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THE DIFFERENCE MATRICES OF THE CLASSES OF A SHARMA–KAUSHIK PARTITION

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Abstract. Sharma-Kaushik partitions have been used to define distances between vectors with n-coordinates. In this paper, "difference matrices" for the partitioning classes have been introduced and investigated. It has been shown that the difference matrices are circulant and that the entries of a product of matrices is an extended intersection number of a distance scheme. The sum of the entries of each row or columns of the product matrix has been obtained.

The algebra of matrices generated by the difference matrices of the classes of an SK-partition have another natural basis. The relationship between these two bases has been given.

1. Introduction

Sharma-Kaushik partitions were introduced by Sharma and Kaushik [10], who defined matrics in terms of these partitions. Metrics so obtained were used in the study of error-corresponding codes by Kaushik [3, 4, 5, 6, 7, 8], Sharma and Dial [9] and Sharma and Kaushik [11]. Sharma and Kaushik [12] also studied the algebra of Sharma-Kaushik partitions.

Matrices such as incidence matrices have proven useful in the study of graphs and other combinatorial structures. In this paper, we introduce and study "difference matrices" for the classes of a Sharma-Kaushik partition.

In Section 2, we present definitions used in this paper. In Section 3, we prove that difference matrices are circulant and that the entries of the product of difference matrices are extended intersection numbers of a distance scheme. We also obtain the sum of the entries in each row or column. Section 5 is devoted to the algebra of matrices generated by the set of difference matrices of the classes of a Sharma-Kaushik partition.

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2. Definitions and notations

Delsarte's [2] definition of association schemes, adopted by us, is as follows:

Definition. Given a set X with at least two elements, and a set of relations $R = \{R_0, R_1, \ldots, R_N\}$, where N is a positive integer, (X, R) is called an association scheme if

1. $R_0 = \{(x, x) \mid x \in X\}$.
2. For each $i = 0, 1, \ldots, N$

$$R_i^{-1} = \{(y, x) \mid (x, y) \in R_i\} \in R\,.$$

3. For any three integers i, j, $k = 0, 1, \ldots, N$, there exists a number c_{ijk} such that

$$\bigl|\{z \in X \mid (x, z) \in R_i, (z, y) \in R_j\}\bigr| = c_{ijk} \quad \text{for any} \quad (x, y) \in R_k.$$

Also $c_{ijk} = c_{jik}$.

In this paper, X shall be the ring of integers modulo q, $q \geq 2$, i.e.,

$$X = F_q = \{0, 1, \ldots, q-1\}, \quad \text{addition} \quad (+) \quad \text{mod } q\,.$$

SK-partitions, introduced by Sharma and Kaushik [7], are defined next.

Definition. Given F_q, $q \geq 2$, a partition

$$P = \{B_0, B_1, \ldots, B_{m-1}\}$$

of F_q is called an SK-partition if

1. $B_0 = \{0\}$, and $q - a \in B_i$ if $a \in B_i$, $i = 1, 2, \ldots, m-1$.
2. If $a \in B_i$ and $b \in B_j$, $i, j = 0, 1, m \ldots, m-1$, and if j precedes i in the order of the partition P, written as $i > j$, then

$$\min\{a, q-a\} > \min\{b, q-b\}\,.$$

3. If $i > j$ $(i, j = 0, 1, \ldots, m-1)$ and $i \neq m-1$, then

$$|B_i| \geqslant |B_j| \quad \text{and} \quad |B_{m-1}| \geqslant \frac{1}{2}|B_{m-2}|\,,$$

where $|B_i|$ stands for the size of the set B_i.

Weight of an element with respect to an SK-partition P.

Given an SK-partition P of F_q, the weight $W_P(a)$ of any element a of F_q is given by

$$W_p(a) = i\,, \quad \text{if} \quad a \in B_i\,, \qquad i = 0, 1, \ldots, m-1\,.$$

Also, if F_q^n is the direct product of n copies of F_q and $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in F_q^n$, then the weight of $\mathbf{x}$ with respect to P is given by

$$W_P(x) = \sum_{i=1}^{n} W_P(x_i)\,.$$

If $\mathbf{y} = (y_1, y_2, \ldots, y_n) \in F_q^n$, then the distance between $\mathbf{x}$ and $\mathbf{y}$ with respect to P is given by

$$d_P(\mathbf{x}, \mathbf{y}) = W_P(\mathbf{x} - \mathbf{y})\,.$$

Definition. Given an SK-partition P of F_q, let

$$R^{d,n,P} = \left\{R_0^{d,n,P}, R_1^{d,n,P}, \ldots, R_{n(m-1)}^{d,n,P}\right\}\,,$$

where

$$R_i^{d,n,P} = \left\{(\mathbf{x}, \mathbf{y}) \in (F_q^n)^2 \mid d_P(\mathbf{x}, \mathbf{y}) = i\right\}\,, \qquad i = 0, 1, \ldots, m(n-1)$$

$(F_q^n, R^{d,n,P})$ is called the *distance scheme* over F_q^n.

Definition. An extended intersection number of $(F_q, R^{d,l,P})$ is given by

$$\begin{aligned} C^{d,l,P}_{i_1,i_2,\ldots,i_a}(x,y) = \big|\big\{&(z_1, z_2, \ldots, z_h) \in F_q^h \mid (x_1, z_1) \in R^{d,l,P}_{i_l}, \\ &(z_1, z_2) \in R^{d,l,P}_{i_2}, \ldots, (z_h, y) \in R^{d,l,P}_{i_{h+1}}\big\}\big| \end{aligned}$$

Notation. If T is a $q \times q$ matrix, the entry in the $(x+1)$-th row and the $(y+1)$-th column is called the (x, y) entry of T, $(x, y = 0, 1, \ldots, q-1)$, (cf. Delsarte [2]).

Definition. Given an SK-partition $P = \{B_0, B_1, \ldots, B_{m-1}\}$ of F_q (the ring of integers modulo q), the $q \times q$ matrix $\overline{B}_i$ $(i = 0, 1, \ldots, m-1)$ is called the *difference matrix* of B_i, if the (x, y) entry of $\overline{B}_i = \begin{cases} 1, & \text{if} \quad x - y \in B_i \\ 0, & \text{otherwise.} \end{cases}$

3. Difference matrices

Theorem 3.1. *The difference matrix $\overline{B}_i$ $(i = 0, 1, \ldots, m-1)$ is a circulant matrix, for any SK-partition $P = \{B_0, B_1, \ldots, B_{m-1}\}$ of F_q.*

Proof. Since

$$x -_q y = (x +_q 1) - (y +_q 1)$$

the (x, y) entry and the $(x +_q 1, y +_q 1)$ entries of $\overline{B}_i$ $(i = 0, 1, \ldots, m-1)$ are equal for x, $y = 0, 1, \ldots, m-1$. □

Corollary. *For any positive integer a, $\overline{B}_{i_1} \times \overline{B}_{i_1} \times \cdots \times \overline{B}_{i_n}$ $(i_1, i_2, \ldots, i_n = 0, 1, \ldots, m-1)$ is a circulant matrix.*

Proof. From the fact that the product of any two circulant matrices is circulant (Davis [1]), it follows that $\overline{B}_{i_1} \times \overline{B}_{i_2} \times \cdots \times \overline{B}_{i_n}$ $(i_1, i_2, \ldots, i_n = 0, 1, \ldots, m-1)$ is circulant. □

We next show the relationship between difference matrices and extended intersection numbers (cf. Sharma and Sookoo [13]).

Theorem 3.2. *Let $P = \{B_0, B_1, \ldots, B_{m-1}\}$ be an SK-partition of F_q. The (x, y) entry of $\overline{B}_{i_1} \times \overline{B}_{i_2} \times \cdots \times \overline{B}_{i_a}$ $(i_1, i_2, \ldots, i_a = 0, 1, \ldots, m-1)$ is equal to the extended intersection number $C^{d,l,P}_{i_1,i_1,\ldots,i_a}, \ldots, i_a(x, y)$ of $(F_q, R^{d,l,P})$ for $a \in \{2, 3, \ldots\}$.*

Proof. First we establish the result for $a = 2$. The (x, y) entry of

$$\begin{aligned}
&\overline{B}_{i_1} \times \overline{B}_{i_2} \\
&= \sum_{z=0}^{q-1} \left[(\text{the } (x,z) \text{ entry of } \overline{B}_{i_1}) \times (\text{the } (z,y) \text{ entry of } \overline{B}_{i_2})\right] \\
&= \left|\left\{z \in F_q \mid \text{ the } (x,z) \text{ entry of } \overline{B}_{i_1} \text{ is one of the } (z,y) \text{ entry of } \overline{B}_{i_2} \text{ is one}\right\}\right| \\
&= \left|\left\{z \in F_q \mid (x-z) \in B_{i_q}, (z-y) \in B_{i_2}\right\}\right|.
\end{aligned}$$

Assuming that the result is true for a $(a > 2)$, we have

$$\begin{aligned}
&C^{d,l,P}_{i_1,i_2,\ldots,i_a}(x, z_a) \\
&\qquad = \left|\left\{(z_1, z_2, \ldots, z_{a-1}) \in F_q^{a-1} \mid (x, z_1) \in R^{d,l,P}_{i_1}, (z_1, z_2) \in R^{d,l,P}_{i_2},\right.\right. \\
&\qquad\qquad \left.\left.(z_2, z_3) \in R^{d,l,P}_{i_3}, \ldots, (z_{a-1}, z_a) \in R^{d,l,P}_{i_a}\right\}\right|.
\end{aligned}$$

Next we have the (x, y) entry of

$$\begin{aligned}
&\overline{B}_{i_1} \times \overline{B}_{i_2} \times \cdots \times \overline{B}_{i_{a+1}} \\
&= \sum_{z_a \in F_q} \left[(\text{the } (x, z_a) \text{ entry of } \overline{B}_{i_1} \times \overline{B}_{i_2} \times \cdots \times \overline{B}_{i_a})\right. \\
&\qquad \left.\times (\text{the } (z_a, y) \text{ entry of } \overline{B}_{i_{a+1}})\right] \\
&= \sum_{z_a \in F_q} \left[\left|\left\{(z_1, z_2, \ldots, z_{a-1}) \in F_q^{a-1} \mid (x, z_1) \in R^{d,l,P}_{i},\right.\right.\right. \\
&\qquad\qquad \left.\left.(z_1, z_2) \in R^{d,l,P}_{i_2}, (z_2, z_3) \in R^{d,l,P}_{i_3}, \ldots, (z_{a-1}, z_a) \in R^{d,l,P}_{i_a}\right\}\right| \\
&\qquad \left.\times (\text{the } (z_a, y) \text{ entry of } \overline{B}_{i_{a+1}})\right] \\
&= \left|\left\{(z_1, z_2, \ldots, z_a) \in F_q^a \mid (x, z_1) \in R^{d,l,P}_{i_1}, (z_1, z_2) \in R^{d,l,P}_{i_2},\right.\right. \\
&\qquad \left.\left.(z_2, z_3) \in R^{d,l,P}_{i_3}, \ldots, (z_a, y) \in R^{d,l,P}_{i_{a+1}}\right\}\right| = C^{d,l,P}_{i_1,i_2,i_{a+1}}(x, y)
\end{aligned}$$

showing that the result is true for a+1. By induction the result holds for $a \in \{2, 3, \ldots\}$. □

We next look at the sums of the entries in the rows and columns of $\overline{B}_{i_1} \times \overline{B}_{i_2} \times \cdots \times \overline{B}_{i_a}$ to determine the relationship these sums bear to the SK-partition P.

Theorem 3.3. *Let $P = \{B_0, B_1, \ldots, B_{m-1}\}$ be an SK-partition of F_q, and let $\overline{B}_i$ be the difference matrix of B_i $(i = 0, 1, \ldots, m-1)$. For any positive integer $a > 1$, the sum of the entries of each row and column of $\overline{B}_{i_1} \times \overline{B}_{i_2} \times \cdots \times \overline{B}_{i_a}$ $(i_1, i_2, \ldots, i_a = 0, 1, \ldots, m-1)$ is $|B_{i_1}| \times |B_{i_2}| \times \cdots \times |B_{i_a}|$.*

Proof. From the previous theorem, it is clear that for $x \in \{0, 1, \ldots, m-1\}$, the sum of the entries in the x-th row of

$$\begin{aligned}
&\overline{B}_{i_1} \times \overline{B}_{i_2} \times \cdots \times \overline{B}_{i_a} \\
&\quad = \sum_{y \in F_q} C^{d,l,P}_{i_1, i_2, \ldots, i_a}(x, y) \\
&\quad = \sum_{y \in F_q} \big|\big\{b_1, b_2, \ldots, b_a \in F_q^a \mid x - y = b_1 + b_2 + \cdots + b_a \\
&\qquad \text{and } W_P(b_\alpha) = i_\alpha \ (\alpha = 1, 2, \ldots, a)\big\}\big| \\
&\qquad\quad \text{(refer Theorem 4 of Sharma and Sookoo [13])} \\
&\quad = \big|\big\{(b_1, b_2, \ldots, b_a) \in F_q^a \mid b_\alpha \equiv B_{i_\alpha} \ (\alpha = 1, 2, \ldots, a)\big\}\big| \\
&\quad = \big|B_{i_1}\big| \times \big|B_{i_1}\big| \times \cdots \times \big|B_{i_a}\big| \,.
\end{aligned}$$

Since B_i $(i = 0, 1, \ldots, m-1)$ is symmetric, the sum of the entries of each column is also $\big|B_{i_1}\big| \times \big|B_{i_2}\big| \times \cdots \times \big|B_{i_a}\big|$. □

4. Algebra generated by the difference matrices of the classes of an SK-partition

The following definitions and notation (cf. Delsarte [2]) are now required.

Definition. Let $P = \{B_0, B_1, \ldots, B_{m-1}\}$ be an SK-partition of F_q and let $\overline{B}_i$ $(i = 0, 1, \ldots, m-1)$ be the difference matrices of P. The *Bose-Mesner algebra* of P is the algebra generated by the $\overline{B}_i$ $(i = 0, 1, \ldots, m-1)$.

Definition. Given an SK-partition $\{P = B_0, B_1, \ldots, B_{m-1}\}$ of F_q, the matrix τ_k is called the *representer matrix* of B_k $(k = 0, 1, \ldots, m-1)$ if

$$\begin{aligned}
\tau_k(x, x) &= \begin{cases} 1\,, & x \in B_k,\ x \in F_q \\ 0\,, & \text{otherwise} \end{cases} \\
\tau_k(x, y) &= 0\,, \quad x \neq y,\ x, y, \in F_q
\end{aligned}$$

Remark. Clearly $\tau_r \tau_s = \delta_{r,s} \tau_r$, $0 \leqslant r$, $s \leqslant n$ where $\delta_{r,s}$ is the Kronecker symbol.

Notation. Given $q \times q$ matrix S and the partition $P = \{B_0, B_1, \ldots, B_{m-1}\}$ of F_q, let

$$J_k = q^{-1} S \tau_k S^* \qquad (k = 0, 1, \ldots, m-1)$$

where τ_k is the representer matrix of B_k, and S^* is the conjugate transpose of S.

We make use of the following matrix (cf. Wallis, Street and Wallis [14]).

Definition. The matrix S of order q is

$$\begin{pmatrix} 1 & 1 & 1 & 1 & \dots & 1 \\ 1 & \omega & \omega^2 & & & \omega^{(n-1)} \\ 1 & \omega^2 & \omega^4 & & & \omega^{2(n-1)} \\ \vdots & \vdots & \vdots & & & \vdots \\ 1 & \omega^{(n-1)} & \omega^{2(n-1)} & & & \omega^{(n-1)(n-1)} \end{pmatrix}$$

where $\omega = \cos \frac{2\pi}{q} + i \sin \frac{2\pi}{q}$.

Remark. Let $F = q^{-1/2} S$. F is unitary. Also S diagonalizes B ($i = 0, 1, \dots, m-1$).

This can be shown as follows:

From the history of circulant matrices (refer Davis [1]), we know that if C is a circulant matrix, then

$$C = F^* \Lambda F\,,$$

where Λ is a diagonal matrix having the eigenvalues of C on its main diagonal. Hence

$$\overline{B}_i = F^* \Lambda_i F\,,$$

where Λ_i is the diagonal matrix of eigenvalues of $\overline{B}_i$. Therefore

$$\overline{B}_i = \left(\frac{1}{\sqrt{q}} S\right) \Lambda_i \left(\frac{1}{\sqrt{q}} S^*\right) = \frac{1}{q} S \Lambda_i S^*\,. \qquad \square$$

In the following theorem, which is based on Theorem 2.2 of Delsarte [2], we show that $\{J_k \mid k = 0, 1, \dots, m-1\}$ is a basis of the Bose-Mesner algebra and we show the relationship between this basis and $\{\overline{B}_k \mid k = 0, 1, \dots, m-1\}$.

Theorem 4.1. *Let $P = \{B_0, B_1, \dots, B_{m-1}\}$ be an SK-partition of F_q. There exists an SK-partition $P' = \{B'_o, B'_1, \dots, B'_{m-1}\}$ of F_q such that the set of J'_is ($i = 0, 1, \dots, m-1$) form a basis of the Bose-Mesner algebra of P, where $J_i = q^{-1} S \tau_i S^*$ and τ_i is the representer matrix of B'_i. Also, if $\overline{B}_i$ ($i = 0, 1, \dots, m-1$) is the difference matrix of B_i and λ_k^i ($i = 0, 1, \dots, m-1$) are the eigenvalues of $\overline{B}_k$, then*

$$\overline{B}_k = \sum_{i=0}^{m-1} \lambda_k^i J_i \qquad (k = 0, 1, \dots, m-1)\,.$$

Proof. Let A be the Bose-Mesner algebra of P. Every matrix in A is circulant, since the sum or product of circulant matrices is also circulant (refer Davis [1]). Hence each matrix B in A is diagonalized by S. Therefore there is a diagonal matrix Λ_B having the eigenvalues of B on the main diagonal such that $B = q^{-1} S \Lambda_B S^*$ As B varies over A, Λ_B varies over a subset A' of the algebra Q of $q \times q$ matrices with complex entries. It is easy to show that A' is a sub-algebra of Q and that

I: $A' \to A$ is an isomorphism if
$\mathrm{I}(\Lambda) = q^{-1} S \Lambda S^*, \quad \forall \Lambda \in A'$.

Let $B^{(e)} A$ have the maximal number n' of distinct eigenvalues, the eigenvalues of $B^{(e)}$ being $\lambda_0, \lambda_1, \dots, \lambda_{n'-1}$ and let $\Lambda_{B^{(e)}}$ be the matrix in A' corresponding to $B^{(e)}$. There exists a partition P' of F_q into n' classes B'_ψ $(\psi = 0, 2, \dots, n'-1)$ such that

$$\Lambda_{B^{(e)}} = \sum_{\psi=0}^{n'-1} \lambda_\psi \tau_\psi$$

where τ_ψ is the representer matrix of B'_ψ.

If $\Lambda_{B^{(e)}}$ has the eigenvalues λ_i in rows $j_i^i, j_2^i, \dots, j_a^i$, then B'_i consists of the following elements of $F_q : j_1^i, j_2^i, \dots, j_a^i$.

We show that the τ_ψ's are all in A', by showing that each is a linear combination of powers of $\Lambda_{B^{(e)}}$.

Since the Λ_i are all distinct, there exist polynomials $f_r(z)$ such that

$$f_r(\lambda_i) = \delta_{r,i} \quad \text{for} \quad r, i = 0, 1, \dots, n'-1\,.$$

Let $f_r(z) = \dfrac{(z-\lambda_0)(z-\lambda_1)\dots\widehat{(z-\lambda_r)}\dots(z-\lambda_{n'-1})}{(\lambda_r-\lambda_0)(\lambda_r-\lambda_1)\dots\widehat{(\lambda_r-\lambda_r)}\dots(\lambda_r-\lambda_{n'-1})}$ where the factors with the cap are the factors missing.

It is easy to see that τ_ψ's are mutually orthogonal and so

$$\left(\Lambda_{B^{(e)}}\right)^t = \lambda_0^t \tau_0 + \lambda_1^t \tau_1 + \dots + \lambda_{n'-1}^t \tau_{n'-1}$$

for any natural number t. Hence

$$f_r\left(\Lambda_{B^{(e)}}\right) = f_r(\lambda_0)\tau_0 + f_r(\lambda_1)\tau_1 + \dots + f_r(\lambda_{n'-1})\tau_{n'-1} = \tau_r\,,$$

since $f_r(\lambda_i) = \delta_{r,i}$.

Since $\Lambda_{B^{(e)}} \in A'$, τ_ψ $(\psi \in \{0, 1, \dots, n'-1\})$ is also in A', as it is a linear combination of powers of $\Lambda_{B^{(e)}}$. Since the τ_ψ's are linearly independent, $n' \leqslant m$, because the dimension of A is m.

We now show that the τ_ψ generate the whole of A', so that $n' = m$. Let Λ_B be an arbitrary element of A' having n'' distinct eigenvalues, α_ψ $(\psi = 0, 1, \dots, n''-1)$.

Clearly there exists a partition Q'' of F_q with n'' classes $\ni$

$$\Lambda_B = \sum_{\psi=0}^{n-1} \alpha_\psi \overline{\tau}_\psi$$

where $\{\overline{\tau}_\psi \mid \psi = 0, 1, \dots, n''-1\}$ is the set of representer matrices of Q''.

It is easy to show that $\overline{\tau}_\psi$ is a function of Λ_B in the same way that we showed that τ_r is a function of $\Lambda_{B^{(e)}}$. Hence $\overline{\tau}_\psi \in A'$.

We show by contradiction that each $\overline{\tau}_\psi$ $(\psi = 1, 2, \ldots, n'')$ is a linear combination of the τ_ψ's $(\psi = 1, 2, \ldots, n')$.

Suppose that there exists some $\underline{\psi}$ such that $\overline{\tau}_{\underline{\psi}}$ is not a linear combination of the τ_ψ. Clearly

$$\Lambda = \tau_1 + 2\tau_2 + \cdots + n'\tau_{n'} + 10n'\overline{\tau}_\psi$$

has more than n' distinct numbers on the main diagonal. So the matrix in A corresponding to Λ has more than n' eigenvalues, contradicting the maximality of n'. Hence each $\overline{\tau}_\psi$ $(\psi = 1, 2, \ldots, n'')$ is a linear combination of the τ_ψ $(\psi = 1, 2, \ldots, n')$.

Thus each element in A can be expressed in terms of the τ_ψ $(\psi = 1, 2, \ldots, n')$. Therefore $m \leqslant n'$.

Since we have already shown that $n' \leqslant m$, we have $n' = m$. So the τ_ψ form a basis of A' and so the J_ψ are a basis of A.

Finally,

$$\overline{B}_k = q^{-1} S \Lambda_{\overline{B}_k} S^*\,,$$

where $\Lambda_{\overline{B}_k}$ is the diagonal matrix of eigenvalues of $\overline{B}_k$. Therefore

$$\overline{B}_k = q^{-1} S \left(\sum_{i=0}^{m-1} \lambda_k^i \tau_i \right) S^*$$

where the λ_k^i are the eigenvalues of $\overline{B}_k$. Hence

$$\overline{B}_k = \sum_{i=0}^{m-1} \lambda_k^i q^{-1} S \tau_i S^* = \sum_{i=0}^{m-1} \lambda_k^i J_i\,. \qquad \square$$

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