwinors change chirality, we want to consider an exchange operator

$$\begin{aligned} E :&= \alpha^0(\iota(\partial_t)\varepsilon(dt) - \varepsilon(dt)\iota(\partial_t)) \\ &= \alpha^0(1 - 2\varepsilon(dt)\iota(\partial_t))\,, \end{aligned}$$

where ι is the interior multiplication and ε is the exterior multiplication. It is immediate that $E^2 = \mathrm{Id}$. Because of the α^0 factor, E reverses chirality. To see that E takes twistors to twistors, note that, for a twistor Φ,

$$\iota(\partial_t)\varepsilon(dt) - \varepsilon(dt)\iota(\partial_t) : \Phi_\lambda \mapsto \Phi_\lambda - 2\delta_\lambda{}^0\Phi_0\,.$$

Thus

$$\begin{aligned} \alpha^\lambda(E\Phi)_\lambda &= \alpha^\lambda\alpha^0(\Phi_\lambda - 2\delta_\lambda{}^0\Phi_0) \\ &= -2g^{\lambda 0}(\Phi_\lambda - 2\delta_\lambda{}^0\Phi_0) + 2\alpha^0\alpha^\lambda\delta_\lambda{}^0\Phi_0 \\ &= \underbrace{-2\Phi^0}_{2\Phi_0} + 4\underbrace{g^{00}}_{-1}\Phi_0 + 2\underbrace{\alpha^0\alpha^0}_{1}\Phi_0 \\ &= 0, \end{aligned}$$

as desired.

We want to convert the relation (3.4) for EA. So we will eventually need $\mathcal{L}_T E$. We have:

$$\begin{aligned} \mathcal{L}_T E = \mathcal{L}_T\left\{\alpha(dt)(1 - 2\varepsilon(dt)\iota(\partial_t))\right\} \\ = \{-\varpi\alpha(dt) + \alpha(d(Tt))\}(1 - 2\varepsilon^0\iota_0) \\ - 2\alpha^0\{\varepsilon(dt)\iota([T,\partial_t]) + \varepsilon(d(Tt)\iota(\partial_t)\}. \end{aligned}$$

But

$$\begin{aligned} Tt &= \cos\rho\sin t, \\ d(Tt) &= -\sin\rho\sin t\,d\rho + \cos\rho\cos t\,dt, \\ [T,\partial_t] &= -\cos\rho\cos t\,\partial_t + \sin t\sin\rho\,\partial_\rho\,. \end{aligned}$$

This reduces the above to

$$\begin{aligned} \mathcal{L}_T E &= \sin t\alpha(d\omega)(1 - 2\varepsilon^0\iota_0) - 2\sin t\alpha^0(\varepsilon^0\iota(Y) + \varepsilon(d\omega)\iota_0) \\ &= \sin t\sin\rho\{-\alpha^1(1 - 2\varepsilon^0\iota_0) - 2\alpha^0(\varepsilon^0\iota_1 - \varepsilon^1\iota_0)\}. \end{aligned} \tag{3.5}$$

By Kosmann ([8], eq(16)), the Lie and covariant derivatives on spinors are related by

$$\mathcal{L}_X - \nabla_X = -\tfrac{1}{4}\nabla_{[a}X_{b]}\gamma^a\gamma^b = -\tfrac{1}{8}(dX_\flat)_{ab}\gamma^a\gamma^b\,.$$

Note that

$$\begin{aligned} T_\flat &= -\cos\rho\sin t\,dt + \cos t\sin\rho\,d\rho, \\ dT_\flat &= 2\sin\rho\sin t\,d\rho\wedge dt\,. \end{aligned}$$

and

$$d\varpi = -T_{\flat,\mathrm{R}}\,,$$

where ♭,R is the musical isomorphism in the "Riemannian" metric. According to the above,

$$\mathcal{L}_T - \nabla_T = -\frac{1}{2}\sin\rho\sin t\alpha^1\alpha^0 \tag{3.6}$$

on spinors.

On a 1-form η,

$$\langle(\mathcal{L}_T - \nabla_T)\eta, X\rangle = -\langle\eta, (\mathcal{L}_T - \nabla_T)X\rangle\,,$$

since $\mathcal{L}_T - \nabla_T$ kills scalar functions. But by the symmetry of the pseudo-Riemannian connection,

$$[T, X] - \nabla_T X = -\nabla_X T\,.$$

We conclude that

$$(\mathcal{L}_T - \nabla_T)\eta = \langle\eta, \nabla T\rangle\,,$$

where in the last expression, $\langle\cdot,\cdot\rangle$ is the pairing of a 1-form with the contravariant part of a $\begin{pmatrix}1\\1\end{pmatrix}$-tensor:

$$((\mathcal{L}_T - \nabla_T)\eta)_\lambda = \eta_\mu\nabla_\lambda T^\mu\,.$$

Combining this with what we derived above for spinors (3.6), for a spinor-1-form Φ_λ, we have

$$((\mathcal{L}_T - \nabla_T)\Phi)_\lambda = \Phi_\mu\nabla_\lambda T^\mu - \frac{1}{2}\sin\rho\sin t\alpha^1\alpha^0\Phi_\lambda\,.$$

But ∇T *a priori* has projections in 3 irreducible bundles, TFS^2, Λ^0, and Λ^2 (after using the musical isomorphisms). By conformality, the TFS^2 part is gone. We expect a Λ^0 part, essentially ϖ. We also found the Λ^2 part above,

$$dT_\flat = 2\sin\rho\sin t\, d\rho\wedge dt\,.$$

More precisely, tracking the normalizations,

$$(\nabla T_\flat)_{\lambda\mu} = (\nabla T_\flat)_{(\lambda\mu)} + (\nabla T_\flat)_{[\lambda\mu]} = (\varpi g + \frac{1}{2}dT_\flat)_{\lambda\mu}\,.$$

Now note that

$$\begin{aligned}\Phi_\mu\nabla_\lambda T^\mu &= \varpi g_\lambda{}^\mu\Phi_\mu + \frac{1}{2}((dT_\flat)_{\nu\mu}\varepsilon^\nu\iota^\mu\Phi)_\lambda\\ &= \varpi\Phi_\lambda + \frac{1}{2}((((dT_\flat)_{01}\varepsilon^0\iota^1 + (dT_\flat)_{10}\varepsilon^1\iota^0)\Phi)_\lambda\\ &= \varpi\Phi_\lambda + \frac{1}{2}((-2\sin\rho\sin t\varepsilon^0\iota^1 + 2\sin\rho\sin t\varepsilon^1\iota^0)\Phi)_\lambda\\ &= \varpi\Phi_\lambda - \sin\rho\sin t((\varepsilon^0\iota^1 - \varepsilon^1\iota^0)\Phi)_\lambda\\ &= \varpi\Phi_\lambda - \sin\rho\sin t((\varepsilon^0\iota_1 + \varepsilon^1\iota_0)\Phi)_\lambda\,.\end{aligned}$$

As a result,

$$\begin{aligned}\mathcal{L}_T - \nabla_T &= \varpi - \sin\rho\sin t\left(\tfrac{1}{2}\alpha^1\alpha^0 + \varepsilon^0\iota_1 + \varepsilon^1\iota_0\right)\\ &=: \varpi - \sin\rho\sin tP\\ &=: \varpi - \mathcal{P}\,,\end{aligned}$$

and

$$\tilde{\mathcal{L}}_T - \nabla_T = -\mathcal{P}\,.$$

An explicit calculation using (3.5) gives

$$(\mathcal{L}_T E)E = -2\mathcal{P}\,.$$

Since $E^2 = \mathrm{Id}$, we conclude that

$$\mathcal{L}_T E = -2\mathcal{P}E\,.$$

With the above, the intertwining relation for EA becomes

$$\begin{aligned}\left(\tilde{\mathcal{L}}_T + \left(\tfrac{n}{2}+r\right)\varpi\right)EA &= E\left(\tilde{\mathcal{L}}_T + \left(\tfrac{n}{2}+r\right)\varpi\right)A + (\mathcal{L}_T E)A\\ &= EA\left(\tilde{\mathcal{L}}_T + \left(\tfrac{n}{2}-r\right)\varpi\right) - 2\mathcal{P}EA,\end{aligned}$$

so that, with $B = EA$,

$$B\left(\nabla_T + \left(\frac{n}{2}-r\right)\varpi - \mathcal{P}\right) = \left(\nabla_T + \left(\frac{n}{2}+r\right)\varpi + \mathcal{P}\right)B\,.$$

To see what P does, let us define two convenient operations.

$$\psi_j \overset{\mathbf{expa}}{\longmapsto} \begin{pmatrix} u & \Xi\psi_j/\sqrt{-1}\\ -\Xi u/\sqrt{-1} & \psi_j\end{pmatrix} \overset{\mathbf{slot}}{\longmapsto} \psi_j\,,$$

where $u = \gamma^k\psi_k$.
Note that

$$\begin{aligned}&\psi_j \overset{\mathbf{expa}}{\longmapsto} \begin{pmatrix} u & \Xi\psi_j/\sqrt{-1}\\ -\Xi u/\sqrt{-1} & \psi_j\end{pmatrix} \overset{\iota_0}{\longmapsto} \begin{pmatrix} u\\ -\Xi u/\sqrt{-1}\end{pmatrix}\\ &\overset{\varepsilon^1}{\longmapsto} \begin{pmatrix} 0 & \varepsilon^1 u\\ 0 & -\Xi\varepsilon^1 u/\sqrt{-1}\end{pmatrix} \overset{\mathbf{slot}}{\longmapsto} -\Xi\varepsilon^1 u/\sqrt{-1}\,.\end{aligned}$$

As for the $\varepsilon^0\iota_1$ term, anything in the range of ε^0 has a **slot** of 0.

Finally,

$$\begin{aligned}&\psi_j \overset{\mathbf{expa}}{\longmapsto} \begin{pmatrix} u & \Xi\psi_j/\sqrt{-1}\\ -\Xi u/\sqrt{-1} & \psi_j\end{pmatrix} \overset{\alpha^0}{\longmapsto} \begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix} u & \Xi\psi_j/\sqrt{-1}\\ -\Xi u/\sqrt{-1} & \psi_j\end{pmatrix}\\ &= \begin{pmatrix} -\Xi u/\sqrt{-1} & \psi_j\\ u & \Xi\psi_j/\sqrt{-1}\end{pmatrix} \overset{\alpha^1}{\longmapsto} \begin{pmatrix} -\Xi\gamma^1 u/\sqrt{-1} & \gamma^1\psi_j\\ -\gamma^1 u & -\Xi\gamma^1\psi_j/\sqrt{-1}\end{pmatrix}.\end{aligned}$$

So

$$\begin{aligned}&\mathbf{slot}\,P\,\mathbf{expa} : \psi_j \mapsto -\tfrac{1}{2}\Xi\gamma^1\psi_j/\sqrt{-1} - \Xi(\varepsilon^1 u)_j/\sqrt{-1}\\ &= -\tfrac{\Xi}{\sqrt{-1}}\left(\tfrac{1}{2}\gamma^1\psi_j + (\varepsilon^1 u)_j\right) = -\tfrac{\Xi}{\sqrt{-1}}\left(\tfrac{1}{2}\gamma^1\psi_j + \delta_j{}^1 u\right).\end{aligned}$$

Up to a factor of a complex unit, **slot** P **expa** is

$$\frac{1}{2}\gamma^1\psi_j + \delta_j{}^1\gamma^k\psi_k\,.$$

We can also get this expression by successively taking the commutator of ϖ with ∂_t and the operator $\mathcal{D}$ defined by

$$\mathbf{slot}\,\mathcal{D}\,\mathbf{expa} : \psi_j \mapsto \frac{1}{2}\gamma^k\nabla_k\psi_j + \gamma^k\nabla_j\psi_k\,.$$

That is,

$$\mathcal{P} = \Xi\sqrt{-1}[\partial_t, [\mathcal{D}, \varpi]]\,.$$

Recall that $\mathcal{P} = \sin\rho\sin tP$.
After some straightforward computation, we get the block matrix for $\mathcal{D}$ relative to the decomposition $\{\langle\theta\rangle, \{\tau\}, [\eta]\}$ (2.3) as follows.

$$\begin{pmatrix} \dfrac{n+1}{2(n-1)}J_\theta & \dfrac{n-2}{4} - \dfrac{n-2}{(n-1)^2}J_\tau^2 & 0 \\ -n & \dfrac{n-3}{2(n-1)}J_\tau & 0 \\ 0 & 0 & \dfrac{1}{2}L \end{pmatrix},$$

where J_θ and J_τ are the Dirac eigenvalues of θ and τ on S^{n-1}, respectively and L is the Rarita-Schwinger eigenvalue of $[\eta]$ on S^{n-1}.

The spectrum generating relation takes the following form:

$$[N, \varpi] = 2\left(\nabla_T + \frac{n}{2}\varpi\right),$$

where $\nabla^{*,\mathrm{R}}\nabla := N$ is the Riemannian Bochner Laplacian. Therefore the relation (3.4) becomes
(3.7)

$$B\left(\frac{1}{2}[N,\varpi] - r\varpi - \Xi\sqrt{-1}[\partial_t,[\mathcal{D},\varpi]]\right) = \left(\frac{1}{2}[N,\varpi] + r\varpi + \Xi\sqrt{-1}[\partial_t,[\mathcal{D},\varpi]]\right)B\,.$$

As explained in detail in ([3]), the recursive numerical spectral data come from the compressed relation of the above.

4. Projections into isotypic summands

Let us denote the $K = \mathrm{Spin}(2)\times\mathrm{Spin}(n)$-type with highest weight as follows:

$$\mathcal{V}_\Xi(f; j, \tfrac{1}{2}+q, \tfrac{1}{2}, \ldots, \tfrac{1}{2}, \tfrac{\varepsilon}{2}) := (f)\otimes \underbrace{(j, \tfrac{1}{2}+q, \tfrac{1}{2}, \ldots, \tfrac{1}{2}, \tfrac{\varepsilon}{2})}_{n/2 \text{ entries}},$$

where $j \in \frac{1}{2}+q+\mathbb{N}$, $\varepsilon = \pm 1$, $q = 0$ or 1, and (f) is a Spin(2)-type generated by the function $e^{\sqrt{-1}ft}$ on S^1 factor.

Proper conformal vector fields and corresponding conformal factors map such a K-type to a sum of different K-types under the classical selection rule ([3]).
Consider a Ξ spinor $\begin{pmatrix}\varphi\\ \psi\end{pmatrix}$. Since $\varphi = \Xi\psi/\sqrt{-1}$, we have

$$\alpha^0\begin{pmatrix}\bullet\\ \psi\end{pmatrix} = \begin{pmatrix}\bullet\\ \Xi\psi/\sqrt{-1}\end{pmatrix}.$$

Here $\bullet$ denotes a top entry that is computable from the bottom entry, but whose value is not needed at the moment.
In addition,

$$\sin t\begin{pmatrix}\bullet\\ \psi\end{pmatrix} = \begin{pmatrix}\bullet\\ \sin t\psi\end{pmatrix} = \begin{pmatrix}\bullet\\ -[\partial_t,\cos t]\psi\end{pmatrix},$$

$$\mathrm{Proj}^f_{f'} \sin t \begin{pmatrix} \bullet \\ \psi \end{pmatrix} = \begin{pmatrix} \bullet \\ \frac{f'-f}{\sqrt{-1}} \cos t|^f_{f'} \psi \end{pmatrix},$$

$$\sin\rho\alpha^1 \begin{pmatrix} \bullet \\ \psi \end{pmatrix} = \begin{pmatrix} \bullet \\ -\sin\rho\gamma^1\psi \end{pmatrix} = \begin{pmatrix} \bullet \\ [D, \cos\rho]\psi \end{pmatrix},$$

$$\mathrm{Proj}^a_b \sin\rho\alpha^1 \begin{pmatrix} \bullet \\ \psi \end{pmatrix} = \begin{pmatrix} \bullet \\ -\mathrm{Proj}^a_b \sin\rho\gamma^1\psi \end{pmatrix} = \begin{pmatrix} \bullet \\ (J_b - J_a)\cos\rho|^a_b\, \psi \end{pmatrix},$$

where $D = \gamma^i\nabla_i$ is the Dirac operator on S^{n-1}, a and b (resp., f and f') are abbreviated labels for the Spin(n)-types (resp., Spin(2)-types) in question and J_a (resp., J_b) is the Dirac eigenvalue on a (resp., b).

For the compressed relations of $\varpi = \cos t \cos\rho$ between Clifford range part, twistor range part, and divergence part (2.3), we note that $\cos\rho$ is the conformal factor corresponding to the conformal vector field $\sin\rho\partial_\rho$ on S^{n-1}. Clifford range piece is essentially spinor on S^{n-1} while twistor range piece and divergence piece are twistors on S^{n-1}. So, for example, $\varpi\langle\theta\rangle$ is a sum of Clifford pieces only. Thus we have:

$$\varpi \begin{pmatrix} \langle\theta\rangle \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \langle|\varpi|\theta\rangle \\ 0 \\ 0 \end{pmatrix} \overset{\mathrm{Proj}}{\longmapsto}: \begin{pmatrix} \langle\tilde\theta\rangle \\ 0 \\ 0 \end{pmatrix},$$

$$\varpi \begin{pmatrix} 0 \\ \{\tau\} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ |\varpi|\{\tau\} \\ |\varpi|\{\tau\} \end{pmatrix} = \begin{pmatrix} 0 \\ C\{|\varpi|\tau\} \\ |\varpi|\{\tau\} \end{pmatrix} \overset{\mathrm{Proj}}{\longmapsto}: \begin{pmatrix} 0 \\ C\{\tilde\tau\} \\ [\eta] \end{pmatrix}, \tag{4.8}$$

$$\varpi \begin{pmatrix} 0 \\ 0 \\ [\eta] \end{pmatrix} = \begin{pmatrix} 0 \\ |\varpi|[\eta] \\ |\varpi|[\eta] \end{pmatrix} \overset{\mathrm{Proj}}{\longmapsto}: \begin{pmatrix} 0 \\ \{\bar\tau\} \\ [\tilde\eta] \end{pmatrix},$$

where C is a quantity we will compute in the following lemma.

Lemma 4.1. *Let* $\alpha = \mathcal{V}_\Xi(f; j, \frac{1}{2}, \cdots, \frac{1}{2}, \frac{\varepsilon}{2})$ *and* $\beta = \mathcal{V}_\Xi(f'; j', \frac{1}{2}, \cdots, \frac{1}{2}, \frac{\varepsilon'}{2})$, $\varepsilon = \pm1$. *Then we have*

$$|_\beta\varpi|_\alpha\{\tau\} = C_{ba}\{|_\beta\varpi|_\alpha\tau\},$$

where

$$C_{ba} = \frac{1}{\lambda_b(\mathcal{T}^*\mathcal{T})}\left(\frac{1}{2}J_b^2 + \frac{1}{2}J_a^2 - \frac{J_bJ_a}{n-1} - \frac{n(n-1)}{4}\right),$$

J_a *(resp.,* J_b*) is the Dirac eigenvalue on Spin(*n*)-type at* α *(resp., Spin(*n*)-type at* β*),* $\lambda_b(\mathcal{T}^*\mathcal{T})$ *is the eigenvalue of* $\mathcal{T}^*\mathcal{T}$ *on Spin(*n*)-type at* β*, and* $\mathcal{T}$ *is the twistor operator (with adjoint* $\mathcal{T}^*$*) over* S^{n-1}.

Proof. It suffices to show that

$$|_b\omega|_a\mathcal{T}\tau = C_{ba}\cdot\mathcal{T}(|_b\omega|_a\tau),$$

where $\omega = \cos\rho$. Let D be the Dirac operator on S^{n-1}. Then

$$\begin{aligned}[D^2, \omega]\tau &= [\nabla^*\nabla, \omega]\tau \quad \text{by Bochner identity} \\ &= (\nabla^*\nabla\omega)\tau - 2\nabla^k\omega\nabla_k\tau = (n-1)\omega\tau + 2\sin\rho\nabla_1\tau,\end{aligned}$$

Also

$$\begin{aligned}
\mathcal{T}^*(\omega\mathcal{T}\tau) &= -\nabla^j(\omega\nabla_j\tau + \tfrac{1}{n-1}\omega\gamma_j D\tau) \\
&= \sin\rho\nabla_1\tau + \omega\nabla^*\nabla\tau + \tfrac{1}{n-1}\sin\rho\gamma_1 D\tau - \tfrac{1}{n-1}\omega D^2\tau \\
&= \tfrac{1}{2}\left([D^2,\omega] - (n-1)\omega\right)\tau + \omega\left(D^2 - \tfrac{(n-1)(n-2)}{4}\right)\tau + \tfrac{1}{n-1}[\omega, D]D\tau \\
&\quad -\tfrac{1}{n-1}\omega D^2\tau \quad \text{by the above and Bochner identity} \\
&= \tfrac{1}{2}D^2(\omega\tau) + \tfrac{1}{2}\omega D^2\tau - \tfrac{1}{n-1}D(\omega D\tau) - \tfrac{n(n-1)}{4}\omega\tau.
\end{aligned}$$

Therefore

$$\begin{aligned}
|_b\omega|_a\mathcal{T}\tau &= \mathcal{T}\left(\frac{1}{\lambda_b(\mathcal{T}^*\mathcal{T})}\mathcal{T}^*(|_b\omega|_a\mathcal{T}\tau)\right) \\
&= \mathcal{T}\left(\frac{1}{\lambda_b(\mathcal{T}^*\mathcal{T})}\left(\frac{1}{2}J_b^2 + \frac{1}{2}J_a^2 - \frac{1}{n-1}J_bJ_a - \frac{n(n-1)}{4}\right)|_b\omega|_a\tau\right).
\end{aligned}$$

□

Remark 1. Eigenvalues of D and $\mathcal{T}^*\mathcal{T}$ on S^{n-1} are known due to Branson ([4]).

With the above (4.8) at hand, we get

(4.9)

$$|_\beta[\mathcal{D},\varpi]|_\alpha\langle\theta\rangle = \begin{pmatrix} (\mathcal{D}_{11}^\beta - \mathcal{D}_{11}^\alpha)\langle\tilde\theta\rangle \\ (\mathcal{D}_{21}^\beta - C_{ba}\mathcal{D}_{21}^\alpha)\{\tilde\theta\} \\ -\mathcal{D}_{21}^\alpha[\eta] \end{pmatrix}, \text{ where } \begin{cases} \langle\tilde\theta\rangle = |_\beta\varpi|_\alpha\langle\theta\rangle \\ [\eta] = |_\beta\varpi|_\alpha\{\theta\} \end{cases},$$

$$|_\beta[\mathcal{D},\varpi]|_\alpha\{\tau\} = \begin{pmatrix} (C_{ba}\mathcal{D}_{12}^\beta - \mathcal{D}_{12}^\alpha)\langle\tilde\tau\rangle \\ C_{ba}(\mathcal{D}_{22}^\beta - \mathcal{D}_{22}^\alpha)\{\tilde\tau\} \\ (\mathcal{D}_{33}^\beta - \mathcal{D}_{22}^\alpha)[\eta] \end{pmatrix}, \text{ where } \begin{cases} \{\tilde\tau\} = |_\beta\varpi|_\alpha\{\tau\} \\ [\eta] = |_\beta\varpi|_\alpha\{\tau\} \end{cases}, \text{ and}$$

$$|_\beta[\mathcal{D},\varpi]|_\alpha[\eta] = \begin{pmatrix} \mathcal{D}_{12}^\beta\langle\bar\tau\rangle \\ (\mathcal{D}_{22}^\beta - \mathcal{D}_{33}^\alpha)\{\bar\tau\} \\ (\mathcal{D}_{33}^\beta - \mathcal{D}_{33}^\alpha)[\tilde\eta] \end{pmatrix}, \text{ where } \begin{cases} \{\bar\tau\} = |_\beta\varpi|_\alpha[\eta] \\ [\tilde\eta] = |_\beta\varpi|_\alpha[\eta] \end{cases}.$$

Here we use subscripts to refer to the specific entries of the $\mathcal{D}$ and superscripts to indicate where these entries are computed.

Let us now consider the compressed relation of (3.7) between K-types related by the selection rule.

Case 1: Multiplicity 2 ↔ 1

$$\alpha = \mathcal{V}_\Xi(f; j, \frac{1}{2}, \cdots, \frac{1}{2}, \frac{\varepsilon}{2}) \leftrightarrow \beta = \mathcal{V}_\Xi(f'; j, \frac{3}{2}, \frac{1}{2}, \cdots, \frac{1}{2}, \frac{\varepsilon}{2}).$$

Note that the operator B in block form looks

$$B = \begin{pmatrix} B_{11} & B_{12} & 0 \\ B_{21} & B_{22} & 0 \\ 0 & 0 & B_{33} \end{pmatrix}.$$

With

$$|_\alpha N|_\beta = f^2 - f'^2 - (n-2)$$

and (4.9), we get $\alpha \to \beta$ transition quantities

$$\beta \to \alpha : \quad \begin{pmatrix} B^\alpha_{11} & B^\alpha_{12} \\ B^\alpha_{21} & B^\alpha_{22} \end{pmatrix} \begin{pmatrix} A_1 \\ E^- \end{pmatrix} = B^\beta_{33} \begin{pmatrix} -A_1 \\ E^+ \end{pmatrix} \text{ and}$$

$$\alpha \to \beta : \quad \begin{pmatrix} A_2 & -E^- \end{pmatrix} \begin{pmatrix} B^\alpha_{11} & B^\alpha_{12} \\ B^\alpha_{21} & B^\alpha_{22} \end{pmatrix} = B^\beta_{33} \begin{pmatrix} -A_2 & -E^+ \end{pmatrix} ,$$

where

$$\begin{aligned} A_1 &:= \Xi(f - f')\mathcal{D}^\alpha_{12} , \\ A_2 &:= -\Xi(f - f')\mathcal{D}^\alpha_{21} , \\ E^- &:= \frac{1}{2}(f^2 - f'^2) - \frac{n-2}{2} - r + \Xi(f - f')(\mathcal{D}^\alpha_{22} - \mathcal{D}^\beta_{33}) , \\ E^+ &:= \frac{1}{2}(f^2 - f'^2) - \frac{n-2}{2} + r - \Xi(f - f')(\mathcal{D}^\alpha_{22} - \mathcal{D}^\beta_{33}) . \end{aligned}$$

In particular, we can write all 2×2 entries of B^α in terms of B^α_{21} and B^β_{33}:

$$\begin{aligned} B^\alpha_{11} &= (E^- B^\alpha_{21} - A_2 B^\beta_{33})/A_2 , \\ B^\alpha_{12} &= -A_1 B^\alpha_{21}/A_2 , \text{ and} \\ B^\alpha_{22} &= (-A_1 B^\alpha_{21} + E^+ B^\beta_{33})/E^- . \end{aligned} \tag{4.10}$$

Thus if we can express B^α_{21} in terms of B^β_{33}, we can completely determine all entries in the 2×2 block.

Case 2: Multiplicity 2 $\leftrightarrow$ 2

$$\alpha = \mathcal{V}_\Xi(f; j, \frac{1}{2}, \cdots, \frac{1}{2}, \frac{\varepsilon}{2}) \to \beta = \mathcal{V}_\Xi(f'; j' \frac{1}{2}, \cdots, \frac{1}{2}, \frac{\varepsilon'}{2}) .$$

Here we have

$$|_\beta N|_\alpha = f'^2 - f^2 + J_b^2 - J_a^2 .$$

So using (4.9), we get the transition quantities

$$\begin{pmatrix} B^\beta_{11} & B^\beta_{12} \\ B^\beta_{21} & B^\beta_{22} \end{pmatrix} \begin{pmatrix} F_1^- & G_2 \\ G_1 & C_{ba} F_2^- \end{pmatrix} = \begin{pmatrix} F_1^+ & -G_2 \\ -G_1 & C_{ba} F_2^+ \end{pmatrix} \begin{pmatrix} B^\alpha_{11} & B^\alpha_{12} \\ B^\alpha_{21} & B^\alpha_{22} \end{pmatrix} , \tag{4.11}$$

where

$$\begin{aligned} F_1^- &:= \frac{1}{2}(f'^2 - f^2) + \frac{1}{2}(J_b^2 - J_a^2) - r + \Xi(f' - f)(\mathcal{D}^\beta_{11} - \mathcal{D}^\alpha_{11}) , \\ F_1^+ &:= \frac{1}{2}(f'^2 - f^2) + \frac{1}{2}(J_b^2 - J_a^2) + r - \Xi(f' - f)(\mathcal{D}^\beta_{11} - \mathcal{D}^\alpha_{11}) , \\ F_2^- &:= \frac{1}{2}(f'^2 - f^2) + \frac{1}{2}(J_b^2 - J_a^2) - r + \Xi(f' - f)(\mathcal{D}^\beta_{22} - \mathcal{D}^\alpha_{22}) , \\ F_2^+ &:= \frac{1}{2}(f'^2 - f^2) + \frac{1}{2}(J_b^2 - J_a^2) + r - \Xi(f' - f)(\mathcal{D}^\beta_{22} - \mathcal{D}^\alpha_{22}) , \\ G_1 &:= \Xi(f' - f)(\mathcal{D}^\beta_{21} - C_{ba}\mathcal{D}^\alpha_{21}) , \text{ and} \\ G_2 &:= \Xi(f' - f)(C_{ba}\mathcal{D}^\beta_{12} - \mathcal{D}^\alpha_{12}) . \end{aligned}$$

Therefore we get determinant quotients of B on multiplicity 2 part.
Note the following diagram of reachable multiplicity 2 isotypic summands from $\mathcal{V}_\Xi(f;j,\frac{1}{2},\cdots,\frac{1}{2},\frac{\varepsilon}{2})$ under the selection rule:

$$\begin{array}{ccccc}
\mathcal{V}_\Xi(f-1;j+1,\frac{1}{2},\cdots,\frac{1}{2},\frac{\varepsilon}{2}) & & & & \mathcal{V}_\Xi(f+1;j+1,\frac{1}{2},\cdots,\frac{1}{2},\frac{\varepsilon}{2}) \\
& \nwarrow & & \nearrow & \\
\mathcal{V}_\Xi(f-1;j,\frac{1}{2},\cdots,\frac{1}{2},-\frac{\varepsilon}{2}) & \leftarrow & \bullet & \rightarrow & \mathcal{V}_\Xi(f+1;j,\frac{1}{2},\cdots,\frac{1}{2},-\frac{\varepsilon}{2}) \\
& \swarrow & & \searrow & \\
\mathcal{V}_\Xi(f-1;j-1,\frac{1}{2},\cdots,\frac{1}{2},\frac{\varepsilon}{2}) & & & & \mathcal{V}_\Xi(f+1;j-1,\frac{1}{2},\cdots,\frac{1}{2},\frac{\varepsilon}{2}).
\end{array}$$

The determinant quotients corresponding to the above diagram are:
(4.12)

$$\begin{pmatrix}
\frac{(-f+J+1-\Xi+r+\frac{\varepsilon}{2}\Xi)(-f+J+1+\Xi+r+\frac{\varepsilon}{2}\Xi)}{(-f+J+1-\Xi-r-\frac{\varepsilon}{2}\Xi)(-f+J+1+\Xi-r-\frac{\varepsilon}{2}\Xi)} & \frac{(f+J+1-\Xi+r-\frac{\varepsilon}{2}\Xi)(f+J+1+\Xi+r-\frac{\varepsilon}{2}\Xi)}{(f+J+1-\Xi-r+\frac{\varepsilon}{2}\Xi)(f+J+1+\Xi-r+\frac{\varepsilon}{2}\Xi)} \\
\frac{(-f+\frac{1}{2}-\Xi+r-\varepsilon\Xi J)(-f+\frac{1}{2}+\Xi+r-\varepsilon\Xi J)}{(-f+\frac{1}{2}-\Xi-r+\varepsilon\Xi J)(-f+\frac{1}{2}+\Xi-r+\varepsilon\Xi J)} & \frac{(f+\frac{1}{2}-\Xi+r+\varepsilon\Xi J)(f+\frac{1}{2}+\Xi+r+\varepsilon\Xi J)}{(f+\frac{1}{2}-\Xi-r-\varepsilon\Xi J)(f+\frac{1}{2}+\Xi-r-\Xi J)} \\
\frac{(-f-J+1-\Xi+r-\frac{\varepsilon}{2}\Xi)(-f-J+1+\Xi+r-\frac{\varepsilon}{2}\Xi)}{(-f-J+1-\Xi-r+\frac{\varepsilon}{2}\Xi)(-f-J+1+\Xi-r+\frac{\varepsilon}{2}\Xi)} & \frac{(f-J+1-\Xi+r+\frac{\varepsilon}{2}\Xi)(f-J+1+\Xi+r+\frac{\varepsilon}{2}\Xi)}{(f-J+1-\Xi-r-\frac{\varepsilon}{2}\Xi)(f-J+1+\Xi-r-\frac{\varepsilon}{2}\Xi)}
\end{pmatrix},$$

where $J=\varepsilon J_a$.

And these data can be put into the following Gamma function expression:

$$\frac{1}{4}\bullet\frac{\Gamma\left(\frac{1}{2}(f+J+r-\frac{\varepsilon}{2}\Xi)\right)\Gamma\left(\frac{1}{2}(-f+J+r+\frac{\varepsilon}{2}\Xi)\right)}{\Gamma\left(\frac{1}{2}(f+J-r+\frac{\varepsilon}{2}\Xi)\right)\Gamma\left(\frac{1}{2}(-f+J-r-\frac{\varepsilon}{2}\Xi)\right)}$$

$$\bullet\frac{\Gamma\left(\frac{1}{2}(f+J+2+r-\frac{\varepsilon}{2}\Xi)\right)\Gamma\left(\frac{1}{2}(-f+J+2+r+\frac{\varepsilon}{2}\Xi)\right)}{\Gamma\left(\frac{1}{2}(f+J+2-r+\frac{\varepsilon}{2}\Xi)\right)\Gamma\left(\frac{1}{2}(-f+J+2-r-\frac{\varepsilon}{2}\Xi)\right)}.$$

Case 3: Multiplicity 1 $\leftrightarrow$ 1

$$\alpha=\mathcal{V}_\Xi(f;j,\frac{3}{2},\frac{1}{2},\cdots,\frac{1}{2},\frac{\varepsilon}{2})\leftarrow\beta=\mathcal{V}_\Xi(f';j'\frac{3}{2},\frac{1}{2},\cdots,\frac{1}{2},\frac{\varepsilon'}{2}).$$

Again we have

$$|_\alpha N|_\beta=f^2-f'^2+J_a^2-J_b^2.$$

And the transition quantities are

$$B^\alpha_{33}P^-=P^+B^\beta_{33}, \tag{4.13}$$

where

$$P^-:=\tfrac{1}{2}(f^2-f'^2)+\tfrac{1}{2}(J_a^2-J_b^2)-r+\Xi(f-f')(\mathcal{D}^\alpha_{33}-\mathcal{D}^\beta_{33})\text{ and}$$
$$P^+:=\tfrac{1}{2}(f^2-f'^2)+\tfrac{1}{2}(J_a^2-J_b^2)+r-\Xi(f-f')(\mathcal{D}^\alpha_{33}-\mathcal{D}^\beta_{33}).$$

The diagram of reachable multiplicity 1 isotypic summands from

$$\mathcal{V}_\Xi(f;j,\frac{3}{2},\frac{1}{2},\cdots,\frac{1}{2},\frac{\varepsilon}{2})$$

under the selection rule looks:

$$\begin{array}{ccccc}
\mathcal{V}_\Xi(f-1;j+1,\frac{3}{2},\frac{1}{2},\cdots,\frac{1}{2},\frac{\varepsilon}{2}) & & & & \mathcal{V}_\Xi(f+1;j+1,\frac{3}{2},\frac{1}{2},\cdots,\frac{1}{2},\frac{\varepsilon}{2}) \\
 & \nwarrow & & \nearrow & \\
\mathcal{V}_\Xi(f-1;j,\frac{3}{2},\frac{1}{2},\cdots,\frac{1}{2},-\frac{\varepsilon}{2}) & \leftarrow & \bullet & \rightarrow & \mathcal{V}_\Xi(f+1;j,\frac{3}{2},\frac{1}{2},\cdots,\frac{1}{2},-\frac{\varepsilon}{2}) \\
 & \swarrow & & \searrow & \\
\mathcal{V}_\Xi(f-1;j-1,\frac{3}{2},\frac{1}{2},\cdots,\frac{1}{2},\frac{\varepsilon}{2}) & & & & \mathcal{V}_\Xi(f+1;j-1,\frac{3}{2},\frac{1}{2},\cdots,\frac{1}{2},\frac{\varepsilon}{2})\,.
\end{array}$$

And the eigenvalue quotients are:

$$\begin{pmatrix}
\dfrac{-f+J+1+r+\frac{\varepsilon}{2}\Xi}{-f+J+1-r-\frac{\varepsilon}{2}\Xi} & \dfrac{f+J+1+r-\frac{\varepsilon}{2}\Xi}{f+J+1-r+\frac{\varepsilon}{2}\Xi} \\[2ex]
\dfrac{-f+\frac{1}{2}+r-\varepsilon\Xi J}{-f+\frac{1}{2}-r+\varepsilon\Xi J} & \dfrac{f+\frac{1}{2}+r+\varepsilon\Xi J}{f+\frac{1}{2}-r-\varepsilon\Xi J} \\[2ex]
\dfrac{-f-J+1+r-\frac{\varepsilon}{2}\Xi}{-f-J+1-r+\frac{\varepsilon}{2}\Xi} & \dfrac{f-J+1+r+\frac{\varepsilon}{2}\Xi}{f-J+1-r-\frac{\varepsilon}{2}\Xi}
\end{pmatrix},$$

where $J=\varepsilon J_a$.

Thus, following the normalization on the multiplicity 2 part, we get the spectral function on the multiplicity 1 part:

(4.14)

$$Z(r;f,J,\Xi\varepsilon)=\frac{\varepsilon}{2}\Xi\frac{\Gamma\left(\frac{1}{2}(f+J+1+r-\frac{\varepsilon}{2}\Xi)\right)\Gamma\left(\frac{1}{2}(-f+J+1+r+\frac{\varepsilon}{2}\Xi)\right)}{\Gamma\left(\frac{1}{2}(f+J+1-r+\frac{\varepsilon}{2}\Xi)\right)\Gamma\left(\frac{1}{2}(-f+J+1-r-\frac{\varepsilon}{2}\Xi)\right)}\,.$$

In particular,

$$Z(\frac{1}{2},f,J,\Xi\varepsilon)=-\frac{1}{4}(f-\Xi\varepsilon J)=\frac{1}{4}\sqrt{-1}\ \mathrm{eig}(E\mathcal{R};f,J,\Xi\varepsilon)\,,$$

where $E\mathcal{R}$ is the exchanged Rarita-Schwinger operator.

5. Interface between multiplicity 1 and 2 parts

Consider the following diagram:

$$\begin{array}{ccc}
\alpha_1=\mathcal{V}_\Xi(f;j,\frac{1}{2},\cdots,\frac{1}{2},\frac{\varepsilon}{2}) & \rightarrow & \alpha_2=\mathcal{V}_\Xi(f+1;j+1,\frac{1}{2},\cdots,\frac{1}{2},\frac{\varepsilon}{2}) \\
\updownarrow & & \updownarrow \\
\beta_1=\mathcal{V}_\Xi(f+1;j,\frac{3}{2},\frac{1}{2},\cdots,\frac{1}{2},\frac{\varepsilon}{2}) & \leftarrow & \beta_2=\mathcal{V}_\Xi(f;j+1,\frac{3}{2},\frac{1}{2},\cdots,\frac{1}{2},\frac{\varepsilon}{2})\,.
\end{array}$$

Then (4.11) reads

$$B^{\alpha_2}M_1=M_2B^{\alpha_1}\,.$$

So

$$\mathrm{det}B^{\alpha_2}=\frac{\mathrm{det}M_2}{\mathrm{det}M_1}\mathrm{det}B^{\alpha_1}\,.$$

Note that $\dfrac{\det M_2}{\det M_1}$ is a determinant quotient computed in (4.12).
From (4.10), we get a relation between B_{12} and B_{33}:

$$\begin{aligned}\det\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22}\end{pmatrix} &= B_{11}B_{22} - B_{12}B_{22} \\ &= -\frac{1}{A_2E^-}B_{33}\left(B_{33}A_2E^+ - (E^-E^+ + A_1A_2)B_{21}\right) .\end{aligned}$$

We can also compare $(2,1)$ entries of both sides in (4.11). Applying (4.10) and (4.13) to the both relations, we can finally write B_{21} in terms of B_{33} with a "big" help from computer algebra package.

2×2 block on

$$\mathcal{V}_\Xi(f;j,\frac{1}{2},\cdots,\frac{1}{2},\frac{\varepsilon}{2})$$

in terms of $(3,3)$

$$\mathcal{V}_\Xi(f+1;j,\frac{3}{2},\frac{1}{2},\cdots,\frac{1}{2},\frac{\varepsilon}{2})$$

is:

$$\begin{pmatrix} \dfrac{4C_1C_2}{(n-1)C_3C_4} - 1 & \dfrac{-2(n-2)\Xi C_5C_2}{(n-1)^2C_3C_4} \\ \dfrac{8n\Xi C_2}{C_3C_4} & \dfrac{-4C_5C_2}{(n-1)C_1C_3C_4} + \dfrac{C_6}{C_1}\end{pmatrix} \bullet Z(r;f+1,J,\Xi\varepsilon)\,, \tag{5.15}$$

where

$$\begin{aligned} C_1 &= 2fn - 2f - 2n + 1 + n^2 + 2rn - 2r - 2\Xi J_a\,, \\ C_2 &= 2fr + \Xi J_a\,, \\ C_3 &= n - 1 + 2r\,, \\ C_4 &= (2f + 2r - \Xi + 2J_a)(2f + 2r + \Xi - 2J_a)\,, \\ C_5 &= (n - 1 + 2J_a)(n - 1 - 2J_a)\,, \text{ and} \\ C_6 &= 2fn - 2f - 2n + 1 + n^2 - 2rn + 2r + 2\Xi J_a\,. \end{aligned}$$

Remark 2. In particular, if $r = \frac{1}{2}$ and $(3,3)$ entry

$$\sqrt{-1}f - \sqrt{-1}\Xi\varepsilon J$$

of the exchanged Rarita-Schwinger operator is put into the above formula, we recover the other 2×2 entries

$$\begin{pmatrix} -\dfrac{n-2}{n}\sqrt{-1}\left(f + \dfrac{n+1}{n-1}\Xi\varepsilon J\right) & -\dfrac{2\sqrt{-1}\Xi}{n(n-1)}\left(\dfrac{(n-1)(n-2)}{4} - \dfrac{n-2}{n-1}J^2\right) \\ 2\sqrt{-1}\Xi & \sqrt{-1}f - \dfrac{n-3}{n-1}\sqrt{-1}\Xi\varepsilon J \end{pmatrix}$$

of the operator ([7]).

References

[1] Branson, T., *Group representations arising from Lorentz conformal geometry*, J. Funct. Anal. **74** (1987), 199–291.

[2] Branson, T., *Nonlinear phenomena in the spectral theory of geometric linear differential operators*, Proc. Symp. Pure Math. **59** (1996), 27–65.

[3] Branson, T., *Stein-Weiss operators and ellipticity*, J. Funct. Anal. **151** (1997), 334–383.

[4] Branson, T., *Spectra of self-gradients on spheres*, J. Lie Theory **9** (1999), 491–506.

[5] Branson, T., Ólafsson, G. and Ørsted, B., *Spectrum generating operators, and intertwining operators for representations induced from a maximal parabolic subgroups*, J. Funct. Anal. **135** (1996), 163–205.

[6] Hong, D., *Eigenvalues of Dirac and Rarita-Schwinger operators*, Clifford Algebras and their Applications in Mathematical Physics, Birkhäuser, 2000.

[7] Hong, D., *Spectra of higher spin operators*, Ph.D. Dissertation, University of Iowa, 2004.

[8] Kosmann, Y., *Dérivées de Lie des spineurs*, Ann. Mat. Pura Appl. **91** (1972), 317–395.

[9] Ørsted, B., *Conformally invariant differential equations and projective geometry*, J. Funct. Anal. **44** (1981), 1–23.

Thomas Branson
Department of Mathematics, University of Iowa
Iowa City IA 52242 USA
E-mail: branson@math.uiowa.edu

Doojin Hong
Department of Mathematical Sciences
Seoul National University, Seoul Korea
E-mail: doojin@math.snu.ac.kr