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Complex structures on the Iwasawa manifold

Georgios Ketsetzis and Simon Salamon

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Abstract. We identify the space of left-invariant oriented complex structures on the complex Heisenberg group, and prove that it has the homotopy type of the disjoint union of a point and a 2-sphere.

Introduction

It is well known that every even-dimensional compact Lie group has a left-invariant complex structure [12], [14]. By contrast, not all nilpotent groups admit left-invariant complex structures. In 6 real dimensions there are 34 isomorphism classes of simply-connected nilpotent Lie groups, and the study [11] reveals that 18 of these admit invariant complex structures. The complex Heisenberg group G possesses a particularly rich structure in this regard, since it has a 2-sphere of abelian complex structures in addition to its standard bi-invariant complex structure J_0.

The Iwasawa manifold $\mathbb{M} = \Gamma\backslash G$ is a compact quotient of G, and any left-invariant tensor on G induces a tensor on $\mathbb{M}$. As explained in §2, studies of Dolbeault cohomology suggest that the moduli space of complex structures on $\mathbb{M}$ is determined by the space of left-invariant complex structures on G. The set of such structures compatible with a standard metric g and orientation is the union of $\{J_0\}$ and the 2-sphere already mentioned [1]. The present paper shows that this description remains valid at the level of homotopy when one no longer insists on compatibility with g. This requires a new approach, in which complex structures are described by a basis of $(1,0)$-forms in echelon form (see Proposition 2.3). Similar techniques can be applied to other Lie groups and nilmanifolds, though we refer the reader to [10] for related studies.

We work mainly with the Lie algebra $\mathfrak{g}$ of G, and regard left-invariant differential forms on G as elements of $\bigwedge^k \mathfrak{g}^*$. A special feature of the space $\mathscr{C}(\mathfrak{g})$ of all invariant complex structures on $\mathbb{M}$ is that any J in $\mathscr{C}(\mathfrak{g})$ is compatible with the fibration of $\mathbb{M}$ as a T^2 bundle over T^4. Algebraically, this amounts to asserting that the 4-dimensional kernel $\mathbb{D}$ of $d : \mathfrak{g}^* \to \bigwedge^2 \mathfrak{g}^*$ is necessarily J-invariant. As we show in §2, the essential features of an invariant complex structure J are captured by its restriction to $\mathbb{D}$, and are described by a complex 2×2 matrix X. In this way, topological questions are related to properties of the eigenvalues of $X\overline{X}$ and some matrix analysis described in [6].

The orientation of the restriction of an almost complex structure J to $\mathbb{D}$ determines two connected components of $\mathscr{C}(\mathfrak{g})$ that we study separately. We establish global complex coordinates on the component $\mathscr{C}_+$ containing the complex structure induced by J_0, and show that it has the structure of a contractible complex 6-dimensional manifold. By exploiting an SU(2) action on the second component $\mathscr{C}_-$, we prove that this retracts onto the 2-sphere of negatively-oriented orthogonal almost complex structures on $\mathbb{D}$.

1 Preliminaries

The Iwasawa manifold $\mathbb{M}$ is defined as the quotient $\Gamma\backslash G$, where

$$G = \left\{ \begin{pmatrix} 1 & z_1 & z_3 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{pmatrix} : z_i \in \mathbb{C} \right\}$$

is the complex Heisenberg group and Γ is the lattice defined by taking z_1, z_2, z_3 to be Gaussian integers, acting by left multiplication. We shall regard $\mathbb{M}$ as a real manifold of dimension 6, and we let $\mathfrak{g}$ denote the real 6-dimensional Lie algebra associated to G.

An *invariant* complex structure on $\mathbb{M}$ is by definition one induced from a left-invariant complex structure on the real Lie group underlying G. Such a structure is invariant by the action of the centre Z of G, that persists on $\mathbb{M}$ (Z consists of matrices for which $z_1 = 0 = z_2$). The set of such structures can be identified with the set $\mathscr{C}(\mathfrak{g})$ of almost complex structures on the real Lie algebra $\mathfrak{g}$ that satisfy the Lie algebraic counterpart

$$[JX, JY] = [X, Y] + J[JX, Y] + J[X, JY]$$

of the Newlander–Nirenberg integrability condition.

The natural complex structure J_0 of G, for which z_1, z_2, z_3 are holomorphic, is a point of $\mathscr{C}(\mathfrak{g})$ that satisfies the stronger condition $[JX, Y] = J[X, Y]$. It induces a bi-invariant complex structure of G that therefore passes to a G-invariant complex structure on $\mathbb{M}$. We shall denote by $\mathscr{C}^+(\mathfrak{g})$ the subset consisting of complex structures inducing the same orientation as J_0.

The 1-forms

$$\omega^1 = dz_1, \quad \omega^2 = dz_2, \quad \omega^3 = -dz_3 + z_1\, dz_2, \tag{1}$$

are left-invariant on G. Define a basis $\{e^1, \dots, e^6\}$ of *real* 1-forms by setting

$$\omega^1 = e^1 + ie^2, \quad \omega^2 = e^3 + ie^4, \quad \omega^3 = e^5 + ie^6. \tag{2}$$

These 1-forms are pullbacks of corresponding 1-forms on the quotient $\mathbb{M}$, which we denote by the same symbols. They satisfy

$$\begin{cases} de^i = 0, \quad 1 \leqslant i \leqslant 4, \\ de^5 = e^{13} + e^{42}, \\ de^6 = e^{14} + e^{23}. \end{cases} \tag{3}$$

Here, we make use of the notation $e^{ij} = e^i \wedge e^j$.

Let $T^k \cong \mathbb{R}^k/\mathbb{Z}^k$ denote a real k-dimensional torus. Then $\mathbb{M}$ is the total space of a principal T^2-bundle over T^4. The mapping $p : \mathbb{M} \to T^4$ is induced from $(z_1, z_2, z_3) \mapsto (z_1, z_2)$. The space of invariant 1-forms annihilating the fibres of p is

$$\mathbb{D} = \langle e^1, e^2, e^3, e^4 \rangle = \ker(d : \mathfrak{g}^* \to \textstyle\bigwedge^2 \mathfrak{g}^*),$$

and this 4-dimensional subspace of $\mathfrak{g}^*$ will play a crucial role in the theory.

Theorem 1.1. *Let J be any invariant complex structure on $\mathbb{M}$. Then p induces a complex structure $\hat{J}$ on T^4 such that $p : (\mathbb{M}, J) \to (T^4, \hat{J})$ is holomorphic.*

Proof. Let J be an element of $\mathscr{C}(\mathfrak{g})$. The essential point is that $\mathbb{D}$ is J-invariant. Once this is established, it suffices to define $\hat{J}$ to be the T^4-invariant complex structure determined on cotangent vectors by $J|_{\mathbb{D}}$. The pullback of a $(1,0)$-form on T^4 is then an invariant $(1,0)$-form on $\mathbb{M}$.

Let Λ denote the space of $(1,0)$-forms relative to J. Then

$$\dim(\langle e^1, e^2, e^3, e^4, e^5 \rangle_c \cap \Lambda) = 2.$$

If $\dim(\mathbb{D}_c \cap \Lambda) = 2$ then $J\mathbb{D} = \mathbb{D}$, as required. If not, there exists a $(1,0)$-form $\delta + e^5$ with $\delta \in \mathbb{D}$. This implies that

$$de^5 \in \Lambda^{2,0} \oplus \Lambda^{1,1},$$

and consequently that $de^5 \in \Lambda^{1,1}$. Similarly for e^6, and thus

$$\omega^1 \wedge \omega^2 = d\omega^3 = de^5 + i de^6 \in \Lambda^{1,1},$$

implying that $J\omega^1 \wedge J\omega^2 = \omega^1 \wedge \omega^2$ and hence

$$\langle J\omega^1, J\omega^2 \rangle = \langle \omega^1, \omega^2 \rangle.$$

Thus, the subspace $\langle \omega^1, \omega^2 \rangle$ is J-invariant, and $J\mathbb{D} = \mathbb{D}$. □

Decreeing the 1-forms e^i to be orthonormal determines a left-invariant metric

$$g = \sum_{i=1}^{6} e^i \otimes e^i \tag{4}$$

on G. This induces metrics on T^4 and $\mathbb{M}$ (that we also denote by g) for which p is a Riemannian submersion. The subset $\mathscr{C}^+(\mathfrak{g}, g)$ of $\mathscr{C}^+(\mathfrak{g})$ corresponding to g-orthogonal oriented complex structures is now easy to describe in terms of Theorem 1.1.

Lemma 1.2. *The restriction of the mapping* $J \mapsto \hat{J}$ *to* $\mathscr{C}^+(\mathfrak{g}, g)$ *is injective.*

Proof. We need to describe the set of invariant orthogonal complex structures on T^4 in terms of 2-forms on $\mathbb{D}$. First recall that an element of $\mathscr{C}^+(\mathfrak{g}, g)$ is determined by the corresponding *fundamental 2-form* γ satisfying $\gamma(X, Y) = g(JX, Y)$. Given J in $\mathscr{C}^+(\mathfrak{g}, g)$, both $\mathbb{D}$ and $\mathbb{D}^\perp = \langle e^5, e^6 \rangle$ are J-invariant and there exists an orthonormal basis $\{f^1, f^2, Jf^1, Jf^2\}$ of $\mathbb{D}$ for which

$$\gamma = f^1 \wedge Jf^1 + f^2 \wedge Jf^2 \pm e^5 \wedge e^6. \tag{5}$$

Then the fundamental 2-form of $\hat{J}$ is

$$\hat{\gamma} = f^1 \wedge Jf^1 + f^2 \wedge Jf^2. \tag{6}$$

The fact that the overall orientation of J on $\mathfrak{g}$ is positive then determines uniquely the sign in (5). $\square$

To continue the discussion in the above proof, fix either a plus or minus sign. Then

$$e^{12} \pm e^{34}, \quad e^{13} \pm e^{42}, \quad e^{14} \pm e^{23} \tag{7}$$

constitutes a basis of the 3-dimensional subspace $\bigwedge^2_\pm \mathbb{D}$ giving rise to the celebrated decomposition

$$\bigwedge^2 \mathbb{D} = \bigwedge^2_+ \mathbb{D} \oplus \bigwedge^2_- \mathbb{D}. \tag{8}$$

This determines a double covering $\mathrm{SO}(4) \to \mathrm{SO}(3)_+ \times \mathrm{SO}(3)_-$, and there exist corresponding *subgroups* $\mathrm{SU}(2)_+$, $\mathrm{SU}(2)_-$ of SO(4) acting trivially on $\bigwedge^2_- \mathbb{D}$, $\bigwedge^2_+ \mathbb{D}$ respectively.

The 2-form (6) belongs to the disjoint union

$$\mathscr{S}_+ \sqcup \mathscr{S}_-, \tag{9}$$

where $\mathscr{S}_\pm$ is a 2-sphere in $\bigwedge^2_\pm \mathbb{D}$. The choice of sign depends on whether $\hat{J}$ is positively or negatively oriented and is duplicated in (5). For example J_0 has fundamental 2-form

$$\gamma = e^{12} + e^{34} + e^{56},$$

and $\hat{\gamma} = e^{12} + e^{34} \in \mathscr{S}_+$. The product $\mathscr{S}_+ \times \mathscr{S}_-$ may be identified with the Grassmannian of oriented 2-planes in $\mathbb{R}^4$ and this was the origin of the concept of self-duality [2], [13]. Notice that (3) implies that $\operatorname{Im} d$ lies in the subspace $\bigwedge^2_+ \mathbb{D}$ of self-dual 2-forms; from this point of view $\mathbb{M}$ is an 'instanton' over the torus T^4.

The main result of [1] may now be summarized by

Theorem 1.3. *The space $\mathscr{C}^+(\mathfrak{g}, g)$ is the disjoint union of $\{J_0\}$ and the 2-sphere of all g-orthogonal almost complex structures J on $\mathfrak{g}$ for which $\hat{J} \in \mathscr{S}_-$.*

Let

$$\mathscr{Z}'_- = \{J \in \mathscr{C}^+(\mathfrak{g}, g) : \hat{J} \in \mathscr{S}_-\}$$

denote the 2-sphere featuring in this theorem; we use the notation of [1]. Consider SO(4) as a subgroup of $GL(6, \mathbb{R})$ by letting it act trivially on e^5, e^6. Since $d(\mathfrak{g}^*)$ is spanned by 2-forms in $\bigwedge^2_+ \mathbb{D}$, the subgroup $SU(2)_-$ is a group of Lie algebra automorphisms of $\mathfrak{g}$, acting transitively on $\mathscr{Z}'_-$. This observation will be important in §4.

2 Deformation of J_0

The main purpose of what follows is to generalize Theorem 1.3 by removing the orthogonality constraint. We begin by decomposing the space of *all* almost complex structures on $\mathbb{D}$ as

$$\mathscr{A}_+ \sqcup \mathscr{A}_-,$$

where $\mathscr{A}_\pm$ consists of those structures inducing a $\pm$ orientation on $\mathbb{D}$. This is the extension of (9) in the non-metric situation, and

$$\mathscr{A}_\pm \cong \frac{GL^+(4, \mathbb{R})}{GL(2, \mathbb{C})} \supset \frac{SO(4)}{U(2)} \cong \mathscr{S}_\pm.$$

We then set

Definition 2.1. Let $\mathscr{C}_\pm = \{J \in \mathscr{C}^+(\mathfrak{g}) : \hat{J} \in \mathscr{A}_\pm\}$.

In contrast to $\mathscr{A}_\pm$, the definition of $\mathscr{C}_\pm$ incorporates the requirement of integrability. If the overall orientation of $\mathfrak{g}$ is not fixed, we obtain

$$\mathscr{C}(\mathfrak{g}) = \mathscr{C}_+ \sqcup \mathscr{C}_- \sqcup (-\mathscr{C}_+) \sqcup (-\mathscr{C}_-),$$

where $-\mathscr{C}_\pm = \{-J : J \in \mathscr{C}_\pm\}$. Signs that appear as *sub*scripts refer exclusively to the orientation on $\mathbb{D}$.

In order to gain a greater understanding of the subsets $\mathscr{C}_+, \mathscr{C}_-$, we now describe a completely different set-theoretic partition of $\mathscr{C}^+(\mathfrak{g})$, in which J_0 plays the role of an origin. We use the notation (1), with $\bar\omega^{123} = \bar\omega^1 \wedge \bar\omega^2 \wedge \bar\omega^3$ etc.

Definition 2.2. Let $\mathscr{C}_0^\bullet$ be the open subset of $\mathscr{C}^+(\mathfrak{g})$ consisting of complex structures admitting a basis $\{\alpha^1, \alpha^2, \alpha^3\}$ of $(1,0)$-forms for which $\alpha^{123} \wedge \bar\omega^{123} \neq 0$, and let $\mathscr{C}_0^\infty$ be the complement $\mathscr{C}^+(\mathfrak{g})\backslash\mathscr{C}_0^\bullet$.

The zero subscript emphasizes that comparisons are being made with reference to J_0, elements of $\mathscr{C}_0^\infty$ are 'infinitely far' from J_0 in the sense that the coefficients in (10) below become unbounded.

Proposition 2.3. *If* $J \in \mathscr{C}_0^\bullet$ *then there exists a basis* $\{\alpha^i\}$ *of* $(1,0)$*-forms and* $a, b, c, d, x, y \in \mathbb{C}$ *such that*

$$\begin{cases} \alpha^1 = \omega^1 + a\bar\omega^1 + b\bar\omega^2, \\ \alpha^2 = \omega^2 + c\bar\omega^1 + d\bar\omega^2, \\ \alpha^3 = \omega^3 + x\bar\omega^1 + y\bar\omega^2 + u\bar\omega^3, \end{cases} \tag{10}$$

where $u = -ad + bc$.

Proof. Theorem 1.1 implies that α^1, α^2 can be chosen so that their real and imaginary components span $\mathbb{D}$. The condition $\alpha^{123} \wedge \bar\omega^{123} \neq 0$ ensures that $\omega^1, \omega^2, \omega^3$ appear with non-zero coefficients. We thus obtain the description (10) for some $u \in \mathbb{C}$. The equation relating a, b, c, d, u is a direct consequence of the integrability condition

$$d\alpha^3 \wedge \alpha^{123} = 0$$

expressing the fact that $d\alpha^3$ has no $(0,2)$-component. □

For the remainder of this section, we focus on $\mathscr{C}_0^\bullet$. In (10), $\hat J$ is the almost complex structure on $\mathbb{D}$ with $(1,0)$-forms

$$\begin{cases} \alpha^1 = \omega^1 + a\bar\omega^1 + b\bar\omega^2, \\ \alpha^2 = \omega^2 + c\bar\omega^1 + d\bar\omega^2, \end{cases} \tag{11}$$

and is conveniently represented by the matrix

$$X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{12}$$

The characteristic polynomial of $X\bar X$ has the form

$$c(x) = x^2 - \gamma x + \delta,$$

where

$$\gamma = \mathrm{tr}(X\overline{X}) = |a|^2 + |d|^2 + 2\,\mathrm{Re}(b\bar{c}),$$
$$\delta = \det(X\overline{X}) = |u|^2.$$

Let λ, μ denote the roots of $c(x)$. An inspection of the coefficients γ, δ shows that λ, μ are either both real of the same sign or complex conjugates. The following result is less obvious.

Lemma 2.4. *If λ, μ are real and non-positive, then $\lambda = \mu \leqslant 0$.*

Proof. Let $z = b\bar{c} \in \mathbb{C}$. The inequality $0 \leqslant |z| + \mathrm{Re}(z)$ implies that

$$0 \leqslant |bc| + \mathrm{Re}(b\bar{c}) \leqslant |bc - ad| + |ad| + \mathrm{Re}(b\bar{c}).$$

Thus,

$$0 \leqslant 2|ad - bc| + |a|^2 + |d|^2 + 2\,\mathrm{Re}(b\bar{c}) = 2\sqrt{\delta} + \gamma.$$

If $\gamma < 0$ then the discriminant $\gamma^2 - 4\delta$ is negative and the roots of $c(x)$ are not real. □

A direct calculation reveals that

$$\alpha^{12} \wedge \bar{\alpha}^{12} = (1 - \gamma + \delta)\omega^{12} \wedge \bar{\omega}^{12} = c(1)\omega^{12} \wedge \bar{\omega}^{12}.$$

Whence

Proposition 2.5.

$$\alpha^{12} \wedge \bar{\alpha}^{12} = 4(1 - \lambda)(1 - \mu)e^{1234},$$
$$\alpha^{123} \wedge \bar{\alpha}^{123} = -8i(1 - \lambda)(1 - \mu)(1 - \lambda\mu)e^{12\cdots 6}.$$

This proposition implies that the sign of $(1 - \lambda)(1 - \mu)$ corresponds to that of $\mathscr{C}_\pm$. Moreover, $\lambda\mu \geqslant 0$, so that

(i) $J \in \mathscr{C}_+ \cap \mathscr{C}_0^\bullet \Rightarrow 0 \leqslant \lambda\mu < 1$,

(ii) $J \in \mathscr{C}_- \cap \mathscr{C}_0^\bullet \Rightarrow \lambda\mu > 1$.

Note that $\mu = \bar{\lambda}$ implies that $c(1) = |1 - \lambda|^2 > 0$, and is only admissible for $J \in \mathscr{C}_+$. The possibilities for the unordered pair $\{\lambda, \mu\}$ are illustrated schematically in Figure 1. The two labelled regions correspond to Condition (i), with the origin a common point of intersection. The semi-circular region corresponds to $\mathrm{Im}\,\lambda > 0$ and $|\lambda| < 1$. By contrast, points (λ, μ) south-east of the diagonal line $\lambda = \mu$ represent those of the real plane in the usual way: the triangular region bounds points arising from $\mathscr{C}_+$ with $\lambda > \mu \geqslant 0$ and the shaded region represents points satisfying (ii).

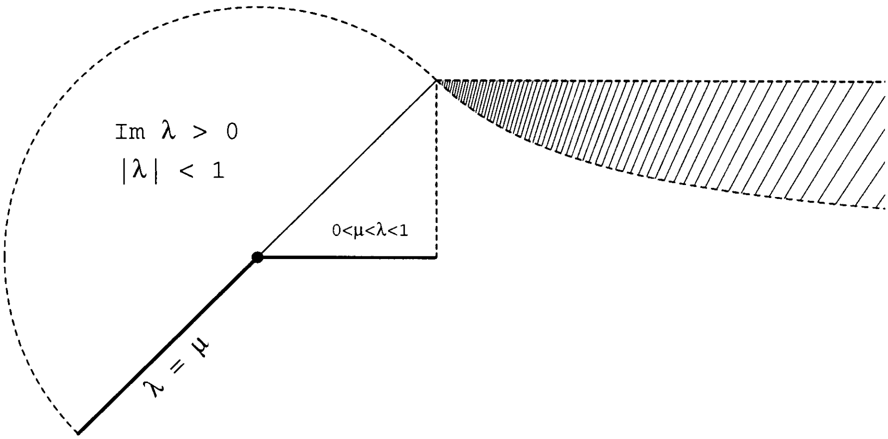

Figure 1

Remark. The similarity class of $X\overline{X}$ is invariant by the action

$$X \mapsto g^{-1} X \bar{g}, \quad g \in \mathrm{GL}(2, \mathbb{C}),$$

that characterizes the relation of 'consimilarity' [6]. This action is known to be transitive on the set of X corresponding to a fixed similarity class of $X\overline{X}$, provided $X\overline{X} \neq 0$. In particular, if $X\overline{X}$ is diagonalizable with λ, μ positive then there exists $g \in \mathrm{GL}(2, \mathbb{C})$ such that

$$X = g^{-1} \begin{pmatrix} \sqrt{\lambda} & 0 \\ 0 & \sqrt{\mu} \end{pmatrix} \bar{g}$$

(the sign of the square roots can be changed by modifying X) [6, 4.6.11]. Points in the open triangular region therefore represent $\mathrm{GL}(2, \mathbb{C})$ orbits of $\mathscr{A}_+$ that consist of projections (via Theorem 1.1) of complex structures in $\mathscr{C}_+$. On the other hand, $\mathscr{C}_-$ contains elements for which λ is infinite and the corresponding eigenvector of $X\overline{X}$ determines a point of $\mathbb{CP}^1$ that re-appears as a 2-sphere in Theorem 4.5 below.

We are focussing attention on the set $\mathscr{C}(\mathfrak{g})$ of left-invariant complex structures on G. The right action of G induces a transitive action on $\mathbb{M}$ and an induced action on $\mathscr{C}^+(\mathfrak{g})$. Given an element J of $\mathscr{C}^+(\mathfrak{g})$, let $R_G(J)$ denote the orbit of J induced by this action.

Proposition 2.6. *If J is given by* (10) *and* (12), *then*

$$\dim_{\mathbb{C}} R_G(J) = \begin{cases} 0 & \textit{if } X = 0, \\ 1 & \textit{if } u = 0 \textit{ and } X \neq 0, \\ 2 & \textit{otherwise}. \end{cases}$$

Proof. Right translation leaves invariant the 1-forms ω^1, ω^2 in (1), but maps ω^3 to $\omega^3 + p\omega^1 + q\omega^2$ for arbitrary $p, q \in \mathbb{C}$. Thus

$$\begin{aligned}\alpha^3 &\mapsto \alpha^3 + p\omega^1 + q\omega^2 + u(\overline{p\omega}^1 + \overline{q\omega}^2)\\ &= \alpha^3 + p(\alpha^1 - a\overline{\omega}^1 - b\overline{\omega}^2) + q(\alpha^2 - c\overline{\omega}^1 - d\overline{\omega}^2) + u(\overline{p\omega}^1 + \overline{q\omega}^2)\\ &= \alpha^3 + p\alpha^1 + q\alpha^2 + (u\bar{p} - ap - qc)\overline{\omega}^1 + (u\bar{q} - bp - dq)\overline{\omega}^2.\end{aligned}$$

This has the effect of replacing (x, y) by $(x + u\bar{p} - ap - cq, y + u\bar{q} - bp - dq)$ in (10). If $X = 0$ then J is unchanged, and $R_G(J) = \{J\}$. The remaining cases follow from the fact that $u = 0$ if and only if $ap + cq$ is proportional to $bp + dq$. □

A point of the moduli space of complex structures on $\mathbb{M}$ consists of an equivalence class of a complex structure (invariant or not) under the action of the diffeomorphism group. A neigbourhood of it at a smooth point J can be identified with a subset of $H^1(\mathbb{M}, \mathcal{O}(T_J))$, where $T = T_J$ denotes the holomorphic tangent bundle of J. This vector space is isomorphic to the corresponding cohomology group of the Dolbeault complex

$$0 \to \Omega^{0,0}(T) \to \Omega^{0,1}(T) \to \Omega^{0,2}(T) \to \Omega^{0,3}(T) \to 0. \tag{13}$$

Now, at least if J has rational coefficients relative to the basis $\{e^i\}$, it is known that the cohomology of (13) coincides with that of the finite-dimensional subcomplex formed by restricting to left-invariant forms of type (p, q) [3], [4].

The cohomology of the invariant subcomplex is easily computed in the case of $\mathbb{M}$, using the techniques of [11]. In all cases, $\ker \bar{\partial} : \Omega^{0,1}(T) \to \Omega^{0,2}(T)$ has dimension 6, whereas $\bar{\partial}(\Omega^{0,0}(T))$ has dimension $0, 1, 2$, consistent with the above proposition. This phenomenon leads to the jumping of Hodge numbers at J_0 described in [8]. In any case, it implies that the true moduli space of complex structures on $\mathbb{M}$ has dimension 4 at generic points. It is also suggests that every point is represented by an invariant complex structure, though the moduli space is singular at J_0 and other boundary points in Figure 1.

3 Study of $\mathscr{C}_+$

In recovering J from $\hat{J}$, we need only worry about the coefficients of α^3 in (10), for which u is determined by a, b, c, d and x, y are arbitrary complex numbers. Connectivity properties of $\mathscr{C}^+(\mathfrak{g})$ are determined by those of its dense subset $\mathscr{C}_0^\bullet$, and one expects the topology to be captured by that of the domains (i), (ii) characterizing the choice of $\{\lambda, \mu\}$. Results in this and the next sections will confirm that $\mathscr{C}_+$ and $\mathscr{C}_-$ are the connected components of $\mathscr{C}^+(\mathfrak{g})$.

Proposition 3.1. $\mathscr{C}_+ \cap \mathscr{C}_0^\infty = \varnothing$.

Proof. Let $J \in \mathscr{C}_0^\infty$. Suppose that $\{\alpha^1, \alpha^2, \alpha^3\}$ is a basis of $(1,0)$-forms of J with the real and imaginary components of α^1, α^2 spanning $\mathbb{D}$. Consider the two cases:

(i) $\alpha^{12} \wedge \bar{\omega}^{12} \neq 0$. This implies that $\alpha^3 \in \langle \bar{\omega}^1, \bar{\omega}^2, \bar{\omega}^3 \rangle$. The positive overall orientation of J then forces $\hat{J} \in \mathscr{A}_-$, and J cannot be in the same connected component as J_0.

(ii) $\alpha^{12} \wedge \bar{\omega}^{12} = 0$ and $\alpha^3 \notin \langle \bar{\omega}^1, \bar{\omega}^2, \bar{\omega}^3 \rangle$. Then $\langle \alpha^1, \alpha^2 \rangle \cap \langle \bar{\omega}^1, \bar{\omega}^2 \rangle \neq \{0\}$, and $\hat{J}$ has a non-zero $(1,0)$-form $A\bar{\omega}^1 + B\bar{\omega}^2$, which (without losing generality) we may take to equal α^1. If

$$\alpha^2 = P\omega^1 + Q\omega^2 + C\bar{\omega}^1 + D\bar{\omega}^2,$$

then the integrability of J forces $AD - BC = 0$, and (subtracting a multiple of α^1) we may suppose that $C = D = 0$. But then $\hat{J} \in \mathscr{A}_-$, and again $J \notin \mathscr{C}_+$. □

Theorem 3.2. *$\mathscr{C}_+$ is isomorphic to $\mathscr{U} \times \mathbb{C}^2$ where $\mathscr{U}$ is a star-shaped subset of $\mathbb{C}^4$.*

Proof. Proposition 3.1 implies that $\{\hat{J} : J \in \mathscr{C}_+\}$ is a subset of

$$\mathscr{U} = \{X \in \mathbb{C}^4 : (1-\lambda)(1-\mu) > 0, 0 \leqslant \lambda\mu < 1\},$$

using the notation (12). We shall show that if $X \in \mathscr{U}$ then $tX \in \mathscr{U}$ for any $t \in [0,1]$, a fact that is illustrated by Figure 1. Indeed, if the eigenvalues λ, μ of $X\overline{X}$ are complex conjugates, then the defining condition for $\mathscr{U}$ is $|\lambda| < 1$, which becomes $t^2|\lambda| < 1$ and remains valid. Suppose now that $\lambda, \mu \in \mathbb{R}$. Then $(1-\lambda)(1-\mu)$ becomes

$$f = (1 - t^2\lambda)(1 - t^2\mu),$$

an expression with roots $t_1^2 = 1/\lambda$ and $t_2^2 = 1/\mu$. If λ, μ are both negative then f has no real roots and is always strictly positive. If λ, μ are both positive then at least one of $1/\lambda, 1/\mu$ is greater than 1, and $(1-\lambda)(1-\mu) > 0$ implies that both are greater than 1. It follows that $f > 0$ for all $t \in [0,1]$, as required.

The restriction of p to $\mathscr{C}_+$ is a trivial bundle, whose fibre is obtained by varying only x, y, and $\mathscr{C}_+$ can be identified with $\mathscr{U} \times \mathbb{C}^2$. □

The complex structure induced on $\mathscr{C}_0^\bullet$ and $\mathscr{U} \times \mathbb{C}^2$ by the coefficients in (10) obviously coincides with that induced by the natural inclusion

$$\mathscr{C}(\mathfrak{g}) \to \mathbb{G}\mathrm{r}_3(\mathbb{C}^6)$$

obtained by mapping an invariant complex structure J to the span of a $(3,0)$-form α^{123}. This is also the natural complex structure induced from that of the potential tangent space $H^1(\mathbb{M}, \mathcal{O}(T_J))$ to the moduli space [11]. From this point of view, as a complex manifold, $\mathscr{C}_+$ can be identified with an open set of the quadric in $\mathbb{C}^7$ defined by the equation $u = -ad + bc$.

Remark. A completely different approach to describing complex structures on a 6-dimensional nilmanifold is based on properties of a $(3,0)$-form $\alpha^{123} = \varphi + i\psi$. The real component φ is a closed 3-form belonging to the open orbit $\mathcal{O}$ of elements of $\bigwedge^3 \mathfrak{g}^* \cong \mathbb{R}^{20}$ with stabilizer isomorphic to $\mathrm{SL}(3,\mathbb{C})$. As a consequence, any element φ of $\mathcal{O}$ determines a corresponding almost complex structure J_φ and $\psi = J_\varphi \varphi$ [5]. The kernel of $d : \bigwedge^3 \mathfrak{g}^* \to \bigwedge^4 \mathfrak{g}^*$ has dimension 15, and $d(J_\varphi \varphi) = 0$ turns out to be a single cubic equation in the coefficients of φ. This provides a description

$$\mathscr{C}(\mathfrak{g}) \cong \{\varphi \in \ker d \cap \mathcal{O} : d(J_\varphi \varphi) = 0\}/\mathbb{C}^*.$$

More details will appear elsewhere.

Consider an element $J \in \mathscr{C}_+$ whose restriction to $\mathbb{D}$ is g-orthogonal and therefore an element of $\mathscr{S}_+$. A point of $\mathscr{S}_+$ at 'finite' distance from $\widehat{J_0}$ is given by (11) with $g(\alpha^i, \alpha^i) = 0$ for $i = 1, 2$ and $g(\alpha^1, \alpha^2) = 0$. This implies that $a = d = 0$ and $b = -c$. It follows that the space of $(1,0)$-forms of J has a basis

$$\begin{cases} \alpha^1 = \omega^1 + b\overline{\omega}^2, \\ \alpha^2 = -b\overline{\omega}^1 + \omega^2, \\ \alpha^3 = \omega^3 + x\overline{\omega}^1 + y\overline{\omega}^2 - b^2\overline{\omega}^3. \end{cases} \tag{14}$$

Thus $\{\lambda, \mu\} = \{-|b|^2\}$, and $|b| < 1$.

Corollary 3.3. *$\{\hat{J} : J \in \mathscr{C}_+\} \cap \mathscr{S}_+$ is an open hemisphere.*

Proof. From (14), an element $\hat{J} \in \mathscr{S}_+$ has $(1,0)$-forms

$$\begin{cases} \alpha^1 = e^1 + ie^2 + be^3 - ibe^4, \\ \alpha^2 = -be^1 + ibe^2 + e^3 + ie^4. \end{cases}$$

Setting

$$A = \frac{1 - |b|^2}{1 + |b|^2}, \quad B = i\frac{\overline{b} - b}{1 + |b|^2}, \quad C = -\frac{b + \overline{b}}{1 + |b|^2}$$

gives $A^2 + B^2 + C^2 = 1$ and

$$\frac{\alpha^1 - \overline{b}\alpha^2}{1 + |b|^2} = e^1 + i(Ae^2 + Be^3 + Ce^4).$$

In the notation (7) with plus signs, the fundamental 2-form of $\hat{J}$ equals

$$e^1 \wedge (Ae^2 + Be^3 + Ce^4) + \cdots = A\omega_1 + B\omega_2 + C\omega_3.$$

The condition $|b| < 1$ translates into $A > 0$, that describes a hemisphere in $\mathscr{S}_+$. $\square$

Example. The almost complex structure I on $\mathbb{D}$ with space of $(1,0)$-forms

$$\langle e^1 + ie^3,\ e^4 + ie^2\rangle = \langle \omega^1 + i\bar{\omega}^2, \omega^2 - i\bar{\omega}^1\rangle$$

has $b = i$ in (14) and is a point on the equator $A = 0$ of $\mathscr{S}_+$. If I were to equal $\hat{J}$ with $J \in \mathscr{C}_+$, then J has a $(1,0)$-form of type

$$\alpha^3 = \omega^3 + x\omega^1 + y\omega^2 + \bar{\omega}^3,$$

with the final coefficient $+1$ necessary to satisfy the integrability condition. But then $\alpha^{12} \wedge \bar{\alpha}^{12} \wedge \alpha^3 \wedge \bar{\alpha}^3 = 0$, which is impossible.

4 Study of $\mathscr{C}_-$

We have remarked (Theorem 1.3) that the imposition of the standard metric implies that J_0 is the only orthogonal structure in its component $\mathscr{C}_+$. Whilst J_0 is convenient for the study of $\mathscr{C}_+$, it is less so for $\mathscr{C}_-$. For example, all the points of $\mathscr{Z}_-'$ belong to $\mathscr{C}_0^\infty$, making calculations difficult in the coordinates of (10). We shall therefore reformulate Definition 2.2 with respect to one particular element in $\mathscr{Z}_-'$.

Definition 4.1. Let $J_1 \in \mathscr{C}_-$ denote the complex structure for which $\eta = \omega^1 \wedge \bar{\omega}^2 \wedge \bar{\omega}^3$ is a $(3,0)$-form, $\mathscr{C}_1^\bullet$ be the open subset of $\mathscr{C}^+(\mathfrak{g})$ consisting of complex structures admitting a basis $\{\beta^1, \beta^2, \beta^3\}$ of $(1,0)$-forms for which $\beta^{123} \wedge \bar{\eta} \neq 0$, and $\mathscr{C}_1^\infty$ be the complement $\mathscr{C}^+(\mathfrak{g}) \backslash \mathscr{C}_1^\bullet$.

The analogue of Proposition 2.3 is

Proposition 4.2. *If $J \in \mathscr{C}_1^\bullet$ then β^i may be chosen so that*

$$\begin{cases} \beta^1 = \omega^1 + a\bar{\omega}^1 + b\omega^2, \\ \beta^2 = \bar{\omega}^2 + c\bar{\omega}^1 + d\omega^2, \\ \beta^3 = \bar{\omega}^3 + x\bar{\omega}^1 + y\omega^2 + v\omega^3, \end{cases} \tag{15}$$

where $a, b, c, d, x, y, v \in \mathbb{C}$ and $d = -av$.

Proof. This follows from Theorem 1.1 and the equation $d\beta^3 \wedge \beta^{123} = 0$. □

Because of the equation $d = -av$, v is unconstrained if a happens to vanish, and this contrasts with the situation in the previous section. Let $\mathscr{A}_1^\bullet$ be the set of almost complex structures on $\mathbb{D}$ with a basis of $(1,0)$-forms consisting of

$$\begin{cases} \beta^1 = \omega^1 + a\bar{\omega}^1 + b\omega^2, \\ \beta^2 = \bar{\omega}^2 + c\bar{\omega}^1 + d\omega^2, \end{cases} \tag{16}$$

for some $a, b, c, d \in \mathbb{C}$. The almost complex structure $\widehat{J_1}$ corresponds to $a = b = c = d = 0$.

Example. Recall that the projection $J \mapsto \hat{J}$ maps $\mathscr{Z}'_-$ onto $\mathscr{S}_-$. Elements of $\mathscr{S}_-$ have the form (16) with $a = d = 0$ and $b = -c$, except that $-\widehat{J_1}$ corresponds to b and c infinite. Thus, any element of $\mathscr{S}_- \backslash \{-\widehat{J_1}\}$ equals $\hat{J}$ for some $J \in \mathscr{C}_1^\bullet$ (compare Corollary 3.3).

With respect to the basis $\{e^1, e^2, e^3, e^4\}$ of $\mathbb{D}$, the element $\widehat{J_1}$ is represented by the matrix

$$Q_1 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \in \mathrm{SO}(4).$$

We can then identify $\mathscr{A}_-$ with the orbit $\{X^{-1} Q_1 X : X \in \mathrm{GL}^+(4, \mathbb{R})\}$, any element of which admits a polar decomposition

$$X^{-1} Q_1 X = SP, \tag{17}$$

where S is symmetric positive-definite and $P \in \mathrm{SO}(4)$.

Lemma 4.3. *With the above notation,* $P^2 = -1$, *and the resulting mapping* $r : \mathscr{A}_- \to \mathscr{S}_-$ *defined by* $SP \mapsto P$ *is a retraction.*

Proof. By first diagonalizing S, we may find a symmetric matrix σ for which

$$S = e^\sigma = \sum_{k=0}^{\infty} \frac{1}{k!} \sigma^k.$$

We claim that $e^{t\sigma} P \in \mathscr{A}_-$ for all $t \in [0, 1]$. Since $(SP)^2 = -1$,

$$SP = -P^{-1} S^{-1} = (P^{-1} S^{-1} P)(-P^{-1}),$$

in which $P^{-1} S^{-1} P = P^T S^{-1} P$ is positive-definite symmetric. Uniqueness of the polar decomposition implies that $S = P^{-1} S^{-1} P$ and $P = -P^{-1}$, so $P^2 = -1$. It follows also that $\sigma = -P^{-1} \sigma P$ and

$$e^{t\sigma} P = (P^{-1} e^{-t\sigma} P) P = -P^{-1} (e^{t\sigma})^{-1} = -(e^{t\sigma} P)^{-1},$$

as required. □

Proposition 4.4. $r^{-1}(Q_1) \cap \{\hat{J} : J \in \mathscr{C}_1^\infty\} = \varnothing$.

Proof. Given an element SQ_1 of $r^{-1}(Q_1) \cap \mathscr{A}_1^\bullet$ with $(1,0)$-forms as in (16), we claim that $b = c$. Identifying almost complex structures with 4×4 matrices, $1 + iSQ_1$ annihilates the $(1,0)$-forms β^1, β^2 of (16). Extending the standard metric g on $\mathbb{D}$ to a complex bilinear form,

$$\begin{aligned} 0 &= g((1+iSQ_1)\beta^1, Q_1\beta^2) - g(Q_1\beta^1, (1+iSQ_1)\beta^2) \\ &= 2g(\beta^1, Q_1\beta^2) \\ &= 2i(b-c), \end{aligned}$$

as stated.

In analogy to Proposition 2.5, we have

$$\beta^{12} \wedge \bar{\beta}^{12} = -4(1-\lambda)(1-\mu)e^{1234}, \tag{18}$$

where λ, μ are the eigenvalues of $Y\overline{Y}$ with $Y = \left(\begin{smallmatrix} a & b \\ b & d \end{smallmatrix}\right)$. Since $Y\overline{Y}$ is Hermitian, λ, μ are non-negative. Equation (18) forces λ, μ to lie in the interval $[0,1)$, and Y is bounded. Thus, $r^{-1}(Q_1) \subset \mathscr{A}_1^\bullet$. Finally suppose that $SQ_1 = \hat{J}$ with $J \in \mathscr{C}_1^\infty$. From (15), it must be the case that J has a $(1,0)$-form β^3 belonging to the span of $\bar{\varpi}^1, \omega^2, \omega^3$. But this is impossible, given that $J \in \mathscr{C}^+(\mathfrak{g})$. □

Theorem 4.5. *$\mathscr{C}_-$ has the homotopy type of a 2-sphere.*

Proof. Let $\mathscr{V} = \{J \in \mathscr{C}_- : \hat{J} \in r^{-1}(Q_1)\}$. We first show that this space is contractible. As a consequence of the previous proposition, a complex structure J in $\mathscr{V}$ has a basis of $(1,0)$-forms

$$\begin{cases} \beta^1 = \omega^1 + ta\bar{\varpi}^1 + tb\omega^2 \\ \beta^2 = \bar{\varpi}^2 + tb\bar{\varpi}^1 - t^2av\omega^2 \\ \beta^3 = \bar{\varpi}^3 + tx\bar{\varpi}^1 + ty\omega^2 + tv\omega^3, \end{cases}$$

for some $a, b, x, y, v \in \mathbb{C}$ and (for the moment) $t = 1$. But if we now allow t to vary in the interval $[0,1]$, these forms define a complex structure in $\mathscr{V}$. This process defines a homotopy

$$\mathscr{V} \times [0,1] \to \mathscr{V},$$

with the property that $(J,0)$ maps to J_1 for all J in $\mathscr{V}$.

The fact that all elements of $\mathscr{Z}'_-$ are equivalent under a $SU(2)_-$ action (see the remarks at the end of §1) allows us to extend the above to all the fibres of r over all points of the 2-sphere $\mathscr{Z}'_-$. □

Combined with the theorem in §3, we can conclude

Theorem 4.6. *$\mathscr{C}^+(\mathfrak{g})$ has the same homotopy type as $\mathscr{C}^+(\mathfrak{g}, g)$, where g is the inner product* (4).

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G. Ketsetzis, Mathematical Institute, University of Oxford, 24–29 St Giles', Oxford, OX1 3LB, UK
Email: Georgios.Ketsetzis@comlab.ox.ac.uk

S. Salamon, Mathematics Department, Imperial College, 180 Queen's Gate, London, SW7 2AZ, UK
Email: s.salamon@ic.ac.uk