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On the Finite Field Nullstellensatz for the intersection of two quadric hypersurfaces

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1 Introduction

Let p be a prime and $\mathbb{K}$ the algebraic closure of the finite field $\mathrm{GF}(p)$. We will always work in characteristic p and consider $\mathbb{P}^n$ as a scheme over $\mathrm{GF}(p)$. Let X be an algebraic scheme defined over a finite field $\mathrm{GF}(p^e)$. $X(\mathbb{K})$ will denote the set of all $\mathbb{K}$-points of X. For every power q of p with $q \geqslant p^e$ let $X(q)$ denote the set of all $\mathrm{GF}(q)$-points of X. Hence $X(q) \subseteq X(q')$ if q, q' are p-powers and $q' \geqslant q \geqslant p^e$. $X(\mathbb{K})$ is the union of all $X(q)$, $q \gg 0$ and q a p-power. If X is reduced, then the scheme X is uniquely determined by the algebraic variety $X(\mathbb{K})$ in the sense of Serre (Hilbert Nullstellensatz). If X is not a zero-dimensional scheme, then $X(\mathbb{K})$ is infinite. We fix a p-power q with $q \geqslant p^e$ and we would like to see up to what order the finite set $X(q)$ determines the infinite set $X(\mathbb{K})$.

Now assume that X is projective and that it is equipped with an embedding $X \subset \mathbb{P}^N$ defined over $\mathrm{GF}(q)$. Let k be an integer. We say that the pair $(X, X(q))$ satisfies the Finite Field Nullstellensatz of order k (or just that $FFN(k)$ is true for X and $X(q)$) if every homogeneous form of degree $\leqslant k$ on $\mathbb{P}^N(\mathbb{K})$ vanishing on $X(q)$ vanishes on $X(\mathbb{K})$. Choose homogeneous coordinates $z_0, \dots, z_N$ on $\mathbb{P}^N$. The set $\mathrm{PG}(N, q)$ is the union of $q+1$ hyperplanes; for instance take the hyperplanes $z_0 = cz_N$, $c \in \mathrm{GF}(q)$, and the hyperplane $z_N = 0$. Hence if $X(\mathbb{K}) \neq X(q)$ (and in particular if $\dim(X) > 0$), then the pair $(X, X(q))$ does not satisfy $FFN(q+1)$. A. Blokhuis and G. E. Moorhouse proved $FFN(q-1)$ for an elliptic quadric surface, $FFN(q)$ for a hyperbolic quadric surface and $FFN(q)$ for a smooth quadric hypersurface of $\mathrm{PG}(n, q)$, $q \geqslant 4$ [1]. G. E. Moorhouse proved $FFN(q)$ for Hermitian varieties, q a square [5, Theorem 4.1], and $FFN(q-1)$ for Grassmann varieties [6, §4]. Here we consider the case of the intersection of two quadric hypersurfaces and prove the following result.

Theorem. *Fix an integer $N \geqslant 7$. Let q be a power of p and assume $q \geqslant 6$. Take two linearly independent quadric hypersurfaces Q_1, Q_2 of $\mathbb{P}^N$ defined over $\mathrm{GF}(q)$ and set*

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$Y := Q_1 \cap Q_2$ (the scheme-theoretic intersection). Then $Y(q) \neq \emptyset$. Let U be the linear subspace of $\mathbb{P}^N$ spanned by $Y(q)$. U is defined over $\mathrm{GF}(q)$. *Set $X := Y \cap U$ (the scheme-theoretic intersection). Then $X(q) = Y(q)$ and for every $P \in Y(q)$ there is a line $D \subset X$ defined over* $\mathrm{GF}(q)$ *with $P \in D$. The pair $(X, X(q))$ satisfies $FFN([(q-1)/4])$.*

Notice that since the line D in the statement of the Theorem is defined over $\mathrm{GF}(q)$, we have $\mathrm{card}\, D(q) = q + 1$. Easy examples show that in general the pair $(Y, Y(q))$ does not satisfies $FFN(1)$ (see Remark 5). To get $FFN(1)$ for the pair $(Y, Y(q))$ one should add some assumption and we prefer to avoid to do that; this is the reason for our formulation of the Theorem. We conjecture that if $n \gg 0$, $n := \dim(U)$, then the pair $(X, X(q))$ satisfies $FFN(q)$. For our proof of $FFN([(q-1)/4])$ the existence of $\mathrm{GF}(q)$-lines through each $\mathrm{GF}(q)$-point is very important. We conjecture that similar results are true for the intersection of s quadric hypersurfaces, i.e. we conjecture the existence of an integer $a(s)$ such that if $n \geqslant a(s)$, calling Y the intersection of s nice quadric hypersurfaces of $\mathbb{P}^n(\mathbb{K})$ defined over $\mathrm{GF}(q)$, then the pair $(Y, Y(q))$ satisfies $FFN(q)$. However, we believe that niceness of the quadrics should be a very restrictive assumption.

2 Proof of the theorem

Remark 1. Recall that by the Chevalley–Warning theorem a finite field is C_1 [2, p. 11]. Since $N > 4$, by a theorem of Nagata and Lang which extends the Chevalley–Warning theorem [2, Theorem 3.4] the quadrics Q_1 and Q_2 have a common point over $\mathrm{GF}(q)$, i.e. the scheme defined by $Q_1(\mathbb{K}) \cap Q_2(\mathbb{K})$ has a $\mathrm{GF}(q)$-point.

Remark 2. Let Z be any projective scheme defined over $\mathrm{GF}(q)$. The scheme Z_{red} is a subscheme of Z invariant for the natural action of the Galois group of the extension $\mathbb{K}/\mathrm{GF}(q)$. Since $\mathrm{GF}(q)$ is a perfect field, this implies that Z_{red} is defined over $\mathrm{GF}(q)$. We have $Z(\mathbb{K}) = Z_{\mathrm{red}}(\mathbb{K})$ and $Z(q) = Z_{\mathrm{red}}(q)$.

Remark 3. Fix a p-power $q' \geqslant q$ and let G be the Galois group of the extension $\mathrm{GF}(q')/\mathrm{GF}(q)$. Let A be a reduced projective scheme defined over $\mathrm{GF}(q)$ and assume that over $\mathrm{GF}(q')$ the scheme A is the union of s subschemes $A_1, \dots, A_s$, none of which is decomposable over $\mathrm{GF}(q')$. Then G acts as a permutation group on $\{1, \dots, s\}$ permuting $A_1, \dots, A_s$. The scheme A_i is invariant by this action of G if and only if A_i is defined over $\mathrm{GF}(q)$. For any $g \in G$ and any component A_i the varieties $g(A_i)$ and A_i are isomorphic over $\mathbb{K}$. In particular we have $\dim(g(A_i)) = \dim(A_i)$ and $\deg(g(A_i)) = \deg(A_i)$. Hence if $(\dim(A_1), \deg(A_1)) \neq (\dim(A_i), \deg(A_i))$ for every $i > 1$, then A_1 is defined over $\mathrm{GF}(q)$.

Remark 4. We use the notation of Remark 3. If $P \in A(q)$ we have $g(P) = P$ for every $g \in G$. Hence if $P \in A_1$ we have $P \in g(A_1)$ for every $g \in G$. In particular if P is a smooth point of A, then $g(A_1) = A_1$ for every $g \in G$, i.e. A_1 is defined over $\mathrm{GF}(q)$. Since a line is uniquely determined by two of its points, if A_1 is a line containing two different points of $A(q)$, then A_1 is defined over $\mathrm{GF}(q)$ and hence $\operatorname{card} A_1(q) = q+1$. Similarly, if A_1 is a smooth conic containing at least 3 points of $A(q)$ and no other component of A is contained in the plane $\langle A_1 \rangle$ spanned by A_1, then A_1 is defined over $\mathrm{GF}(q)$ and hence $\operatorname{card} A_1(q) = q+1$.

We separate here one step of the proof of the Theorem, because it may be useful for attacking the conjecture on the intersection of s quadrics. In each case or subcase considered we are able to identify $\langle (X \cap M')(q) \rangle$ and to give a large integer k such that the pair $(X \cap M', (X \cap M')(q))$ satisfies $FFN(k)$ seeing $X \cap M'$ as a subscheme of $\langle (X \cap M')_{\mathrm{red}} \rangle$. In most cases the integer k we found is obviously the best possible one, i.e. $FFN(k+1)$ fails.

Preliminary steps for the proof of the Theorem. Let $M(q) \subset \mathrm{PG}(n,q)$ be a 3-dimensional linear space. Call $M(\mathbb{K})$ the 3-dimensional linear subspace of $\mathbb{P}^n(\mathbb{K})$ spanned by the finite set $M(q)$ and M' the associated scheme. Hence $M'(\mathbb{K}) = M(\mathbb{K})$ and $M'(q) = M(q)$. Set $W := (M' \cap X)_{\mathrm{red}}$. Since X and M are defined over $\mathrm{GF}(q)$, W is defined over $\mathrm{GF}(q)$ (Remark 2). We have $W \neq \emptyset$, because $W(\mathbb{K}) \neq \emptyset$. We fix an integer $k \leqslant q$ and a homogeneous form F of degree k defined over $\mathrm{GF}(q)$ and vanishing on $X(q)$. We distinguish 7 cases and divide some of them into several subcases.

(a) $W = M'$. Hence $W(q) = M(q)$, $\langle W(q) \rangle = M'(\mathbb{K})$. Since $\deg(F) = k \leqslant q$ and F vanishes at each point of $M(q)$, $F \mid M'(\mathbb{K}) \equiv 0$.

(b) Here we assume that W is a quadric surface cone, say with vertex P and the smooth plane conic C defined over $\mathrm{GF}(q)$ as a base. We have $P \in \mathrm{PG}(3,q)$. If C has no $\mathrm{GF}(q)$-point, then $W(q) = \{P\}$ and hence $\langle W(q) \rangle = \{P\}$, while $\langle W(\mathbb{K}) \rangle = M'$. Now assume $C(q) \neq \emptyset$. Hence $\operatorname{card} C(q) = q+1$, $\operatorname{card} W(q) = 1+q+q^2$, $\langle W(q) \rangle = M'$ and if $k \leqslant q/2$ we have $F \mid W(\mathbb{K}) \equiv 0$.

(c) Here we assume that W is a reducible quadric surface, say $W = A \cup B$ with A and B planes. If the two planes A and B are not defined over $\mathrm{GF}(q)$, then only the line $A \cap B$ is defined over $\mathrm{GF}(q)$ and hence $W(q) = (A \cap B)(q)$, $\operatorname{card} W(q) = q+1$ and $\langle W(q) \rangle = A \cap B$. Hence if $W(q)$ is not contained in a line, A and B are defined over $\mathrm{GF}(q)$ and $W(q) = A(q) \cup B(q)$, $\langle W(q) \rangle = M'$ and $\operatorname{card} W(q) = 2(q^2+q+1) - q - 1$. Since $k \leqslant q$, we obtain $F \mid W(\mathbb{K}) \equiv 0$ if $W(q)$ is not contained in a line.

(d) Here we assume that W is a plane. We have $\langle W(q) \rangle = \langle W(\mathbb{K}) \rangle$. Since $k \leqslant q$, we have $F \mid W(\mathbb{K}) \equiv 0$.

(e) Here we assume that W is the disjoint union of a plane A and a non-empty union B of points and curves. Since two quadric surfaces containing A intersect in the union of A plus a line (perhaps contained in A), B is a line. By the last part of Remark 3 both A and B are defined over $\mathrm{GF}(q)$. Hence we have $\langle W(q) \rangle = \langle W(\mathbb{K}) \rangle$ and $F \mid W(\mathbb{K}) \equiv 0$.

(f) From now on, we assume that W has pure dimension one. By the Bezout theorem we have $1 \leqslant \deg(W) \leqslant 4$ and if $\deg(W) = 4$, then W is a reduced complete intersection of two quadric surfaces. In particular W has at most 4 irreducible components. Let A be an irreducible component of W defined over $\mathrm{GF}(q)$. If $\deg(A) = 1$ we have $\operatorname{card} A(q) = q+1$. Since $k \leqslant q$ we have $F \mid A(\mathbb{K}) \equiv 0$. Now assume $\deg(A) = 2$. By [4, pp. 3 and 4] either $A(q) = \emptyset$ or $\operatorname{card} A(q) = q+1$. If $A(q) = \emptyset$, we cannot say anything; however, this case will not arise in the proof of the Theorem, because we will always meet a case with $A(q) \neq \emptyset$. If $\operatorname{card} A(q) = q+1$ we obtain $F \mid A(\mathbb{K}) \equiv 0$ when $k \leqslant q/2$. Now assume $\deg(A) = 3$. Since A is contained in the intersection of two quadric surfaces and W does not contain a plane, A spans M'. Hence A is a rational normal curve of M' and we have $A(\mathbb{K}) \cong \mathbb{P}^1(\mathbb{K})$. The canonical line bundle of a smooth projective curve defined over any field K is defined over K. In particular the canonical line bundle of A is defined over $\mathrm{GF}(q)$. The canonical divisor of $\mathbb{P}^1$ has degree -2, i.e. even degree, while $3 = \deg(A)$ is odd. Hence there is a degree one line bundle on A defined over $\mathrm{GF}(q)$. This implies that A is isomorphic to $\mathbb{P}^1$ over $\mathrm{GF}(q)$. In particular we have $\operatorname{card} A(q) = q+1$. Hence $F \mid W(\mathbb{K}) \equiv 0$ if $3k \leqslant q$. Now assume $\deg(A) = 4$. Hence $A = W$, $p_a(A) = 1$ and A is the complete intersection of two quadrics. First assume A singular. Since $p_a(A) = 1$, we have $\operatorname{card}(\operatorname{Sing}(A)) = 1$, the normalization A' of A is isomorphic to $\mathbb{P}^1$ over $\mathbb{K}$ and A has either an ordinary node or an ordinary cusp. The curve A' is defined over $\mathrm{GF}(q)$ by the universal property of the normalization. If A has a cusp, then the counter-image of $\operatorname{Sing}(A)$ in A' is a unique point of A' and hence it is defined over $\mathrm{GF}(q)$; we have $q+1 = \operatorname{card} A'(q) = \operatorname{card} A(q)$ and hence $F \mid A(\mathbb{K}) \equiv 0$ if $4k \leqslant q$. Now assume that A has an ordinary node. If $A_{\mathrm{red}}(q) \neq \emptyset$, then $A'(q) \neq \emptyset$, i.e. A' is isomorphic to $\mathbb{P}^1$ over $\mathrm{GF}(q)$. Hence $\operatorname{card} A'(q) = q+1$ and $\operatorname{card} A(q) = q$. Since $\deg(A) = 4$, we have $F \mid A(\mathbb{K}) \equiv 0$ if $4k < q$ by the Bezout theorem.

(g) Now we assume the existence of an irreducible component B of W not defined over $\mathrm{GF}(q)$. Since W has pure dimension one and $\deg(W) \leqslant 4$, we have $\deg(B) \leqslant 2$ by Remark 3. First we consider the case $\deg(B) = 2$. Hence over $\mathbb{K}$ the irreducible curve B is a smooth conic and $W = B \cup B'$ with B' a smooth conic (over $\mathbb{K}$). Since $\deg(W) = 4$, we have $W = M' \cap X$, i.e. W is the complete intersection of two quadric surfaces. Hence W spans M', W is connected and $p_a(W) = 1$. In particular we have $1 \leqslant \operatorname{card}(\operatorname{Sing}(W(\mathbb{K}))) \leqslant 2$. By Remark 4 this case cannot occur if $\operatorname{card} W(q) \geqslant 3$. Now assume $\deg(B) = 1$. First assume that W has an irreducible component D with $\deg(D) \geqslant 2$. Since $\deg(W) < \deg(B) + 2\deg(D)$, D is defined over $\mathrm{GF}(q)$. Hence $\operatorname{card} D(q) = q+1$ and $\langle D(q) \rangle$ is a plane. By Remark 4 this case cannot occur if $W(q)$ spans M'. Look at $P \in W(q)$ and assume that P is not contained in a component of W defined over $\mathrm{GF}(q)$. Since $M' \cap X$ is the complete intersection of two quadric surfaces, there cannot be 3 components of W containing P, unless every component of W contains P (Remark 4); hence in this subcase we obtain that all the components of W are defined over $\mathrm{GF}(q)$ (Remark 3), contradiction. If P is contained in a unique component of W, then that component is defined over $\mathrm{GF}(q)$ by the first part of Remark 4. Now we assume that P is contained in exactly two components, say B_1 and B_2, of W, none of them defined over $\mathrm{GF}(q)$. By Remark 3 neither B_1 nor B_2 contain other points of $W(q)$. Since $\langle W(q) \rangle = M'$, we obtain $\deg(W) = 4$ and that the other two components, say A_1 and A_2, of W are defined over $\mathrm{GF}(q)$. Since W is the complete intersection of two quadric surfaces, W is connected and $p_a(W) = 1$. Since B_1 and B_2 are coplanar and W is the complete intersection of two quadrics, neither A_1 nor A_2 can be contained in the plane $\langle B_1 \cup B_2 \rangle$. The plane $\langle B_1 \cup B_2 \rangle$ is defined over $\mathrm{GF}(q)$ because B_1 and B_2 are exchanged by G. Hence the points $A_i \cap \langle B_1 \cup B_2 \rangle$, $i = 1, 2$, are defined over $\mathrm{GF}(q)$. Since W is connected and $P \in B_1 \cap B_2$, we obtain that at least one of the lines B_i, $i = 1, 2$, contains two points of $W(q)$ and hence it is defined over GF(q) (Remark 4). Since the scheme $M' \cap X$ is the complete intersection of two quadric surfaces, we have $h^0(M' \cap X, \mathcal{O}_{M' \cap X}) = 1$ (cf. [3]), i.e. $M' \cap X$ is connected in a very strong sense. In particular $W = (M' \cap X)_{\text{red}}$ cannot be the union of two disjoint lines. Since $\langle W(q) \rangle = M'$, we obtain $\deg(W) \geqslant 3$. First assume $\deg(W) = 3$. Since $W = (M' \cap X)_{\text{red}}$ and $\deg(M' \cap X) = 4$, the scheme $M' \cap X$ contains one line, D, of W with multiplicity two, while the other two lines of W appear with multiplicity one. Hence D is G-invariant, i.e. it is defined over GF(q), contradiction. Now assume $\deg(W) = 4$. If $W(q)$ contains a point contained in only one line $D \subseteq W$, then D must be defined over GF(q), contradiction. Since card $W(q) \geqslant 4$ by assumption, we obtain that at least one line of W contains two points of $W(q)$ and hence it is defined over GF(q) (Remark 4), contradiction.

Proof of the Theorem. We divide the proof into five steps.

Step 1. Since $N \geqslant 6$, we have $Y(q) \neq \emptyset$ by an extension due to Nagata and Lang of the Chevalley–Warning theorem [2, Theorem 3.4 and p. 11]. Set $n := \dim(U)$. U is defined over GF(q) because it is spanned by a subset of PG(N, q). By Remark 2 and the very definitions of U and X, $X(q) = Y(q)$ and $X(q)$ spans U, i.e. the pair $(X, X(q))$ satisfies $FFN(1)$ with respect to U. Fix an integer $k \leqslant q$ and a homogeneous form F of degree k defined over GF(q) and vanishing on $X(q)$. We call again Q_i the restriction of Q_i to U.

Step 2. Fix $P \in X(q)$. First assume that both Q_1 and Q_2 are singular at P, i.e. that they are cones with vertex P. Fix a hyperplane H of $\langle X \rangle$ defined over GF(q) (i.e. spanned by a subset of PG(n, q)) with $P \notin H$. Hence $X \cap H$ is defined inside H by two quadratic equations defined over GF(q). H is the intersection of $\langle X \rangle$ with a hyperplane H' of $\mathbb{P}^N$ defined over GF(q). Since $\dim(H') = N - 1 > 4$, we have $(X \cap H)(q) \neq \emptyset$ [2, Theorem 3.4 and p. 11]. Fix $O \in (X \cap H)(q))$. The line D spanned by $\{P, O\}$ is defined over GF(q). Since $O \in Q_1 \cap Q_2$ and Q_1 and Q_2 are cones with vertex P, then $D \subseteq X$, as wanted. Now assume that Q_1 and Q_2 are smooth at P. Let $T_P Q_i(\mathbb{K}) \subseteq \mathbb{P}^N(\mathbb{K})$ (resp. $T_P Q_i(q) \subseteq \mathbb{P}^N(q)$) be the tangent space of Q_i at P. Since Q_i is smooth at P, $T_P Q_i(\mathbb{K})$ and $T_P Q_i(q)$ are hyperplanes and $T_P Q_i(\mathbb{K})$ is spanned by $T_P Q_i(q)$. Set $Z(\mathbb{K}) := T_P Q_1(\mathbb{K}) \cap T_P Q_2(\mathbb{K})$ and $Z(q) := T_P Q_1(q) \cap T_P Q_2(q)$. Hence $Z(\mathbb{K})$ and $Z(q)$ are projective spaces (respectively over $\mathbb{K}$ and over GF(q)) such that $n - 2 \leqslant \dim Z(\mathbb{K}) = \dim Z(q) \leqslant n - 1$. We will call Z the corresponding linear subspace of $\mathbb{P}^n$. Hence $\dim Z = \dim Z(q)$ and Z is generated by $Z(q)$. Since Q_i is smooth at P, $Q_i \cap T_P Q_i$ is the union of all lines contained in Q_i and passing through P. Furthermore, $Q_i(q) \cap T_P Q_i(q)$ is the union of all lines of GF(q) contained in $Q_i(q)$ and passing through P. Z is the intersection of U with a codimension one or two linear subspace of $\mathbb{P}^N(q)$ defined over GF(q). Since $N - 2 \geqslant 4$, we have $(Z \cap X)(q) \neq \emptyset$ [2, Theorem 3.4 and p. 11]. For any $O \in (Z \cap X)(q)$ the line spanned by P and O is the line we were looking for. Now assume that Q_1 is smooth at P but that Q_2 is singular at P. Take a hyperplane H of $T_P Q_1(\mathbb{K})$ defined over GF(q) with $P \notin H$. Set $Y := X \cap H$. Since X, Z and U are defined over GF(q), Y is defined over GF(q). The scheme Y is defined in H by two quadric hypersurfaces. Since $\dim H = N - 2 \geqslant 4$, we have $Y(q) \neq \emptyset$ [2, Theorem 3.4 and p. 11]. For any

$O \in Y(q)$ the line spanned by P and O is the line we were looking for, because it is contained in $T_P Q_2$, too. In the same way we find the line D if Q_1 is singular at P, but Q_2 is smooth at P.

Step 3. Use the set-up and notation of Step 2. Instead of H (resp. Z) take a hyperplane H_1 (resp. Z_1) of H (resp. Z) defined over $\mathrm{GF}(q)$. Since $N-3 \geqslant 4$, we may take $O \in (X \cap H_1)(q)$ (resp. $O \in (X \cap Z_1)(q)$). Hence we obtain that for every $P \in X(q)$ there are several lines (at least three) contained in X, defined over $\mathrm{GF}(q)$ and containing P.

Step 4. Assume the existence of an integer u with $2 \leqslant u \leqslant n$ and lines $T_i \subset X$, $1 \leqslant i \leqslant u$, defined over $\mathrm{GF}(q)$, such that $T_i \cap T_j \neq \emptyset$ if and only if $|i-j| \leqslant 1$ and $T_1 \cup \cdots \cup T_u$ spans a linear space of dimension u. Assume $k < q/2$. For any integer t with $3 \leqslant t \leqslant n$, we define the following assertion $H(t)$:

$H(t)$: There exists a t-dimensional linear subspace M_t of $\mathbb{P}^N(\mathbb{K})$ spanned by a subset of $X(q)$ (and hence defined over $\mathrm{GF}(q)$) such that $F \mid (X \cap M_t)_{\mathrm{red}}(\mathbb{K}) \equiv 0$.

If $H(n)$ is true, then X satisfies $FFN(k)$. In this step we will prove $H(t)$ for every integer $t \leqslant u$ taking as M_t the linear span of $T_1 \cup \cdots \cup T_u$. First, we use the preliminary step to the proof of the Theorem to check $H(3)$ with $M_3 := \langle T_1 \cup \cdots \cup T_3 \rangle$; we use parts (a), (b), (c) and (d) if $X \cap M_3$ contains a surface and part (g) if $\dim(X \cap M_3) = 1$; indeed, since $\operatorname{card} T_1(q) = q+1$ we avoid the case $W(q) = \{P\}$ in part (b); in case (c) both planes A and B are defined over $\mathrm{GF}(q)$ because $\operatorname{card} T_1 \cup T_2(q) = 2q+1 > q+1 = \operatorname{card}(A \cap B)(q)$. Assume $u \geqslant 4$. We have $\operatorname{card} T_4(q) = q+1$. For every $P \in T_4(q)$ let $A(P)$ be the hyperplane of M_4 spanned by M_2 and P. M_4 is defined over $\mathrm{GF}(q)$ and $M_4 \cap T_4 = \{P\}$. By the previous step we have $F \mid (X \cap A(P))_{\mathrm{red}}(\mathbb{K}) \equiv 0$ for every P. Since $A(P) \cap X$ contains P and $P \notin T_1 \cup T_2$, $(X \cap A(P))_{\mathrm{red}}$ is the union of $T_1 \cup T_2$ and at least another curve containing P. Hence $(X \cap M_4)_{\mathrm{red}}$ contains $T_1 \cup T_2$ and at least $q+1$ other curves, say $C_1, \ldots, C_{q+1}$, such that $F \mid C_i(\mathbb{K}) \equiv 0$ for every i. If X contains M_4, then $F \mid M_4(\mathbb{K}) \equiv 0$ because $\mathrm{PG}(4,q)$ satisfies $FFN(q)$ and $k \leqslant q$. Hence to prove $H(4)$ using M_4 we may assume that X does not contain M_4. In order to obtain a contradiction we assume that F does not vanish at some point of $(X \cap M_4)_{\mathrm{red}}(\mathbb{K})$. First assume that $X \cap M_4$ does not contain a hypersurface of M_4. This is equivalent to assuming that the scheme $X \cap M_4$ is a codimension 2 complete intersection of two quadric hypersurfaces of M_4. Since $\deg X \cap M_4 = 4$, we have $\deg(X \cap M_4)_{\mathrm{red}} \leqslant 4$. Call A_i, $1 \leqslant i \leqslant s$, the irreducible components of $(X \cap M_4)_{\mathrm{red}}$ defined over $\mathbb{K}$, not necessarily over $\mathrm{GF}(q)$ of $(X \cap M_4)_{\mathrm{red}}$. Fix an index i. Either $F \mid A_i(\mathbb{K}) \equiv 0$ or the scheme $\{F = 0\} \cap A_i$ has degree $2\deg(A_i)$ and hence the scheme $(\{F=0\} \cap A_i)_{\mathrm{red}}$ has degree at most $2\deg(A_i)$. Hence if $\{F=0\}$ does not contain an irreducible component of $(X \cap M_4)_{\mathrm{red}}$, then the sum of all degrees of the curves $T_1, T_2, C_1, \ldots, C_{q+1}$ is at most 8. If $q \geqslant 6$ this is impossible. Now assume that $(X \cap M_4)_{\mathrm{red}}$ has some component of dimension 3, say B_j, $1 \leqslant j \leqslant r$, and some component of dimension 2, say A_i, $1 \leqslant i \leqslant s$, with $r \geqslant 1$ and $s \geqslant 0$. Since $X \cap M_4$ is defined by two quadratic equations, $B_1 \cup \cdots \cup B_r$ is either a quadric hypersurface of M_4 (perhaps reducible) or a hyperplane of M_4. First assume that $X \cap M_4$ is a quadric hypersurface of M_4. We must have $X \cap M_4 = B_1 \cup \cdots \cup B_r$. Since $T_1 \cup T_2 \cup T_3 \cup T_4 \subseteq X \cap M_4$, B_1 cannot be a cone with vertex a line R and as base a conic without $\mathrm{GF}(q)$-points, because in this case we would have $\operatorname{card}(X \cap M_4)(q) = \operatorname{card} R(q) =$

$q+1$; hence we have $H(4)$, because the irreducible quadric hypersurfaces of $\mathrm{PG}(4,q)$ with rank at least 4 satisfies $FFN(q-1)$. If $X \cap M_4$ is a reducible quadric hypersurface, then both components of $X \cap M_4$ are defined over $\mathrm{GF}(q)$ because $T_1 \cup T_2 \cup T_3 \cup T_4 \subseteq X \cap M_4$ and each line T_i is defined over $\mathrm{GF}(q)$; in this subcase $H(4)$ is true, because every linear space satisfies $FFN(q)$. Now assume that $B_1 \cup \cdots \cup B_r$ is a hyperplane. We may also assume $s \geqslant 1$, otherwise $F \mid (X \cap M_4)_{\mathrm{red}}(\mathbb{K}) \equiv 0$, because a linear space satisfies $FFN(q)$ and B_1 is defined over $\mathrm{GF}(q)$ by Remark 3. Since $X \cap M_4$ is the intersection of two quadric hypersurfaces of M_4 containg B_1, we have $s = 1$, and A_1 is a plane. Since A_1 is defined over $\mathrm{GF}(q)$ and $k \leqslant q$, we obtain $F \mid (X \cap M_4)_{\mathrm{red}}(\mathbb{K}) \equiv 0$. Now assume $u \geqslant 5$. We will prove $H(5)$. For every $P \in T_5(q) \backslash (T_5(q) \cap M_4(q))$, call $A(P)$ the hyperplane spanned by M_4 and P. The previous proof gives $F \mid (X \cap A(P))_{\mathrm{red}}(\mathbb{K}) \equiv 0$. Since $\mathrm{card}\, T_5(q) \cap M_4(q)) = q$ and $2k < q$, we obtain $H(5)$. If $u \geqslant 6$ we continue in the same way.

Step 5. We are not able to prove that we always may take $u = n$. By Step 2 we may at least take $u \geqslant 3$. Take the maximal integer u such that there is $T_1 \cup \cdots \cup T_u$ and assume $u < n$. Since u is maximal, for every $O \in T_u(q) \backslash T_{u-1}(q)$ every line contained in X and containing O is contained in $\langle T_1 \cup \cdots \cup T_u \rangle$. However, to prove $H(t)$ we need the full force of the existence of $T_1 \cup \cdots \cup T_u$ only for $u = 3$. In the other cases it is sufficient to take another line $D \subset X$, D defined over $\mathrm{GF}(q)$ and D not contained in $\langle T_1 \cup \cdots \cup T_u \rangle$. Such a line exists because $u < n := \dim \langle X(q) \rangle$ and for every $P \in X(q)$ with $P \in \langle T_1 \cup \cdots \cup T_u \rangle$ there is a line $D \subset X$, D defined over $\mathrm{GF}(q)$ with $P \in D$ (Step 1). Since the set $D(q)$ contains at least q points not contained in $\langle T_1 \cup \cdots \cup T_u \rangle$, the proof of $H(t)$ given in Step 4 works for $t = u+1$ using either $M_{+1} = \langle T_1 \cup \cdots \cup T_u \cup D \rangle$ if $D \cap \langle T_1 \cup \cdots \cup T_u \rangle \neq \emptyset$ or M_{u+1} spanned by $T_1 \cup \cdots \cup T_{u-1}$, D and one of the q points of $T_u(q) \backslash T_{u-1}(q)$. Then we continue inductively using at each step a suitable line and adding the new line to the previous configuration of lines (perhaps with several connected components) either q new $\mathrm{GF}(q)$-points or $q+1$ new $\mathrm{GF}(q)$-points and conclude the proof of the Theorem.

Remark 5. Here we show a very trivial case in which $n < N$, i.e. $Y \neq X$ and Y does not satisfy $FFN(1)$. Assume that in the pencil spanned by Q_1 and Q_2 there is a double hyperplane, say Q, with Q_{red} the hyperplane M and, say, $Q \neq Q_1$. For any scheme Z we have $Z(\mathbb{K}) = Z_{\mathrm{red}}(\mathbb{K})$ and in particular $Z(q) = Z_{\mathrm{red}}(q)$. Hence $Y(q) = (M \cap Q_1)(q) \subseteq M(q)$. Notice that this case may occur even if we assume that both Q_1 and Q_2 are smooth.

Remark 6. The existence of multiple components of Y has another drawback. Assume $\dim(Y) = N-2$, i.e. assume that Q_1 and Q_2 have no common components; for instance if Q_1 is irreducible just assume $Q_1 \neq Q_2$. It may occur that $(Q_1 \cap Q_2)_{\mathrm{red}}$ spans $\mathbb{P}^N$ but that $Q_1 \cap Q_2$ has a multiple component. For instance take a $\mathrm{GF}(q)$-plane A and an $(N-3)$-dimensional linear space V defined over $\mathrm{GF}(q)$ with $A \cap V = \emptyset$. Take two smooth conics C_1 and C_2 in V defined over $\mathrm{GF}(q)$ with $\mathrm{card}\, C_1 \cap C_2 = 3$, i.e. tangent at exactly one point. Let Q_i be the quadric cone with vertex V and base C_i. Call q_i any homogeneous equation of Q_i. Even if $Q_1 \cap Q_2$ satisfies $FFN(k)$ we may only say that a degree k polynomial vanishing on $Q_1 \cap Q_2(q)$ vanishes at each point of $(Q_1 \cap Q_2)_{\mathrm{red}}(\mathbb{K})$, not that $F = a_1q_1 + a_2q_2$ with a_i a homogeneous polynomial of degree $k-2$. The latter is the algebraic form of $FFN(k)$ when $\dim(X) = n-2$ and X has no multiple component.

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