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CONFORMALLY FLAT SEMI-SYMMETRIC SPACES

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Abstract. We obtain the complete classification of conformally flat semi-symmetric spaces.

1. Introduction

Conformally flat manifolds represent a classical field of investigation in Riemannian geometry. A survey on conformally flat spaces would be too long a task for this Introduction. For the purpose of this paper, it suffices to recall only some problems related with symmetry. Locally symmetric conformally flat spaces are well-known, they have been classified by P. Ryan [R] (see also [K]), who proved the following

Theorem 1.1 ([R]). *Let M be an n-dimensional conformally flat space with parallel Ricci tensor. Then M has as its simply connected Riemannian covering one of the following spaces:*

$$\mathbb{R}^n, S^n(k), \mathbb{H}^n(-k), \mathbb{R} \times S^{n-1}(k), \mathbb{R} \times \mathbb{H}^{n-1}(-k), S^p(k) \times \mathbb{H}^{n-p}(-k)\,,$$

where by $S^n(k)$ we denote a Euclidean n-sphere with constant curvature $k > 0$, and by $\mathbb{H}^n(-k)$ we denote an n-dimensional simply connected, connected space with constant curvature $-k < 0$.

Semi-symmetric spaces are a well-known and natural generalization of locally symmetric spaces. A *semi-symmetric space* is a Riemannian manifold (M, g) such that its curvature tensor R satisfies the condition

$$R(X, Y) \cdot R = 0\,,$$

for all vector fields X, Y on M, where $R(X, Y)$ acts as a derivation on R [S]. Such a space is called "semi-symmetric" since the curvature tensor R_p of (M, g) at a point $p \in M$ is the same as the curvature tensor of a symmetric space (which may change with the point p). So, locally symmetric spaces are obviously semi-symmetric, but the converse is not true, as was proved by H. Takagi [T]. In any dimension greater

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than two there exist examples of semi-symmetric spaces which are not locally symmetric (we refer to [BKV] for a survey). Nevertheless, semi-symmetry implies local symmetry in several cases and it is an interesting problem, given a class of Riemannian manifolds, to decide whether inside that class semi-symmetry implies local symmetry or not (see for example [B], [BC], [CV]).

In this paper, we classify conformally flat semi-symmetric spaces, generalizing the result of Ryan. To do this, we use the very special geometry of the conformally flat spaces and the local structure of a semi-symmetric space as described by Szabó [S]. We prove the following

Main Theorem. *A conformally flat semi-symmetric space M (of dimension $n > 2$) is either locally symmetric or it is locally irreducible and isometric to a semi-symmetric real cone.*

The paper is organized in the following way. In Section 2, we recall some basic facts and results about conformally flat Riemannian manifolds and semi-symmetric spaces. Then, in the Sections 3 and 4 we prove the main result on conformally flat semi-symmetric spaces, dealing respectively with the locally irreducible case and with the locally reducible case.

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2. Preliminaries

Let (M, g) be a Riemannian manifold of dimension $n > 2$ and R its curvature tensor, taken with the sign convention $R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$, for all vector fields X, Y on M, where ∇ denotes the Levi Civita connection of M. By ϱ, Q and τ we denote respectively the Ricci tensor, the Ricci operator associated to ϱ through g and the scalar curvature of M. Let p be a point of M and $\{e_1, \dots, e_n\}$ an orthonormal basis of the tangent space T_pM. The components of R and ϱ with respect to $\{e_1, \dots, e_n\}$ are denoted respectively by R_{ijkh} and ϱ_{ik}. As is well-known, for a conformally flat space we have

$$\begin{aligned} R_{ijkh} = & \frac{1}{n-2}(g_{ih}\varrho_{jk} + g_{jk}\varrho_{ih} - g_{ik}\varrho_{jh} - g_{jh}\varrho_{ik}) \\ & - \frac{\tau}{(n-1)(n-2)}(g_{ih}g_{jk} - g_{ik}g_{jh}) \, . \end{aligned} \tag{2.1}$$

Moreover, (2.1) characterizes conformally flat Riemannian manifolds of dimension $n \geq 4$, while it is trivially satisfied by any three-dimensional manifold. Conversely, the condition

$$\nabla_i\varrho_{jk} - \nabla_j\varrho_{ik} = \frac{1}{2(n-1)}(g_{jk}\nabla_i\tau - g_{ik}\nabla_j\tau) \, , \tag{2.2}$$

which characterizes three-dimensional conformally flat spaces, is trivially satisfied by any conformally flat Riemannian manifold of dimension greater than three.

We now recall some basic facts about semi-symmetric spaces. Let (M, g) be a smooth, connected Riemannian manifold. As already mentioned in the Introduction, (M, g) is said to be *semi-symmetric* if its curvature tensor R satisfies

$$R(X, Y) \cdot R = 0\,, \tag{2.3}$$

for all vector fields X, Y and where $R(X, Y)$ acts as a derivation on R. This is equivalent to the fact that R_p is, for each $p \in M$, the same as the curvature tensor of a symmetric space. This last space may vary with p. We recall the following

Definition 2.1. The *nullity vector space* of the curvature tensor at a point p of a Riemannian manifold (M, g) is given by

$$E_{0p} = \{X \in T_pM \mid R(X, Y)Z = 0 \text{ for all } Y, Z \in T_pM\}\,.$$

The *index of nullity at* p is the number $\nu(p) = \dim E_{0p}$. The *index of conullity at* p is the number $u(p) = \dim M - v(p)$.

By means of the index of nullity and the index of conullity, Szabó [S] classified locally irreducible semi-symmetric spaces, proving that such a space must be locally isometric to one of the following spaces:

(1) a symmetric space when $\nu(p) = 0$ at each point p, or

(2) a real cone when $\nu(p) = 1$ and $u(p) = n - 1 > 2$ at each point p, or

(3) a Kählerian cone when $\nu(p) = 2$ and $u(p) = n - 2 > 2$ at each point p, or

(4) a Riemannian manifold foliated by Euclidean leaves of codimension two when $\nu(p) = n - 2$ and $u(p) = 2$ at each point p of a dense open subset U of M.

Remark 2.2. Note that real cones also exist in dimension three, as cones over two-dimensional manifolds of constant curvature (see the description of real cones in the following section 3). Such spaces do not appear explicitly in Szabó's classification, since they are special cases of (4), that is, Riemannian manifolds foliated by Euclidean leaves of codimension two.

The following result describes the local structure of a semi-symmetric space (M, g).

Theorem 2.3 ([S])**.** *There exists an open dense subset U of M such that around every point of U the manifold is locally isometric to a Riemannian product of type*

$$\mathbb{R}^k \times M_1 \times \cdots \times M_r\,, \tag{2.4}$$

where $k \geq 0$, $r \geq 0$ and each M_i is either a symmetric space, a two-dimensional manifold, a real cone, a Kählerian cone or a Riemannian space foliated by Euclidean leaves of codimension two.

The decomposition (2.4) may vary in the different connected components U_α of U, while it is constant on each U_α. While proving the Main Theorem, we shall come back to the description of these factors of the local decomposition of a semi-symmetric space. For more details and references we refer to [BKV].

3. Irreducible conformally flat semi-symmetric spaces

We open this Section by proving the following result, which will be used throughout the rest of the paper.

Theorem 3.1. *Let* (M,g) *be a Riemannian manifold satisfying* (2.1)*, of dimension* $n \geq 3$ (that is, either $\dim M = 3$ or M is conformally flat)*. Then, at each point* p *of* M*, the index of nullity is either* $\nu(p) = 0, 1$ *or* n*.*

Proof. Fix a point $p \in M$. If the curvatur tensor R_p at p vanishes, then $\nu(p) = n$ and the conclusion follows. So, we now assume $R_p \neq 0$ and we prove that $\nu(p) \leq 1$. For this purpose, it is enough to show that if $\nu(p) \neq 0$, then $\nu(p) = 1$.

Suppose then $\nu(p) \neq 0$. Let $e_0 \in E_{0p}$ be a unit vector and $\{e_0, e_1, \dots, e_{n-1}\}$ an orthonormal basis of T_pM. Since $e_0 \in E_{0p}$, $R_{0jkh} = 0$ for all j, k, h, from which it also follows that $\varrho_{0k} = -\sum_j R_{0jkj} = 0$ for all k. Therefore, from (2.1) we have

$$0 = R_{0jkh} = \frac{1}{n-2}(\delta_{0h}\varrho_{jk} - \delta_{0k}\varrho_{jh}) - \frac{\tau}{(n-1)(n-2)}(\delta_{0h}\delta_{jk} - \delta_{jh}\delta_{0k}),$$

for any choice of j, k, h. Choosing $k = 0$ and $h \neq 0$, we then get

$$\varrho_{jh} - \frac{\tau}{n-1}\delta_{jh} = 0.$$

Hence, the Ricci tensor at p is described by

$$\begin{cases} \varrho_{ij} = \frac{\tau}{n-1} & \text{if } i = j \geq 1, \\ \varrho_{ij} = 0 & \text{in all the other cases.} \end{cases} \tag{3.1}$$

Clearly, if $\nu(p) > 1$, we can choose at least two mutually orthogonal unit vectors in E_{0p}, say e_0, e_1, and an orthonormal basis $\{e_0, e_1, \dots, e_{n-1}\}$ containing them. But then, since e_1 is a nullity vector, from (3.1) we have

$$0 = \varrho_{11} = \frac{\tau}{n-1},$$

that is, $\tau = 0$ and so, again by (3.1), $\varrho_{ij} = 0$ for all i, j. Then (2.1) yields that $R_p = 0$, contrary to our assumption. Therefore, $\nu(p) = 1$ and this completes the proof. □

Remark 3.2. In the proof of Theorem 3.1, we showed that if (M,g) is a Riemannian manifold satisfying (2.1) and p a point of M with $\nu(p) \neq 0$, then either $R_p = 0$ or ϱ is described by (3.1).

Theorem 3.1 restricts the research of conformally flat semi-symmetric spaces to the ones having index of nullity equal to 0, 1 or n. The most interesting case is the one of a semi-symmetric space having nullity index equal to 1, since if the nullity index is constant and equal to 0 (respectively, to n), then the space is locally symmetric (respectively, flat).

For this reason, we now give a short description of *real cones*, which will turn out to be the only examples of conformally flat semi-symmetric spaces which are not locally symmetric. We refer to [BKV] for more details.

Consider a Riemannian manifold $(\bar{M}, \bar{g})$. Let $\mu(t)$ be the unique solution of the differential equation $\frac{d\mu}{dt} = -\mu^2$ with initial condition $\mu(0) = \mu_0 > 0$, that is, $\mu(t) = (t + (1/\mu_0))^{-1}$. Put $\mathbb{R}_+ = \{x \in \mathbb{R} \mid x > -1/\mu_0\}$ and on the product manifold $\mathbb{R}_+ \times \bar{M}$ consider the Riemannian metric

$$g = dx^0 \otimes dx^0 + \mu(x^0)^{-2}\pi^* g\,,$$

where x^0 is the natural coordinate on $\mathbb{R}_+$ and $\pi : \mathbb{R}_+ \times \bar{M} \to \bar{M}$ the projection on the second factor. The manifold $(\mathbb{R}_+ \times \bar{M}, g)$ is called a *Riemannian cone over* $(\bar{M}, \bar{g})$. Let $T = \partial/\partial x^0$ denote the unit vector field tangent to $\mathbb{R}_+$ in $\mathbb{R}_+ \times \bar{M}$. The curvature tensor of $M = \mathbb{R}_+ \times \bar{M}$ is described by (see [BKV])

$$\begin{aligned} R(X,Y)Z = g(B_0(Y),Z)B_0(X) - g(B_0(X),Z)B_0(Y) \\ + (\pi^*\bar{R})(X,Y)Z\,, \end{aligned} \tag{3.2}$$

for all tangent vectors X, Y, Z to $\bar{M}$, where $B_0(X) := \nabla_X T = \mu(X - g(X,T)T)$.

Any semi-symmetric real cone $(M = \mathbb{R}_+ \times \bar{M}, g)$ is locally isometric to some maximal cone $M_c(\tilde{M}, \mu_0)$, where $(\tilde{M}, \tilde{g})$ is a real space form of constant curvature c [BKV]. Note that at any point of M, $T \in E_0$. If M is locally irreducible and $c \neq 1$, then at each point of M the index of nullity is equal to one and the index of conullity coincides with the dimension of $\bar{M}$. We include the case when $\dim\bar{M} = 2$. In [BKV], this case was excluded, since a three-dimensional real cone is a special case of three-dimensional Riemannian manifold foliated by Euclidean leaves of codimension two (briefly, a *Riemannian manifold of conullity two*).

At any point p of a semi-symmetric real cone M, fix an orthonormal basis of tangent vectors $\{e_0, e_1, \ldots, e_r\}$, with $e_0 = T_p$ and $e_1, \ldots, e_r$ tangent to the real space form $(\tilde{M}^r, \tilde{g})$ $(r = n-1)$. Then, using (3.2) to compute the components of the curvature tensor, we get

$$\begin{cases} R_{ijkh} = 0 & \text{if } 0 \in \{i,j,k,h\}\,, \\ R_{ijkh} = \mu^2(c-1)(\delta_{ik}\delta_{jh} - \delta_{jk}\delta_{ih}) & \text{otherwise.} \end{cases} \tag{3.3}$$

Computing the Ricci components and the scalar curvature of M starting from (3.3), it is easy to check that (2.1) is satisfied and, if $\dim M \geq 4$, this implies that M is conformally flat. If $\dim M = 3$, one can check that (2.2) holds and so, M is conformally flat also in this case. Therefore, a real semi-symmetric cone M is a conformally flat (semi-symmetric) Riemannian manifold, with scalar curvature $\tau = r(r-1)(c-1)\mu^2$. Note that τ cannot be constant, as μ depends on t and so, M is never locally symmetric.

We can now classify locally irreducible conformally flat semi-symmetric spaces, by proving the following

Theorem 3.3. *A locally irreducible conformally flat Riemannian manifold* (M, g) *is semi-symmetric if and only if it is locally symmetric or locally isometric to a* (*semi-symmetric*) *real cone.*

Proof. The "if" part is trivial, since a Riemannian manifold which is locally symmetric or locally isometric to a real cone is clearly semi-symmetric. We now prove the "only if" part. According to Szabó's classification, M must be locally isometric to one of the spaces (1)–(4) listed before Remark 2.2. Since M is locally irreducible, M cannot be flat. So, let p be a point of M with $R_p \neq 0$. Theorem 3.1 then implies that either $\nu(p) = 0$ or 1. This excludes the case (3), while case (4) is only possible when $n = 3$. If $\nu(p) = 0$, then M is locally isometric to an irreducible symmetric space and the conclusion follows. In order to complete the proof, we have to show that a three-dimensional conformally flat Riemannian manifold of conullity two is isometric to a real cone. Note that a more general classification result for three-dimensional locally irreducible pseudo-symmetric spaces of constant type was proved by N. Hashimoto and M. Sekizawa in [HSk]. (This was pointed out to the author by O. Kowalski after the first version of this paper was submitted.) Nevertheless, to keep the paper more self-contained, we shall present here an alternative proof.

Let N be a three-dimensional conformally flat Riemannian manifold of conullity two. We refer to [BKV] for a detailed description of such a space. Here, we just recall that, with respect to a suitable system of coordinates $\{x, y, w\}$, N admits a local orthonormal frame $\{E_1, E_2, E_3\}$ whose dual coframe is of the form

$$\omega^1 = f(w,x,y)\,dw\,, \quad \omega^2 = A(w,x,y)\,dx + C(w,x,y)\,dw\,, \quad \omega^3 = dy + H(w,x)\,dw\,.$$

The curvature tensor of N is given by

$$R = 4k\omega^1 \wedge \omega^2 \otimes \omega^1 \wedge \omega^2\,, \tag{3.4}$$

the Ricci tensor and its covariant derivative are respectively given by

$$\varrho = k(\omega^1 \otimes \omega^1 + \omega^2 \otimes \omega^2)$$

and

$$\begin{aligned}\nabla\varrho = {} & dk \otimes (\omega^1 \otimes \omega^1 + \omega^2 \otimes \omega^2) \\ & - k((a\omega^1 + b\omega^2) \otimes (\omega^1 \otimes \omega^3 + \omega^3 \otimes \omega^1) + \\ & + (c\omega^1 + d\omega^2) \otimes (\omega^2 \otimes \omega^3 + \omega^3 \otimes \omega^2))\,.\end{aligned} \tag{3.5}$$

Assuming that N is conformally flat, (2.2) holds. We can use (3.5) to compute the components of $\nabla\varrho$ with respect to $\{E_1, E_2, E_3\}$ and then apply (2.2). After some routine calculations we get $a = e$, $b = c = 0$ and $E_1(k) = E_2(k) = 0$.

In order to proceed, we need to go deeper into the theory of foliated semi-symmetric spaces as developed in [BKV]. It is shown there that there exist four types of foliated semi-symmetric spaces, according to the number of *asymptotic distributions* they admit. In the three-dimensional situation — the one we are interested in —, asymptotic distributions are given by the solutions of the equation

$$c(\omega^1)^2 + (e-a)\omega^1\omega^2 - b(\omega^2)^2\,. \tag{3.6}$$

Since we found that $a = e$ and $b = c = 0$, it follows that N admits infinitely many asymptotic distributions, that is, N is a *planar* foliated semi-symmetric space. We refer to Section 5.1 of [BKV] for details.

The metric of a locally irreducible three-dimensional planarly foliated semi-symmetric space N is of “cone type”, since it is locally determined by an orthonormal coframe of the form

$$\omega^1 = t(w,x)\, dw\,, \quad \omega^2 = y\, dx \quad \omega^3 = dy$$

(see [BKV, Theorem 6.4]), and its curvature tensor is described by (3.4), with $k = -y^{-2}\big(\frac{t''_{xx}}{t} + 1\big)$. As $E_1(k) = E_2(k) = 0$, $\frac{t''_{xx}}{t}$ is independent of w and x and so, it is constant. Therefore, N turns out to be a Riemannian cone over a two-dimensional Riemannian manifold of constant curvature and this completes the proof. □

4. Reducible conformally flat semi-symmetric spaces

First of all, we need the following Lemma, which clarifies the relation between the index of nullity of a reducible Riemannian manifold and that of its components.

Lemma 4.1. *Let (M, g) be a Riemannian manifold, locally isometric to a Riemannian product $M_1 \times M_2$. Then, at any point $p = (p_1, p_2)$ of M, we have*

$$\nu(p) = \nu(p_1) + \nu(p_2)\,.$$

Proof. As is well known, $T_pM = T_{p_1}M_1 \oplus T_{p_2}M_2$, that is, T_pM splits into the direct sum of $T_{p_1}M_1$ and $T_{p_2}M_2$. Using the fact that the curvature of a Riemannian product is given by $R = R_1 + R_2$, it is easy to show that $E_{0p} = E_{0p_1} \oplus E_{0p_2}$, from which the conclusion follows at once. □

Remark 4.2. One can easily extend the result of Lemma 4.1 to the case when M is locally isometric a Riemannian product $M_1 \times \cdots \times M_k$, obtaining, for any point $p = (p_1, \ldots, p_k)$,

$$\nu(p) = \nu(p_1) + \cdots + \nu(p_k)\,.$$

Note that, in particular, $\nu(p) = 0$ implies $\nu(p_1) = .. = \nu(p_k) = 0$, while $\nu(p) = 1$ implies that there exists an index j such that $\nu(p_j) = 1$ and $\nu(p_i) = 0$ for all $i \neq j$.

Proposition 4.3. *Let M be a semi-symmetric conformally flat Riemannian manifold. If M is locally isometric to a Riemannian product $M' \times \tilde{M}$, with* $\dim M' = 2$, *then M is locally symmetric.*

Proof. Fix a local orthonormal frame $\{e_1, e_2, v_1, \ldots, v_m\}$ of vector fields of M, with e_1, e_2 tangent to M' and $v_1, \ldots, v_m$ to $\tilde{M}$. From (2.1), we get

$$R_{1212} = -\frac{1}{n-2}(\varrho_{22} + \varrho_{11}) + \frac{\tau}{(n-1)(n-2)}\,. \tag{4.1}$$

On the other hand, since $R = R' + \tilde{R}$, where R' and $\tilde{R}$ denote respectively the curvature tensors of M' and $\tilde{M}$, we have

$$R_{1212} = R'_{1212} = -K\,,$$

where K denotes the Gaussian curvature of M'. Moreover, again by $R = R' + \tilde{R}$, we easily get $\varrho_{11} = \varrho_{22} = K$. Therefore, from (4.1) we get

$$\tau = -(n^2 - 5n + 4)K\,.$$

On the other hand, $\tau = \tau' + \tilde{\tau} = 2K + \tilde{\tau}$ and so,

$$\tilde{\tau} = -(n^2 - 5n + 6)K\,. \tag{4.2}$$

Since $\tilde{\tau}$ and K depend respectively of the points of $\tilde{M}$ and M', (4.2) implies that $\tilde{\tau}$ and K are constant, unless $n^2 - 5n + 6 = 0$, which can only occur for ($n = 2$ and) $n = 3$. So, the possible cases are the following:

a) If $n > 3$, then $\tilde{\tau}$ and K are constant. Therefore, $\tau = 2K + \tilde{\tau}$ is constant and (2.2) implies that ϱ is a *Codazzi tensor*. Hence, M is locally symmetric, as was proved by E. Boeckx in [B].

b) If $n = 3$, then M is locally isometric to $\mathbb{R} \times M'$ and $\{e_0 = v_1, e_1, e_2\}$ is a local orthonormal frame of vector fields on M. Applying (2.2), taking into account that $e_0 \in E_0$, we get

$$0 = \nabla_0 \varrho_{10} - \nabla_1 \varrho_{00} = -\frac{1}{4}\nabla_1 \tau\,,$$
$$0 = \nabla_0 \varrho_{20} - \nabla_2 \varrho_{00} = -\frac{1}{4}\nabla_2 \tau\,.$$

Moreover, $\nabla_0 \tau = 0$, since $\tau = \tau_0 + \tau'$ with $\tau_0 = 0$ and τ' constant along $\mathbb{R}$. Therefore, τ is constant and, applying again the result of [B], we can conclude that M is locally symmetric. □

The following result ends the proof of the Main Theorem.

Theorem 4.4. *A locally reducible conformally flat Riemannian manifold is semi-symmetric if and only if it is locally symmetric.*

Proof. Since locally symmetric spaces are always semi-symmetric, it is enough to prove the "only if" part. Let (M, g) be a locally reducible conformally flat semi-symmetric space. According to Theorem 2.3, there exists an open dense subset U of M such that each point $p \in U$ admits a neighborhood which is isometric to a Riemannian product of type (2.4), and such decomposition remains constant on each connected component U_α of U. If we prove that each U_α is locally symmetric, then we can conclude by a continuity argument that M itself is locally symmetric.

We consider the three-dimensional case first. Taking into account (2.4), each U_α is either flat (and hence, locally symmetric), or it is isometric to a Riemannian product $\mathbb{R} \times M'$ and so, U_α is again locally symmetric, as follows from Proposition 4.3.

Next, suppose that $n = \dim M > 3$ and let U_α be a connected component of U, locally isometric to a Riemannian product of type (2.4).

a) If U_α is flat, then it is clearly locally symmetric.

b) Being an open subset of M, U_α itself is semi-symmetric and conformally flat. If one of the factors of the decomposition (2.4) of U_α is a two-dimensional space, then applying Proposition 4.3 we can conclude that U_α is locally symmetric. So, in the sequel we shall assume that U_α is not flat and none of the factors in (2.4) is two-dimensional.

Since U_α is conformally flat and not flat, Theorem 3.1 implies that the nullity index ν on U_α is either equal to 0 or to 1. So, one of the following remaining cases occurs.

c) If $\nu = 0$, then $k = 0$ in (2.4), since each vector tangent to $\mathbb{R}^k$ belongs to the nullity vector space. Moreover, $\nu = 0$ on each M_i (and none of them is two-dimensional), as follows from Lemma 4.1 and Remark 4.2. Therefore, each M_i is a symmetric space, since all the other irreducible semi-symmetric spaces have nullity index at least 1. So, U_α is locally symmetric.

d) If $\nu = 1$, then one of the factors in (2.4) has nullity index 1 and all the others have nullity index 0 (see again Remark 4.2). If $k \neq 0$, then $k = 1$ and (2.4) becomes

$$\mathbb{R} \times M_1 \times \cdots \times M_r \, ,$$

where each M_i has nullity index 0 and is not two-dimensional. Thus, each M_i is a symmetric space and we can conclude that U_α is locally symmetric.

If $k = 0$, then by (2.4) and Remark 4.2 it follows that U_α is locally isometric to a Riemannian product

$$M_1 \times M_S \, ,$$

where M_1 is an irreducible semi-symmetric space with nullity index 1 and M_S is a (reducible or irreducible) symmetric space. We prove that this case cannot occur. In order to do this, we use some well-known curvature formulas for the Riemannian product $M_1 \times M_S$.

First of all, since M_1 is an irreducible semi-symmetric space with nullity index 1, either M is a three-dimensional Riemannian manifold of conullity two, or it is a real cone. In both cases, M_1 satisfies (2.1). We shall denote by n_1 the dimension of M_1 and by R', ϱ' and τ' the curvature tensor, the Ricci tensor and the scalar curvature of M_1, respectively.

Fix a local orthonormal frame $\{e_1, \ldots, e_{n_1}\}$ on M_1, with $e_1 \in E_0$. The Ricci tensor ϱ' is described by (3.1) and from (2.1) we get

$$R'(e_i, e_j, e_i, e_j) = -\frac{1}{n_1 - 2}(\varrho'_{jj} + \varrho'_{ii}) + \frac{1}{(n_1 - 1)(n_1 - 2)}\tau' \, , \tag{4.3}$$

for all i, j. On the other hand, the curvature tensor and the Ricci tensor of the Riemannian product $M_1 \times M_S$ are respectively given by $R = R' + R_S$ and $\varrho = \varrho' + \varrho_S$. Therefore, we have

$$\begin{aligned} R'(e_i, e_j, e_i, e_j) &= R(e_i, e_j, e_i, e_j) \\ &\quad - \frac{1}{n-2}(\varrho_{jj} + \varrho_{ii}) + \frac{1}{(n-1)(n-2)}\tau \, . \end{aligned} \tag{4.4}$$

Taking into account (3.1), from (4.3) and (4.4) we respectively get

$$R'(e_i, e_j, e_i, e_j) = -\frac{\tau'}{(n_1 - 1)(n_1 - 2)} \tag{4.5}$$

and

$$R'(e_i, e_j, e_i, e_j) = -\frac{\tau}{(n-1)(n-2)}\,. \tag{4.6}$$

Comparing (4.5) and (4.6), we get

$$\frac{\tau}{(n-1)(n-2)} = \frac{\tau'}{(n_1-1)(n_1-2)}\,. \tag{4.7}$$

Moreover, on the Riemannian product $M_1 \times M_S$ we have $\tau = \tau' + \tau_S$, where τ_S is constant since M_S is a symmetric space. So, differentiating (4.7) by e_i, $i = 1, \dots, n_1$, we get

$$\frac{\nabla_{e_i}\tau'}{(n-1)(n-2)} = \frac{\nabla'_{e_i}\tau'}{(n_1-1)(n_1-2)}\,. \tag{4.8}$$

Since $\nabla_{e_i} = \nabla'_{e_i}$ for all i and $n \neq n_1$, from (4.8) it follows that $\nabla'_{e_i}\tau' = 0$ for all i, that is, τ' is constant. Since τ_S is also constant, τ is constant and so, as $M_1 \times M_S$ satisfies (2.2), the Ricci tensor ϱ of $M_1 \times M_S$ is a Codazzi tensor, which implies that $M_1 \times M_S$ is locally symmetric [B]. But then, since M_S is a symmetric space, M_1 itself should be locally symmetric, which cannot occur and this ends the proof. □

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