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THE MOVING FRAMES FOR DIFFERENTIAL EQUATIONS II. UNDERDETERMINED AND FUNCTIONAL EQUATIONS

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Abstract. Continuing the idea of Part I, we deal with more involved pseudogroup of transformations $\bar{x} = \varphi(x)$, $\bar{y} = L(x)y$, $\bar{z} = M(x)z$, $\dots$ applied to the first order differential equations including the underdetermined case (i.e. the Monge equation $y' = f(x, y, z, z')$) and certain differential equations with deviation (if $z = y(\xi(x))$ is substituted). Our aim is to determine complete families of invariants resolving the equivalence problem and to clarify the largest possible symmetries. Together with Part I, this article may be regarded as an introduction into the method of moving frames adapted to the theory of differential and functional-differential equations.

Introduction

The most general pointwise transformations of homogeneous linear differential equations with deviating arguments were investigated in [6], [10], [11], [12], [14], [15], [16], for example. It is given by the formula

$$\bar{y}(\varphi(x)) = L(x)y(x)\,, \tag{1}$$

i.e., the transformation consists of a change of the independent variable and the multiplication by a nonvanishing factor L. They coincide with the most general pointwise transformations of homogeneous linear differential equations of the n-th ($n \geq 2$) order without deviations, more see in the monograph [13]. Global transformations of the kind (1) may serve for investigation of oscilatory behavior of solutions from certain classes of linear differential equations because each of global pointwise transformation preserves distribution of zeros of solutions of differential equations, see e.g., [6], [12], [13], [14].

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However, transformations (1) can be applied to certain classes of a nonlinear equations, as well. For instance, we can mention the family of all equations

$$y'(x) = \sum_{i=0}^{n} a_i(x) b_i(y(x)) \Pi_{j=1}^{m} \delta_{i\,j}\big(y(\xi_j(x))\big)\,, \quad x \in \mathbf{j} \subseteq \mathbf{R}\,, \quad (\Pi\delta_j = \delta_1\delta_2\cdots\delta_m)$$

derived in [17]. Here $b_i,\ \delta_{i\,j}$ are nontrivial solutions of Cauchy's functional equation

$$b(uv) = b(u)b(v)\,, \quad (u, v \in \mathbf{R} - \{0\})$$

with the general solutions continuous at a point

$$b(u) = 0\,,\ b(u) = |u|^c\,,\ b(u) = |u|^c \text{ sign } x \quad (c \in \mathbf{R} \quad \text{being arbitrary constant}),$$

see Aczél [1]. The mentioned result was derived (without regularity conditions) by rather artificial functional equations assuming apriori the existence of differential equations of the kind

$$L'(x) = h\big(x, \varphi(x), L(x), L(\eta_1(x)), \dots , L(\eta_m(x))\big)\,,$$
$$\varphi'(x) = g\big(x, \varphi(x), L(x), L(\eta_1(x)), \dots , L(\eta_m(x))\big)$$

for functions L', φ', where the deviations η_i were defined by using equations $\xi_i(\varphi(x)) = \varphi(\eta_i(x))$, $x \in \mathbf{j} \subseteq \mathbf{R}$. We shall see that results of this kind can be (with regularity conditions) systematically obtained by quite other and more natural method. In this paper we solve the symmetry and the equivalence problem (local approach) for the transformations (1) and formulate some results in terms of global transformations. We apply the method of moving frames first on determined equation $y' = f(x, y)$ and then on the Monge equation $y' = f(x, y, z, z')$. A simple arrangement then provides a large hierarchy of classes of functional equations of the first order with deviation which admit the transformations (1).

A determined equations

1. The pseudogroup. Thorough this Part II of our article, the pseudogroup under consideration will consist of all invertible transformations

$$(2) \quad \bar{x} = \varphi(x)\,,\ \bar{y}^i = L_i(x) y^i \quad (i = 1, \dots , m;\ x \in \mathcal{D}(\varphi) \cap \mathcal{D}(L_1) \cap \ldots \cap \mathcal{D}(L_m))$$

applied to the curves $y^i = y^i(x)$ $(i = 1, \dots , m)$ lying in $\mathbf{R}^{m+1}$ which are transformed into the curves $\bar{y}^i = \bar{y}^i(\bar{x}) = L_i(x) y^i(x)$. (The definition domains $\mathcal{D}(y^i)$ of functions y^i are appropriately adapted.) Then the first order derivatives transform in accordance with the rule

$$(3) \quad \bar{y}^{i\,\prime} \varphi' = L_i' y^i + L_i y^{i\,\prime} \quad \Big(y^{i\,\prime} = \frac{dy^i}{dx}\,,\ \bar{y}^{i\,\prime} = \frac{d\bar{y}^i}{d\bar{x}}\,,\ \varphi' = \frac{d\varphi}{dx}\,,\ L_i' = \frac{dL_i}{dx}\Big)$$

written in a slightly abbreviated notation. This provides the *first order prolongation* of formulae (2).

In alternative terms of Part I Remark 3, our prolonged pseudogroup involves invertible transformations $\Phi : \mathcal{D}(\Phi) \to \mathcal{R}(\Phi)$ given by (2), (3), where $\mathcal{D}(\Phi), \mathcal{R}(\Phi) \subseteq \mathbf{R}^{2m+1}$ are certain open subsets with coordinates $x, y^1, \dots , y^m, y^{1\prime}, \dots , y^{m\prime}$. The invertibility means that $\varphi'(x), L_1(x), \dots , L_m(x)$ are nonvanishing functions on the definition domain $\mathcal{D}(\Phi)$ which is the direct product of the above domains $\mathcal{D}(\varphi)$ and $\mathcal{D}(L_i)$.

Passing to moving frames, the *primary pseudogroup* (2) *can be characterized by the property of invariance of forms* $\omega_0 = Adx$ *and* $\omega_i = dy^i/y^i - A_i dx$ $(i = 1, \dots, m)$*, where* $A \neq 0$ *and* A_i *are additional variables.* Indeed, the requirement

$$\omega_0 = Adx = \bar\omega_0 = \bar A d\bar x \quad (A, \bar A \neq 0)$$

ensures that $\bar x = \varphi(x)$ is a function of x and, assuming the invertibility, the transformation rule $A = \bar A\varphi'$ for the new variable A. Analogously,

$$\omega_i = \frac{dy^i}{y^i} - A_i dx = \bar\omega_i = \frac{d\bar y^i}{\bar y^i} - \bar A_i d\bar x \quad \big(\bar x = \varphi(x)\big)$$

implies $d\ln(\bar y^i/y^i) = (\bar A_i\varphi' - A_i)dx$, hence $\bar y^i/y^i = L_i(x)$ is a function of x, and then the rule $L_i'/L_i = \bar A_i\varphi' - A_i$ for the variables A_i.

One can also observe that invariance of the form $\vartheta = B(dy - y'dx)$ with a new variable B $(B \neq 0)$ provides the prolongation transformation (3) for the derivative y'. (A self-evident fact since the equation $\vartheta = 0$ determines the sence of the coefficient $y' = dy/dx$ and analogously $\bar\vartheta = 0$ means that $\bar y' = d\bar y/d\bar x$.)

2. Differential equations. The pseudogroup (2), (3) with $m = 1$ and the abbreviation $y = y^1, L = L_1$ will be applied to the differential equation $y' = f(x, y)$ which is transformed into certain differential equation $\bar y' = \bar f(\bar x, \bar y)$ where

$$\bar f\varphi' = L'y + Lf \tag{4}$$

by using (3). Instead of employing this transformation rule, we shall employ the invariance of the form $\vartheta = B(dy - f\,dx)$. We have moreover the invariant forms $\omega_0 = A\,dx$ and $\omega_1 = dy/y - A_1\,dx$. The dependence

$$\vartheta = yB\omega_1 - B(f - yA_1)\frac{\omega_0}{A} = 0$$

holds true and analogously with the dashed forms. It follows that

$$yB = \bar y\bar B\,,\ B(f - yA_1)/A = \bar B(\bar f - \bar y\bar A_1)/\bar A$$

and we may ensure the equalities

$$yB = \bar y\bar B = 1\,,\ B(f - yA_1)/A = \bar B(\bar f - \bar y\bar A_1)/\bar A = 0$$

by the choice $B = 1/y$, $A_1 = f/y$. So we obtain the invariant form

$$\omega = \frac{1}{y}(dy - f dx) = \bar\omega = \frac{1}{\bar y}(d\bar y - \bar f d\bar x)$$

which replace both ϑ and ω_1. In order to reduce the remaining variable A, the identity $d\omega = d\bar\omega$ may be employed. Clearly is

$$d\omega = dx \wedge d(f/y) = \frac{\omega_0}{A} \wedge (f/y)_y\, y\omega = \frac{y}{A}\,(f/y)_y\,\omega_0 \wedge \omega$$

in terms of invariant forms (and analogously for $d\bar\omega$). Two cases are to be distinguished:

(ι) If $(f/y)_y = 0$, then $f = g(x)y$ and we do not have any invariants.

($\iota\iota$) If $(f/y)_y \neq 0$, then we may choose $A = y\,(f/y)_y$ (and analogously $\bar{A}$) ensuring $d\omega = \omega_0 \wedge \omega = \bar{\omega}_0 \wedge \bar{\omega} = d\bar{\omega}$ and the invariant form $\omega_0 = y\,(f/y)_y\,dx$ (inserting A, we do not change the notation). However, $d\omega_0 = d\bar{\omega}_0$ where

$$d\omega_0 = d(y\,(f/y)_y)_y \wedge dx = (1+I)\omega \wedge \omega_0\,, \quad I = y\frac{(f/y)_{yy}}{(f/y)_y} = y\,(\ln|(f/y)_y|)_y$$

(and analogously for $d\bar{\omega}_0$) and we have the invariant $I = \bar{I}$.

3. Continuation. Assume (ι), hence $f = g(x)y$ and $\bar{f} = \bar{g}(\bar{x})\bar{y}$ are related by the rule (4) which simplifies as $\bar{g}(\bar{x})\varphi' = L'(x)/L(x) + g(x)$. The function $\bar{x} = \varphi(x)$ arbitrarily chosen to apply $\mathcal{D}(g)$ onto $\mathcal{D}(\bar{g})$ and then $L = \exp\{\int(\bar{g}(\varphi)\varphi' - g)dx\} = \exp\{\int \bar{g}(\bar{x})d\bar{x} - g(x)dx\}$ is determined up to a constant factor. In particular (if $g = \bar{g}$) the equation $y' = g(x)y$ admits symmetries isomorphic to the group of all diffeomorphisms of the definition domain $\mathcal{D}(g)$.

Let us turn to the more involved case ($\iota\iota$).

The highest symmetry subcase take place if $I = C = \text{const}$. Then

$$A = a(x)y^{C+1}\,, \quad \frac{f}{y} = \frac{a(x)}{C+1}y^{C+1} + b(x) \qquad (C \neq -1)\,,$$
(5)
$$\frac{f}{y} = a(x)ln|y| + b(x) \qquad (C = -1)$$

by a simple verification. The equivalence of equations $y' = f,\ \bar{y}' = \bar{f}$ is possible if and only if $\bar{I} = C$ is the same constant. It is determined by the completely integrable system

$$\omega_0 = \bar{\omega}_0\,, \quad \omega = \bar{\omega} \tag{6}$$

where

$$d\omega_0 = (1+C)\omega \wedge \omega_0\,, \quad d\omega = \omega_0 \wedge \omega \tag{7}$$

(analogously for dashed forms) and the Frobenius theorem can be applied. In more detail, assuming $C \neq -1$, hence (5_2), the structural formulae (7) can be simplified by introduction of the form $\vartheta = (1+C)\omega + \omega_0$. Clearly $d\vartheta = 0$ therefore ϑ is a total differential and the system (6) can be replaced by the simpler $\omega_0 = \bar{\omega}_0$, $\vartheta = \bar{\vartheta}$. So we have the equations

$$a(x)y^{C+1}\,dx = \bar{a}(\bar{x})\bar{y}^{C+1}\,d\bar{x}\,, \quad \frac{dy}{y} - b(x)\,dx = \frac{d\bar{y}}{\bar{y}} - \bar{b}(\bar{x})\,d\bar{x} \tag{8}$$

$\big(x = \varphi(x), \bar{y} = L(x)y\big)$ for the unknowns φ and L.

The equations (8) can be clarified in terms of better coordinates. First of all, we have

$$\frac{dy}{y} - b(x)\,dx = dz, \quad z = \ln|y| - \int b(x)\,dx \quad \big(y = e^{z+\int b\,dx}\big)$$

(analogously for dashed variables) and then (8_2) reads $dz = d\bar{z}$ whence $\bar{z} = z + k$ ($k = \text{const.}$) In terms of variables x and z, equation (8_1) simplifies as

$$a(x)e^{(C+1)\int b dx}\,dx = e^{(C+1)k}\bar{a}(\bar{x})e^{(C+1)\int \bar{b}\,d\bar{x}}\,d\bar{x}\,.$$

This can be drastically simplified (by the change of variables x and $\bar{x}$) to the equation $dx = e^{(C+1)k}\,d\bar{x}$ whence $\bar{x} = e^{-(C+1)k}x + l$ ($l = \text{const.}$). So the higher symmetry subcase is governed by the *two-parameter Lie group*

$$\bar{z} = z + k\,, \quad \bar{x} = e^{-(C+1)k}x + l \quad (k, l \text{ constants})$$

which is non-Abelian since $C \neq -1$.

Assuming $C = -1$, hence (5_3), the form $\omega_0 = a(x)\,dx$ is a total differential and the system (6) reads

$$a(x)\,dx = \bar{a}(\bar{x})\,d\bar{x}\,, \quad \frac{dy}{y} - (a(x)\ln|y| + b(x))\;dx = \frac{d\bar{y}}{\bar{y}} - \left(\bar{a}(\bar{x})\ln|\bar{y}| + \bar{b}(\bar{x})\right)\;d\bar{x}\,.$$

After a change of variables x and $\bar{x}$, one can ensure $a(x) = \bar{a}(\bar{x}) = 1$, hence the conditions

$$dx = d\bar{x}\,, \quad dz - z\,dx = (b(x) - \bar{b}(\bar{x}))\,dx\,, \quad \left(z = \ln\frac{\bar{y}}{y} = \ln|L|\right)$$

for the sought equivalence transformation. One can then observe that $dz - zdx = e^x d\,(ze^{-x})$, the second condition is simplified as

$$d\left(ze^{-x}\right) = e^{-x}(b(x) - \bar{b}(\bar{x}))\,dx$$

and we obtain the transformations

$$\varphi(x) = \bar{x} = x + \text{const.}, \quad |L(x)| = e^z = e^{e^x \int e^{-x}(b(x) - \bar{b}(x+\text{const.}))\,dx}$$

depending on two parameters.

The middle symmetry subcase takes place if I is not a constant and other invariants are functions of this I. Let F be such an invariant. Clearly

$$(9) \qquad dF = F_x\,dx + F_y\,dy = \frac{1}{A}(F_x + fF_y)\omega_0 + yF_y\omega \quad \left(A = y\,(f/y)_y\right)$$

and it follows that the functions $\partial F/\partial\omega_0 = (F_x + fF_y)/A$, $\partial F/\partial\omega = yF_y$ are invariants, too, in particular $yI_y = G(I)$ is a composed function. It follows easily that either of the possibilities may in principle occur:

$$(10) \qquad I = I(x) \;\; (\text{if} \;\; I_y = G(I) = 0)\,, \quad I = a(b(x)y) \;\; (\text{if} \;\; G(I) \neq 0),$$

where a, b are appropriate functions and $b(x)y$ again is an invariant (hence

$$(11) \qquad b(x)y = \bar{b}(\bar{x})\bar{y} = \bar{b}(\varphi(x))L(x)y$$

with the corresponding dashed objects).

One can verify that (10_1) leads to the contradiction. (Hint: The equation $I(x) = y(ln(f/y)_y)_y$ provides the formulae

$$f/y = c(x)y^{I(x)+1} + d(x)\,, \quad A = C(x)y^{I(x)+1}$$

with appropriate $c(x)$, $d(x)$, $C(x)$ but the invariant $\partial I/\partial\omega_0 = I'/A$ is not a function of the variable x alone.) The second possibility leads to the formula

$$(12) \qquad f/y = c(x)a(b(x)y) + d(x)$$

for appropriate functions $c(x)$, $d(x)$. Conversely, assuming (12), one can find

$$A = c(x)b(x)ya'(b(x)y)\,, \quad I = yb(x)a''(b(x)y)/a'(b(x)y) \tag{13}$$

by direct calculation. Then $\partial I/\partial\omega$ is a function of $b(x)y$ (hence of I) by easy verification, however, $\partial I/\partial\omega_0$ is a function of this kind if and only if the condition

$$\frac{1}{c(x)}\left(\frac{b'(x)}{b(x)} + d(x)\right) = \text{const.} \tag{14}$$

is satisfied (and analogously for dashed variables).

Assuming (13), the equivalence problem leads to *one-parameter Lie group.* Indeed, the system (6) ensuring the equivalence can be replaced by the equivalent requirements $I = \bar{I}, \omega_0 = \bar{\omega}_0$ expressed by

$$b(x) = \bar{b}(\varphi(x))L(x)\,, \quad c(x) = \bar{c}(\varphi(x))\varphi'(x)\,. \tag{15}$$

(Hint. $I = \bar{I}$ implies $dI = d\bar{I}$ which together with $\omega_0 = \bar{\omega}_0$ implies $\omega = \bar{\omega}$, see (9) applied to $F = I$. Moreover $I = \bar{I}$ is equivalent to (15_1) and (15_2) follows by using $\omega_0 = Adx$ with coefficient A given by (13_1).) The requirement (15) simplifies into $\varphi'(x) = 1$ (hence $\varphi(x) = x + \text{const.}$) if coordinates $x, \bar{x}$ are changed appropriately. Then (15_2) uniquely determines $L(x)$.

The lower symmetry subcase takes place if there are two functionaly independent invariants in the family $I, J = \partial I/\partial\omega_0, K = \partial I/\partial\omega$. Then the Pfaffian system (9) determining the equivalence can be replaced by the algebraical requirements

$$I = \bar{I}\,, \quad J = \bar{J}\,, \quad K = \bar{K} \tag{16}$$

as follows by using (9) with F equal to either I, J, K. If (e.g.) I, J are functionaly independent and $K = k(I, J)$, then the compatibility condition $\bar{K} = k(\bar{I}, \bar{J})$ is necessary and sufficient for the existence of equivalence desired.

An underdetermined equation

4. The Monge equation. We are passing to the main topic of this article. The equivalence problem for the underdetermined differential equation $y' = f(x, y, z, z')$ with respect to the pseudogroup of (inverible, local) transformations

$$\bar{x} = \varphi(x)\,, \quad \bar{y} = L(x)y\,, \quad \bar{z} = M(x)z \qquad (\varphi'(x)L(x)M(x) \neq 0) \tag{17}$$

will be investigated. Recalling the prolongation

$$\bar{y}'\varphi' = Ly' + L'y\,, \quad \bar{z}'\varphi' = Mz' + M'z$$

to first order derivatives (see (2), (3), (4) with $m = 2$ and abbreviation $y = y^1$, $z = y^2$, $L = L_1$, $M = L_2$), the transformed equation reads

$$\bar{y}' = \bar{f}(\bar{x}, \bar{y}, \bar{z}, \bar{z}') \quad (\bar{f}\varphi' = Lf + L'y)\,, \tag{18}$$

however, instead of direct use of (18), we shall employ the invariance conditions

$$\omega_i = \bar{\omega}_i \qquad (i = 0, 1, 2) \tag{19}$$

with differential forms

$$\omega_0 = A\,dx\,, \quad \omega_1 = \frac{1}{y}(dy - f\,dx)\,, \quad \omega_2 = \frac{1}{z}(dz - z'\,dx) \tag{20}$$

(where $A \neq 0$ is a new parameter) and the relevant dashed counterparts (e.g., $\bar\omega_2 = (d\bar z - \bar z' d\bar x)/\bar z$) *ensuring both the existence of the transformations of the kind* (17) *together with the prolongation and the formula* (18). (Easy direct proof of this assertion may be omitted, see also Sections 1 and 2 for analogous arguments.)

In accordance with general principles of mowing frames, let us continue with the exterior derivative $d\omega_i = d\bar\omega_i$ of equations (19) expressed in terms of invariant forms. Clearly

$$d\omega_1 = dx \wedge d(f/y) = \frac{\omega_0}{A} \wedge \Big((f/y)_y\, y\omega_1 + (f/y)_z\, z\omega_2 + (f/y)_{z'}\ dz' \Big)\ ,$$

where the last term causes a difficulty. However, the equality $d\omega_2 = d\bar\omega_2$ with

$$d\omega_2 = dx \wedge d\frac{z'}{z} = \frac{\omega_0}{A} \wedge \frac{1}{z}(dz' - \frac{z'}{z}\,dz) = \omega_0 \wedge \omega_3\,, \tag{21}$$

$$\omega_3 = \frac{1}{Az}(dz' - z'\omega_2 - B\omega_0) \tag{22}$$

(where $B \neq 0$ is a new parameter) means that the new form ω_3 is uniquely determined, hence invariant. (In more detail, $\omega_3 = \bar\omega_3$ holds true for the equivalence transformation.) If dz' is expressed in terms of ω_3 by using (22), we obtain better formula

$$d\omega_1 = \omega_0 \wedge \left(\frac{y}{A}\,(f/y)_y\,\omega_1 + \frac{1}{A}\,(z\,(f/y)_z + z'\,(f/y)_{z'})\,\omega_2 + z\,(f/y)_{z'}\,\omega_3 \right) \tag{23}$$

than above (and analogously $d\bar\omega_1$). It follows that the coefficients

$$\frac{y}{A}\,(f/y)_y\ , \quad \frac{1}{A}\,(z\,(f/y)_z + z'\,(f/y)_{z'})\ , \quad I = z\,(f/y)_{z'} \tag{24}$$

are transformed into the dashed couterparts. In particular, we have the invariant $I = \bar I$ (independent of auxiliary variable A).

At this place, covariant derivatives are to be recalled: if (e.g.) $F = F(x, y, z, z')$ is an invariant, then we have developments

$$dF = F_x\,dx + F_y dy + F_z\,dz + F_{z'}\,dz' = \sum \frac{\partial F}{\partial \omega_i}\omega_i\,,$$

where the coefficients

$$\begin{aligned} &\frac{\partial F}{\partial \omega_0} = \frac{1}{A}\,(F_x + fF_y + z'F_z) + BF_{z'}\ , \quad && \frac{\partial F}{\partial \omega_1} = yF_y\,, \\ &\frac{\partial F}{\partial \omega_2} = zF_z + z'F_{z'}\,, && \frac{\partial F}{\partial \omega_3} = AzF_{z'} \end{aligned} \tag{25}$$

are transformed into the dashed couterparts. This may provide new invariants and the procedure can be repeatedly applied.

Returning to (24), we will distinguish three cases.

(ι) Assuming $y\,(f/y)_y \neq 0$, then the coefficient (24_1) can be reduced to unity by appropriate choice of A and we obtain new invariant J by inserting this A into (24_2):

$$A = y\,(f/y)_y = f_y - f/y\,, \quad J = \frac{zf_z + z'f_{z'}}{yf_y - f}\,. \tag{26}$$

Additional invariants will arize by using the invariant form ω_0, see below.

($\iota\iota$) If $y\,(f/y)_y = 0$ identically but $zf_y + z'f_{z'} \neq 0$, then, denoting $f = yg(x,z,z')$, we may reduce (24_2) to unity by the choice

$$A = z\,(f/y)_z + z'\,(f/y)_{z'} = zg_z + z'g_{z'}\,. \tag{27}$$

Clearly $I = zg_{z'}$ and additional invariants will arize from the form ω_0.

($\iota\iota\iota$) If $y\,(f/y)_y = zf_y + z'f_{z'} = 0$ identically, then $f = yg(x, z'/z)$ with $I = zg_{z'}$. Determination of coefficient A and other invariants will be mentioned in Section 10.

If the coefficient $A = A(x,y,z,z')$ is specified (by using formulae like (26_1) or (27_1)), then the exterior derivative

$$d\omega_0 = \frac{dA}{A} \wedge \omega_0 = (K_1\omega_1 + K_2\omega_2 + K_3\omega_3) \wedge \omega_0\,, \tag{28}$$

provides new invariants

$$K_1 = y\frac{A_y}{A}\,, \quad K_2 = z\frac{A_z}{A} + z'\frac{A_{z'}}{A}\,, \quad K_3 = zA_{z'}\,. \tag{29}$$

On this occation, let us recall the remaining structural formula

$$d\omega_1 = \omega_0 \wedge (\omega_1 + J\omega_2 + I\omega_3)\,, \quad d\omega_2 = \omega_0 \wedge \omega_3 \quad (\text{case} \quad (\iota))\,, \tag{30}$$

$$d\omega_1 = \omega_0 \wedge (\omega_2 + I\omega_3)\,, \quad d\omega_2 = \omega_0 \wedge \omega_3 \quad (\text{case} \quad (\iota\iota))\,, \tag{31}$$

determining the already well-known invariants I and J.

5. Case (ι) with constant invariants. We shall investigate the rather special situation when all invariants are constant. Especially the equation $z(f/y)_{z'} = I =$ const. implies that

$$\frac{f}{y} = I\frac{z'}{z} + a(x,y,z) \tag{32}$$

for appropriate function a. Then $A = ya_y$ by using (26_1), therefore $K_3 = 0$ identically and we may denote

$$l = \text{const.} = K_1 = y\frac{A_y}{A} \qquad (\text{hence} \quad A = b(x,z)|y|^l)\,,$$

$$m = \text{const.} = K_2 = z\frac{A_z}{A} \qquad (\text{hence} \quad A = b(x,y)|z|^m)\,,$$

with appropriate factor b (not the same). Altogether taken, $ya_y = A = b(x)|y|^l|z|^m$ with certain nonvanishing factor b whence

$$(33)\qquad \begin{aligned} a &= b(x)\frac{|y|^l}{l}|z|^m + c(x,z) \qquad (\text{if} \quad l \neq 0)\,,\\ a &= b(x)|z|^m \ln|y| + c(x,z) \quad (\text{if} \quad l = 0)\,. \end{aligned}$$

On the other hand, if $A = b(x)|y|^l|z|^m$ and (32) are inserted into (26_2), one can obtain $Jb(x)|y|^l|z|^m = za_z$ whence

$$(34)\qquad \begin{aligned} a &= Jb(x)|y|^l\frac{|z|^m}{m} + d(x,y) \quad (m \neq 0)\,,\\ a &= Jb(x)|y|^l \ln|z| + d(x,y) \quad (m = 0)\,. \end{aligned}$$

By comparing both results (33) and (34), it follows that either of the following possibilities take place

$$\begin{aligned} a &= b(x)\frac{|y|^l}{l}|z|^m + c(x) \qquad (\text{if} \quad l \neq 0,\ m \neq 0 \ \text{ and then } \ \frac{1}{l} = \frac{J}{m})\,,\\ a &= b(x)\ln|y| + Jb(x)\ln|z| + c(x) \qquad (\text{if} \quad l = m = 0)\,. \end{aligned}$$

Insertion into (32) yields the final result

$$(35)\qquad \frac{f}{y} = I\frac{z'}{z} + b(x)\frac{|y|^l}{l}|z|^m + c(x) \qquad (\text{if} \ \ l \neq 0 \ \text{ and } \ m \neq 0)\,,$$

$$(36)\qquad \frac{f}{y} = I\frac{z'}{z} + b(x)\ln|y||z|^J + c(x) \qquad (\text{if} \ \ l = m = 0)\,.$$

Moreover $Jm = l$ and $A = b(x)|y|^l|z|^m$ in the case (35), $A = b(x)$ in the case (36). Consequently $b(x) \neq 0$ but $c(x)$ may be quite arbitrary function. The possibilities $m = 0$, $l \neq 0$ and $m \neq 0$, $l = 0$ do not occur.

Concerning the equivalence transformation between equations $y' = f$ and $\bar{y}' = \bar{f}$ of the kind (35), the equality of invariants provides a necessary but rather poor requirement, therefore all equations (19) must be taken into account. Then, owing to formulae (28), (29) with $K_1 = l$, $K_2 = m$, $K_3 = 0$ interrelated by $Jm = l$, one can observe that $d\omega = 0$ identically, where

$$\omega = \omega_1 + \frac{1}{l}\omega_0 - I\omega_2 = d\ln C(x)\frac{|y|}{|z|^l} \qquad \left(C(x) = e^{\int c(x)dx}\right)\,,$$

by using (20) with function f/y of the kind (35). The original system (19) may be replaced by the conditions $\omega_0 = \bar{\omega}_0$, $\omega = \bar{\omega}$, $\omega_1 = \bar{\omega}_1$, where the last one can be omitted (it provides a mere transformation rule for z'). So we have the conditions

$$b(x)|y|^l|z|^m dx = \bar{b}(\bar{x})|\bar{y}|^l|\bar{z}|^m d\bar{x}\,, \qquad C(x)|y|\,|z|^{-I}\cdot\text{const.} = \bar{C}(\bar{x})|\bar{y}|\,|\bar{z}|^{-I}$$

(const. $\neq 0$) whence

$$(37)\quad b(x) = \bar{b}(\varphi)|L|^l|M|^m\varphi' = \bar{b}(\varphi)(|L|\,|M|^J)^l\,, \qquad C(x)\cdot\text{const.} = \bar{C}(\varphi)|\,|M|^{-I}$$

by using formulae (17). In particular, it follows that

$$b(x)|\bar{C}(\varphi)|^l = \bar{b}(\varphi)\varphi'|C(x)\cdot\text{const.}\,|^l|M|^{l(I+J)}$$

and these conditions can be comfortably discussed.

Analogously, the equivalence transformations of the kind (36) can be determined from the equations $\omega_0 = \bar{\omega}_0$, $\omega_1 - I\omega_2 = \bar{\omega}_1 - I\bar{\omega}_2$. In this way, we obtain the conditions

$$b(x)\,dx = \bar{b}(\bar{x})\,d\bar{x}\,, \quad \text{hence} \quad B(x) + \text{const.} = \bar{B}\big(\varphi(x)\big) \tag{38}$$

(where $B = \int b\,dx$) and

$$d\ln\frac{|y|}{|z|^I} + \big(b(x)\ln|y|\,|z|^J + c(x)\big)\ dx = d\ln\frac{|\bar{y}|}{|\bar{z}|^I} + \big(\bar{b}(\bar{x})\ln|\bar{y}|\,|\bar{z}|^J + \bar{c}(\bar{x})\big)\ d\bar{x}\,,$$

hence

$$\frac{d}{dx}\ln\frac{|L|}{|M|^I} + b(x)\ln|L|\,|M|^J + \bar{c}(\varphi)\varphi' = c(x)\,, \tag{39}$$

by using (17). This is a quite reasonable result.

6. Case $(\iota\iota)$ with constant invariants. Since $zg_{z'} = I = \text{const.}$, it follows that $g = Iz'/z + k(x,z)$ whence $A = zk_z$ (where $k_z \neq 0$) by using (27). Passing to invariants (29), one can easily obtain the condition $k_{zz}/k_z = C/z$ ($C = \text{const.}$) whence

$$k = b(x)\frac{|z|^{C+1}}{C+1} + c(x) \quad (\text{if } \ C \neq -1)\,, \quad k = b(x)\ln|z| + c(x) \quad (\text{if } \ C = -1)$$

for appropriate functions $b(x) \neq 0$ and $c(x)$. So we have two possibilities:

$$\begin{aligned} &\frac{f}{y} = I\frac{z'}{z} + b(x)\frac{|z|^{C+1}}{C+1} \qquad (C \neq -1)\,, \\ &\frac{f}{y} = I\frac{z'}{z} + b(x)\ln|z| + c(x)\,, \end{aligned} \tag{40}$$

where $b(x)$ is a nonvanishing function.

Equivalence transformations between equations $y' = f$ and $\bar{y}' = \bar{f}$ of the kind $(\iota\iota)$ can be determined by using a little adapted equations (19). In more detail, assuming the first possibility (40_1), one can observe that the form

$$\omega = \omega_1 + \frac{\omega_0}{C+1} - I\omega_2 = d\left(\ln C(x)\frac{|y|}{|z|^I}\right) \qquad \left(C(x) = e^{\int c(x)\,dx}\right)$$

is a total differential and the equivalences are determined by means of simplified equations $\omega_0 = \bar{\omega}_0$, $\omega = \bar{\omega}$. Analogously, assuming (41_2), then already the form $\omega_0 = c(x)\,dx$ is (locally) a total differential, moreover the form

$$\omega = \omega_1 - I\omega_2 = d\left(\ln|y|\,|z|^I + E(x)\right) - c(x)\ln|z|\,dx$$

not depending on the derivative could be advantageously employed. We shall not state the final requirements for φ, L, M since they do not differ much from previous formulae (37), (38), (39).

The similarity of final results in cases (ι) and $(\iota\iota)$ is not a self–evident fact because of quite dissimilar initial data, e.g., coefficient A in the case $(\iota\iota)$ is quite other than in (ι) and corresponds (in a certain) sense to the invariant J. It would be desirable to invent a universal approach involving both (ι) and $(\iota\iota)$.

7. Case ($\iota\iota$) with constant invariants. Recall that $f = yg(x,t)$, where $t = z'/z$, and we have the invariant const. $= I = g_t$, hence $g = It + c(x)$. System (19) determining the equivalence transformations can be replaced by the simpler $\omega_0 = \bar{\omega}_0$, $\omega_1 - I\omega_2 = \bar{\omega}_1 - I\bar{\omega}_2$ which reads

$$Adx = \bar{A}d\bar{x}\,, \quad d\ln C(x)\frac{|y|}{|z|^I} = d\ln \bar{C}(\bar{x})\frac{|\bar{y}|}{|\bar{z}|^I} \qquad \left(C(x) = e^{\int c(x)\,dx}\right).$$

Using explicit formulae (17) of transformations, one can obtain only the single interrelation (37_2) for the functions φ, L, M. It follows that there exists a large family of equivalences since the function $\bar{x} = \varphi(x)$ can be in principle quite arbitrary.

8. Nonconstant invariants. We shall thoroughly discuss the highest possible symmetry problem with nonconstant invariants: let all invariants be composed functions of the kind $G(F)$, where F is a certain "basical" nonconstant invariant and the letter G will (systematically) denote various functions of one independent variable. The "basical" invariant F can be made more explicit by using covariant derivative (25). For instance the requirement $yF_y = G(F)$ regarded as a differential equation has the general solution

$$F = \mathcal{G}\big(a(x,z,z')y\big)$$

and it follows that we may assume either of the simplified versions

$$F = a(x,z,z')y \qquad (\text{if} \quad G \neq 0)\,,$$
$$F = a(x,z,z') \qquad (\text{if} \quad G = 0)\,.$$

We shall, however, use the more advantageous transcription

$$F = a(x,z,t)y\,, \quad F = a(x,z,t) \quad (\text{where} \quad t = \frac{z'}{z})\,, \tag{41}$$

from now on. With this notation, assumptions (41) substituted into the requirement

$$\partial F/\partial\omega_2 = zF_z + z'F_{z'} = zF_z + 0F_t = G(F)$$

read

$$z\frac{a_z}{a} = \frac{G(ay)}{ay}\,, \quad za_z = G(a)\,, \tag{42}$$

respectively. Identity (42_1) corresponding to (41_1) clearly implies $G(ay)/(ay) = C = \text{const.}$, whence

$$a = b(x,t)|z|^C\,, \qquad C \in \mathbf{R}\,. \tag{43}$$

Identity (42_2) may be regarded for a differential equation with the general solution

$$a = \mathcal{G}(b(x,t)z) \quad (\text{if} \quad G \neq 0)\,,$$
$$a = b(x,t) \qquad (\text{if} \quad G = 0)$$

and it follows that the assumptions (41) can be still improved: the "basical" invariant F can be taken of either kind

$$F = b(x,t)|z|^C y\,, \quad F = b(x,t)z\,, \quad F = b(x,t)\,. \tag{44}$$

With this preliminary result, the last requirement $AzF_{z'} = AF_t = G(F)$ concerning covariant derivatives is either triviality (if $b_t = 0$) or provides the universal formula

$$A = \frac{b}{b_t} H(F) \qquad (\text{if} \quad b_t \neq 0) \tag{45}$$

for the coefficient A (where $H(u) = G(u)/u$ is an arbitrary function).

Let us deal with invariants (29) corresponding to the coefficient A.

First assume $b_t \neq 0$ in order to employ formula (45). Requirements $yA_y/A = G(F)$ and $zA_z/A + z'A_{z'}/A = G(F)$ are always satisfied. Requirement $zA_{z'} = A_t = G(F)$ provides the universal condition

$$(b/b_t)_t = G(F) \tag{46}$$

(G is changed) for all possibilities (44). Assuming (44_1) or (44_2), then $G(F) = E = \text{const.}$ (by using $\partial/\partial y$, $\partial/\partial z$ in (46)) and one can obtain the formulae

$$\begin{aligned} &b = c(x)|Et + e(x)|^{1/E}, \quad &&\frac{b}{b_t} = Et + e(x) \quad &&(\text{if} \quad E \neq 0), \\ &b = c(x)e^{t/e(x)}, &&\frac{b}{b_t} = e(x) &&(\text{if} \quad E = 0) \end{aligned} \tag{47}$$

where $c(x) \neq 0, e(x)$ $\big(e(x) \neq 0$ in (47_2)$\big)$ are arbitrary. Assuming (44_3), then

$$b = K\big(c(x)t + e(x)\big) \qquad (K' \neq 0,\ c(x) \neq 0), \tag{48}$$

so we may simplify by putting $F = c(x)t + e(x)$ and $A = H(F)/c(x)$ in this case. Thus

$$\begin{array}{lll} A = (Et + e(x))H(F), & F = c(x)|Et + e(x)|^{1/E}|z|^C y, & E \neq 0,\ C \in \mathbf{R}, \\ A = (Et + e(x))H(F), & F = c(x)|Et + e(x)|^{1/E} z, & E \neq 0, \\ A = e(x)H(F) & F = c(x)e^{t/e(x)}|z|^C y, & e(x) \neq 0,\ C \in \mathbf{R}, \\ A = e(x)H(F) & F = c(x)e^{t/e(x)} z, & e(x) \neq 0, \\ A = H(F)/c(x), & F = c(x)t + e(x) & \end{array} \tag{49}$$

$\big(c(x) \neq 0\big)$ are our results for $b_t \neq 0$.

Second, assume $b_t = 0$, hence $b = b(x)$ in formulae (44). Let us pass to the (as yet unknown) coefficient $A = A(x, y, z, t)$ ($t = z'/z$). We will use invariants (29)

$$K_1 = \frac{yA_y}{A} = G(F), \quad K_1 = \frac{zA_z}{A} = G(F), \quad K_3 = A_t = G(F), \tag{50}$$

where G denote various functions.

(a) We have $yF_y = F, zF_z = CF$ for the invariant $F = b(x)|z|^C y$, $C \in \mathbf{R}$. Then $A = h(x, z, t)H(F)$ is a solution of (50_1) with certain nonvanishing functions h, H. Substituting A into (50_2) we obtain the condition

$$zh_z/h = G(F) - CFH'(F)/H(F) = D = \text{const.} \in \mathbf{R}$$

(use $\partial/\partial y$ for the last equation). Thus $h = c(x,t)|z|^D$ with a nonzero function c, $A = c(x,t)|z|^D H(F)$, $D \in \mathbf{R}$. The remaining condition $c_t|z|^D H(F) = K(F)$ (see (50_3)) is then

$$c_t|z|^D = K(F)/H(F) = E = \text{const.}$$

(use $\partial/\partial y$ for the last equation). Hence either

$$A = c(x)|z|^D H(F)\,, \quad D \in \mathbf{R} \qquad \text{(if} \quad E = 0) \tag{51}$$

or

$$A = (Et + e(x))H(F) \qquad \text{(if} \quad E \neq 0) \tag{52}$$

for the invariant $F = b(x)|z|^C y$, $C \in \mathbf{R}$.

(b) Similarly, $F_y = 0$, $zF_z = F$ in the subcase $F = b(x)z$. The condition (50_1) is equivalent to $A = h(x,z,t)|y|^{H(F)}$, (50_2) is corresponding to

$$zh_z/h = G(F) - FH'(F)\ln|y| \;\text{ and }\; H'(F) = 0\,, \;\text{ i.e., }\; H(F) = H = \text{const.} \in \mathbf{R}\,.$$

Therefore $zh_z = G(F)$ implies $h = c(x,t)K(F)$, $A = c(x,t)K(F)|z|^H$ with certain nonzero functions c, K. From (50_3) we get $c_t|y|^H = E(F)$ and either $c_t = 0$ for $H \neq 0$ (use $\partial/\partial y$) or $E(F) = E = \text{const.} \in \mathbf{R}$ for $H = 0$ (use $\partial/\partial z$). Thus

$$\begin{array}{llll} & A = c(x)K(F)|y|^H\,, & F = b(x)z & H \in \mathbf{R} - \{0\}\,, \\ \text{and} & A = (Et + e(x))K(F)\,, & F = b(x)z\,, & E \neq 0 \\ \text{and} & A = c(x)K(F) & F = b(x)z\,, & (\text{if } E = 0) \end{array} \tag{53}$$

for certain nonvanishing functions $c(x)$, K, respectively.

(c) The requirement $yA_y/A = H(b(x))$, corresponding to (50_1) in the subcase $F = b(x)$, implies $A = b(x,z,t)|y|^{Hb(x)}$ for certain functions $h \neq 0, H$. The next condition (50_2) is $zh_z/h = K(b(x))$ with the solution $h = c(x,t)|z|^{K(b(x))}$, i.e., $A = c(x,t)|y|^{H(b(x))}|z|^{K(b(x))}$ with certain functions $c \neq 0, K, H$. The condition (50_3) we read

$$c_t(x,t)|y|^{H(b(x))}|z|^{K(b(x))} = G\big(b(x)\big)$$

and either $c_t = G(b(x))$ for $H(b(x)) = K(b(x)) = 0$ or $c_t = 0$ in another subcases. Thus either

$$A = G(F)t + e(x)\,, \quad G(F)^2 + e(x)^2 \neq 0\,, \qquad F = b(x) \tag{54}$$

or

$$A = c(x)|y|^{H(b(x))}|z|^{K(b(x))}\,, \quad H(F)^2 + K(F)^2 \neq 0\,, \qquad F = b(x) \tag{55}$$

for nonvanishing functions b, c, respectively.

Before passing to remaining invariants I, J and determination of the crucial function f, let us review our achievements

$$
\begin{aligned}
&(56)\quad A = (Et + e(x))H(F)\,, && F = c(x)|Et + e(x)|^{1/E}|z|^C y\,, && (E \neq 0, C \in \mathbf{R})\\
&(57)\quad A = (Et + e(x))H(F)\,, && F = c(x)|Et + e(x)|^{1/E} z\,, && (E \neq 0)\\
&(58)\quad A = (Et + e(x))H(F)\,, && F = b(x)|z|^C y\,, && (C \in \mathbf{R})\\
&(59)\quad A = (Et + e(x))H(F)\,, && F = b(x)z\,, && \\
&(60)\quad A = e(x)H(F)\,, && F = c(x)e^{t/e(x)}|z|^C y\,, && (e(x) \neq 0, C \neq 0)\\
&(61)\quad A = e(x)H(F)\,, && F = c(x)e^{t/e(x)} y\,, && (e(x) \neq 0)\\
&(62)\quad A = e(x)H(F)\,, && F = c(x)e^{t/e(x)} z\,, && (e(x) \neq 0)\\
&(63)\quad A = e(x)H(F)\,, && F = c(x)|z|^C y\,, && (C \in \mathbf{R})\\
&(64)\quad A = e(x)H(F)\,, && F = c(x)z\,, && \\
&(65)\quad A = e(x)|z|^D H(F)\,, && F = b(x)|z|^C y\,, && (D \neq 0, C \in \mathbf{R})\\
&(66)\quad A = e(x)|y|^D H(F)\,, && F = b(x)z\,, && (D \neq 0)\\
&(67)\quad A = H(F)/c(x)\,, && F = c(x)t + e(x)\,, && \\
&(68)\quad A = e(x)\,, && F = b(x)\,, && \\
&(69)\quad A = G(F)t + e(x)\,, && F = b(x)\,, && (G(F) \neq 0)\\
&(70)\quad A = e(x)|y|^{G(F)}\,, && F = b(x)\,, && (G(F) \neq 0)\\
&(71)\quad A = e(x)|z|^{G(F)}\,, && F = b(x)\,, && (G(F) \neq 0)\\
&(72)\quad A = e(x)|y|^{H(F)}|z|^{G(F)}\,, && F = b(x)\,, && (G(F) \neq 0\,,\quad H(F) \neq 0)
\end{aligned}
$$

with $t = z'/z$, nonvanishing functions A, F and constants C, D, E.

9. Continuation. We are eventually passing to the concluding step, to invariants I, J and function f. Employing the above results, the reasoning will be quite simple but rather lengthy since the needful interrelations differ according to cases (ι)–$(\iota\iota\iota)$.

Let us begin with assumption (ι). Then the strategy is as follows: denoting for a moment $\mathcal{F}(x, y, z, t) = f(x, y, z, z')/y$, where $t = z'/z$, we have the conditions

$$
y\mathcal{F}_y = A\,, \quad \mathcal{F}_t = I\,, \quad \frac{z}{A}\mathcal{F}_z = J\,, \tag{73}
$$

where A is given by either formula (56)–(72) and I, J are composed functions of the kind $G(F)$ with well-known argument F. Then (73_1) determines the function $\mathcal{F}$ modulo a summand $k(x, z, t)$ and these $\mathcal{F}$, k can be easily corrected by using

(73_2), (73_3). We state the final result:

$$f/y = (Et + e(x))\mathcal{H}(F) + mt + l(x)\,, \quad \text{(see (56) for } A, F) \tag{74}$$

$$f/y = (Et + e(x))\mathcal{H}(F) + l(x)\,, \quad \text{(see (58) for } A, F) \tag{75}$$

$$f/y = e(x)(\mathcal{H}(F) + \ln|z|^m) + nt + l(x)\,, \quad \text{(see subcases (60), (61), (63) for } A, F) \tag{76}$$

$$f/y = He(x)(\mathcal{J}(F) + \ln|y|) + nt + l(x)\,,\ H(F) \equiv H = \text{const.}, \quad \text{(see (62), (64) for } A, F) \tag{77}$$

$$f/y = e(x)|z|^D(\mathcal{J}(F) + \text{const.}) + nt + l(x)\,, \quad \text{(see (65) for } A, F) \tag{78}$$

$$f/y = (1/D)e(x)|y|^D H(F) + nt + l(x)\,, \quad \text{(see (66) for } A, F) \tag{79}$$

$$f/y = (1/c(x))(\tilde{\mathcal{I}}(F) + H\ln|y||z|^k) + l(x)\,,\ H(F) \equiv H\,, \quad \text{(see (67) for } A, F) \tag{80}$$

$$f/y = e(x)\ln|z|^{\mathcal{J}(F)}|y| + I(F)t + l(x)\,, \quad \text{(see (68) for } A, F) \tag{81}$$

$$f/y = (e(x)/G(F))|y|^{G(F)} + I(F)t + l(x)\,, \quad \text{(see (70) for } A, F) \tag{82}$$

$$f/y = (e(x)/H(F))|y|^{H(F)}|z|^{K(F)} + I(F)t + l(x)\,, \quad \text{(see (72) for } A, F) \tag{83}$$

with $t = z'/z$, constants k, m, n, D, H, functions $\mathcal{H}(F) = \int (H(F)/F)dF$, $\mathcal{J}(F) = \int (J(F)/F)dF$, $\tilde{\mathcal{I}}(F) = \int I(F)dF$, leaving contradictions (57), (59), (69), (71) out of considerations.

We continue with assumption ($\iota\iota$). Then $f = g(x, z, z')y$ and denoting $\mathcal{G}(x, z, t) = g(x, z, z')$ for a moment (with the usual $t = z'/z$), we have two conditions

$$z\mathcal{G}_z = A\,, \quad \mathcal{G}_t = I \tag{84}$$

replacing the previous (73) and calculations simplify: (84_1) determines the function $\mathcal{G}$ and (84_2) provides the necessary correction. Omitting details and contradiction subcases, the final result reads

$$f/y = (Et + e(x))\mathcal{H}(F) + mt + l(x)\,, \quad \text{(see (57) for } A, F) \tag{85}$$

$$f/y = e(x)\ln|z|^H + mt + l(x)\,,\ H(F) \equiv H \neq 0\,, \quad \text{(see (61), (63) for } A, F) \tag{86}$$

$$f/y = e(x)\mathcal{H}(F) + mt + l(x)\,, \quad \text{(see (62), (64) for } A, F) \tag{87}$$

$$f/y = (1/c(x))(\tilde{\mathcal{I}}(F) + \ln|z|^H) + l(x)\,,\ H(F) \equiv H \neq 0 \quad \text{(see (67) for } A, F) \tag{88}$$

$$f/y = e(x)\ln|z| + I(F)t + l(x)\,, \quad \text{(see (68) for } A, F) \tag{89}$$

$$f/y = (e(x)/G(F))|z|^{G(F)} + I(F)t + l(x)\,, \quad \text{(see (71) for } A, F) \tag{90}$$

with $t = z'/z$, nonvanishing functions A, F constants k, m, H, D functions $\mathcal{H}(F) = \int (H(F)/F)\,dF$, $\mathcal{K}(F) = \int (K(F)/F)\,dF$, $\tilde{\mathcal{I}}(F) = \int I(F)\,dF$.

Eventually passing to assumption ($\iota\iota\iota$), we have the only condition $I = zg_{z'}$, where $g = g(x, z'/z) = f/y$. In more correct notation $g(x,t) = g(x, z'/z)$ ($t =$

z'/z) clearly $I = g_t$ is a function of variables x, t and the same should be valid for the "basical" invariant F in this subcase of nonconstant invariants. It follows that only cases (68)–(72) are to be discussed. By using the only poor information $g_t = I(F)$, one can easily conclude that

(91) $f/y = (1/c(x))\tilde{\mathcal{I}}(F) + l(x)\,,$ (see (67) for A, F)

(92) $f/y = I(F)t + l(x)\,,$ (see subcases (68)–(72) for A, F)

$\tilde{\mathcal{I}}(F) = \int I(F)dF$.

10. Remark to case $(\iota\iota)$. The last result deserves a short note concerning the structural formulae by using (91), (92). Let us recall the data: we have invariant forms

$$\omega_0 = A\,dx\,,\quad \omega_1 = dy/y - g\,dx\,,\quad \omega_2 = dz/z - t\,dx \qquad (g = g(x,t),\ t = z'/z)$$

with the corresponding parameter A. Then

$$\begin{aligned} &d\omega_0 = (dA/A) \wedge \omega_0\,,\\ &d\omega_1 = \omega_0 \wedge (dg/A) = g_t\omega_0 \wedge (dt/A) = I(F)\omega_0 \wedge (dt/A)\,,\\ &d\omega_2 = \omega_0 \wedge (dt/A) \end{aligned}$$

and it follows that the proportionality factor $g_t = I$ of forms $d\omega_1, d\omega_2$ in an (already well-known) invariant. On the other hand, $d\omega_2 = \omega_0 \wedge \omega_3$, where $\omega_3 = dt/A + B\omega_0$ (new parameter B) is invariant form. Thus

$$d\omega_0 = (dA/A) \wedge \omega_0\,,\quad d\omega_1 = I(F)\omega_0 \wedge \omega_3\,,\quad d\omega_2 = \omega_0 \wedge \omega_3\,.$$

Moreover, we get $d\omega_0 = H'(F)\omega_3 \wedge \omega_0$ and $d\omega_0 = 0$ and $d\omega_0 = G(F)\omega_3 \wedge \omega_0$ and $d\omega_0 = (H(F)\omega_1 + K(F)\omega_2) \wedge \omega_0$ with invariants $G(F)$, $H(F)$, $H'(F)$, $K(F)$ for (91) and (92), (68) and (92), (69) and (92), (70)–(72), respectively.

11. On the equivalence problem. Recall that the equivalence transformation between given equations $y = f$ and $\bar{y} = \bar{f}$ are determined by the system (19). Assuming formulae (17) with the relevant prolongation, equation $\omega_2 = \bar{\omega}_2$ (being equivalent to the prolongation rule) may be omitted. (It may be nevertheless useful for certain corrections of the remaining equations $\omega_0 = \bar{\omega}_0$ and $\omega_1 = \bar{\omega}_1$.) The presence of invariants F as a rule essentially clarifies the calculations, e.g., the equalities $F = \bar{F}$ can be substituted for a part (or the whole) system (19).

We shall mention the above discussed interesting case of only one "basical" invariant F. Then covariant derivatives (25) also are invariants, in particular $\partial F/\partial\omega_i = G(F)\ \ (i = 1,2,3)$ were systematically taken into account. One could observe that we do not deal with invariant $\partial F/\partial\omega_0$ in the above reasonings. The reason is as follows: either $\partial F/\partial\omega_0$ effectively involves the parameter A (if $F_{z'} \neq 0$, see (25)) and then $\partial F/\partial\omega_0 = 0$ can be achieved, or $\partial F/\partial\omega_0$ is a true invariant but one can observe that the equation $\partial F/\partial\omega_0 = \partial\bar{F}/\partial\bar{\omega}_0$ is a consequence of the system (19) and (the equalities $\partial F/\partial\omega_i = \partial\bar{F}/\partial\bar{\omega}_i$ $(i = 1,2,3)$, hence) the equality $F = \bar{F}$.

With this preparation, let us turn to the proper equivalences and let us deal with the instructive case (74), hence both functions f and $\bar{f}$ should be of this kind.

We have the following equations determining the sought equivalences (17): $\omega_0 = \bar\omega_0$, $\omega_1 = \bar\omega_1$, $F = \bar F$. We have omitted $\omega_2 = \bar\omega_2$ as a consequence of transformation equations. Analogously, the middle equation $\omega_1 = \bar\omega_1$ can be replaced by the transformation rule (18_2). So we may deal with

$$\omega_0 = \bar\omega_0 \Longleftrightarrow A = \bar A\varphi' \Longleftrightarrow (Et + e(x))H(F) = (\bar E\bar t + \bar e(\varphi))\bar H(\bar F)\varphi'\,,$$

equivalent to $E = \bar E$, $\bar H(\bar F) = \bar H(F) = H(F)$,

$$e(x) = \bar e(\varphi)\varphi' + E\frac{M'}{M} \tag{93}$$

in spite of $t = z'/z$, hence $\bar t = \bar z'/\bar z = (t + M'/M)/\varphi'$. Then

$$F = \bar F \Longleftrightarrow c(x)|\varphi'|^{1/E} = \bar c(\varphi)|M|^C L\,, \tag{94}$$

$$\frac{\bar f}{\bar y}\varphi' = \frac{f}{y} + \frac{L'}{L} \Longleftrightarrow l(x) + \frac{L'}{L} = \bar l(\varphi)\varphi' + m\frac{M'}{M} \tag{95}$$

with $\bar{\mathcal H}(\bar F) = \mathcal H(F)$, $\bar m = m$ and exploiting relations (93), (94). Thus (93)–(95) are the necessary and sufficient conditions for the symmetry equivalence problem of the Monge equation

$$y' = f(x,y,z,z') = \{(Et+e(x))\mathcal H(F)+mt+l(x)\}y\,, \quad F = c(x)|Et+e(x)|^{1/E}|z|^C y$$

with respect to the pseudogroup of transformations (17).

Functional differential equations

12. Differential equation with deviation. Let us consider equations

$$y'(x) = f\big(x, y(x), y(\xi(x)), (y(\xi(x)))'\big)\,, \qquad x \in \mathbf j \subseteq \mathbf R\,, \quad \Big(' = \frac{d}{dx}\Big) \tag{96}$$

$$\bar y'(\bar x) = \bar f\big(\bar x, \bar y(\bar x), \bar y(\bar\xi(\bar x)), (\bar y(\bar\xi(\bar x)))'\big)\,, \qquad \bar x \in \mathbf i \subseteq \mathbf R\,, \quad \Big(' = \frac{d}{d\bar x}\Big) \tag{97}$$

on definition intervals $\mathbf i$, $\mathbf j$.

We say that (96) *is globally transformable into* (97) *if there exist two functions* φ, L *such that*

— the function L *is of the class* $C^1(\mathbf j)$ *and is nonvanishing on* $\mathbf j$*;*

— the function φ *is a* C^1*-diffeomorphism of the interval* $\mathbf j$ *onto* $\mathbf i$*;*

and the function

$$\bar y(\bar x) = \bar y(\varphi(x)) = L(x)y(x) \tag{98}$$

is a solution of (97) *whenever* $y(x)$ *is a solution of* (96).

If (96) is globally transformable into (97), then we say that (96), (97) are *equivalent equations* and moreover,

$$\bar\xi(\bar x) = \bar\xi(\varphi(x)) = \varphi(\xi(x)) \tag{99}$$

is satisfied for deviations $\bar\xi(\bar x)$, $\xi(x)$.

We get

$$\bar y(\varphi(x)) = L(x)y(x)\,, \qquad \text{i.e.,}$$
$$\bar y(\bar\xi(\bar x)) = \bar y(\bar\xi(\varphi(x))) = \bar y(\varphi(\xi(x))) = L(\xi(x))y(\xi(x))\,,$$

using (99). Thus

$$\bar z = M(x)z\,,$$

denoting $z(x) = y(\xi(x))$, $M(x) = L(\xi(x))$, $\bar z = \bar y(\bar\xi(\bar x))$ and assuming the equivalence of equations (96)

$$y'(x) = f\big(x, y(x), z(x), (z(x))'\big)\,, \quad x \in \mathbf{j} \subseteq \mathbf{R} \qquad \Big({}' = \frac{d}{dx}\Big)$$

and (97)

$$\bar y'(\bar x) = \bar f\big(\bar x, \bar y(\bar x), \bar z(\bar x), (\bar z(\bar x))'\big)\,, \quad \bar x \in \mathbf{i} \subseteq \mathbf{R}\,, \qquad \Big({}' = \frac{d}{d\bar x}\Big)$$

rewritten as a Monge equations.

Assertion 1. *For f given by* (35), (36), $A = b(x)|y|^l|z|^m$ *and* (40), $A = b(x)|z|^{C+1}$, *respectively.*

Any equation $y' = f(x,y,z,z')$ is globally transformable into some equation $\bar y = \bar f(\bar x, \bar y, \bar z, \bar z')$ if and only if (99) *is satisfied for transformation* (98) *and*

$$A = \bar A\varphi'\,, \quad \frac{\bar f}{\bar y}\varphi' = \frac{f}{y} + \frac{L'}{L}$$

are identities on the whole interval $\mathbf{i}$.

The assertion follows from the definition of global transformation (96) and results of Sections 5, 6, 7. In accordance with the instructive case (74) and results of Section 11 we can formulate the following

Assertion 2. *For f given by* (74) − (83) *and* (85) − (90) *and* (91), (92) *with* $t = z'/z$, *respectively.*

Any equation $y' = f(x,y,z,z')$ is globally transformable into some equation $\bar y = \bar f(\bar x, \bar y, \bar z, \bar z')$ if and only if (99) *is satisfied for transformation* (98) *and*

$$A = \bar A\varphi'\,, \quad F = \bar F\,, \quad \frac{\bar f}{\bar y}\varphi' = \frac{f}{y} + \frac{L'}{L}$$

are identities for the relevant f, A, F on the whole interval $\mathbf{i}$.

13. Example.

Example. Let us deal with the instructive case (74) from the point of view of global transformations. We consider equations

$$\frac{y'(x)}{y(x)} = \big(Et + e(x)\big)\mathcal{H}\Big(c(x)|Et + e(x)|^{1/E}|y(\xi(x))|^C y(x)\Big)$$
$$+\, mt + l(x)\,, \qquad \Big({}' = \frac{d}{dx}\Big) \tag{100}$$

$t = \frac{(y(\xi(x)))'}{y(\xi(x))}$, $x \in \mathbf{j} \subseteq \mathbf{R}$,

$$\frac{\bar{y}'(\bar{x})}{\bar{y}(\bar{x})} = \big(\bar{E}\bar{t} + \bar{e}(\bar{x})\big)\bar{\mathcal{H}}\left(\bar{c}(\bar{x})|\bar{E}\bar{t} + \bar{e}(\bar{x})|^{1/\bar{E}}|\bar{y}(\bar{\xi}(\bar{x}))|^{C}\bar{y}(\bar{x})\right)$$

$$+ m\bar{t} + \bar{l}(\bar{x})\,, \qquad \big(' = \frac{d}{d\bar{x}}\big) \tag{101}$$

$\bar{t} = \frac{(\bar{y}(\bar{\xi}(\bar{x})))'}{\bar{y}(\bar{\xi}(\bar{x}))}$, $\bar{x} \in \mathbf{i} \subseteq \mathbf{R}$ and a global transformation (98) with the property (99) for $\xi(x) \neq x$ and $\bar{\xi}(\bar{x}) \neq \bar{x}$ on the interval $\mathbf{j}$ and $\mathbf{i}$, respectively. Here $e(x)$, $l(x)$, $c(x) \neq 0$, $\mathcal{H}$, $\bar{e}(\bar{x})$, $\bar{l}(\bar{x})$, $\bar{c}(\bar{x}) \neq 0$, $\bar{\mathcal{H}}$ are arbitrary functions, m, C, $E \neq 0$, $\bar{m}$, $\bar{C}$, $\bar{E} \neq 0$ costants. An equation (100) is globally transformable into (101) if and only if the condition

$$\frac{(\bar{y}(\varphi(x)))'}{\bar{y}(\varphi(x))} = \frac{\bar{y}'(\varphi)}{\bar{y}(\varphi)}\varphi'(x) = \frac{L'(x)}{L(x)} + \frac{y'(x)}{y(x)}$$

is satisfied on the whole interval $\mathbf{j}$. We have $\bar{t}\varphi' = M'/M + t$ for $M = L(\xi)$, hence

$$\frac{L'}{L} + \bar{l} \; - m\frac{M'}{M} - \bar{l}(\varphi)\varphi'$$
$$= (\bar{E}t + \bar{E}\frac{M'}{M} + \bar{e}(\varphi)\varphi')\bar{\mathcal{H}}\left(\bar{c}(\varphi)\left|\bar{E}\frac{1}{\varphi'}(\frac{M'}{M} + t) + \bar{e}(\varphi)\right|^{1/\bar{E}}|M|^{C}|y(\xi)|^{C}Ly\right)$$
$$- (Et + e(x))\mathcal{H}\left(c(x)|Et + e(x)|^{1/E}|y(\xi(x))|^{C}y(x)\right)$$

for arbitrary functions $\mathcal{H}$, $\bar{\mathcal{H}}$, i.e., $\frac{L'}{L} + \bar{l} - m\frac{M'}{M} - \bar{l}(\varphi)\varphi' = 0$ and $\bar{E}\frac{M'}{M} + \bar{e}(\varphi)\varphi' = e(x)$. Thus $\mathcal{H} = \bar{\mathcal{H}}$ and

$$\mathcal{H}\left(\bar{c}(\varphi)\left|\bar{E}\frac{1}{\varphi'}(\frac{M'}{M} + t) + \bar{e}(\varphi)\right|^{1/\bar{E}}|M|^{C}|y(\xi)|^{C}Ly)\right)$$
$$=\mathcal{H}\left(c(x)|Et + e(x)|^{1/E}|y(\xi(x))|^{C}y(x)\right)$$

is then equivalent to $c(x)|\varphi'|^{1/E} = \bar{c}(\varphi)|M|^{C}L$. The necessary and sufficient conditions (93)–(95) for the symmetry equivalence problem (local) are together with (99) the necessary and sufficient conditions for the equivalence of the given equations by means of global transformation (98).

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