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Regular hyperbolic fibrations

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Abstract. A hyperbolic fibration is a set of $q-1$ hyperbolic quadrics and two lines which together partition the points of $\mathrm{PG}(3,q)$. The classical example of a hyperbolic fibration comes from a pencil of quadrics; however, several other families are now known. In this paper we begin the development of a general framework to study hyperbolic fibrations for odd prime powers q.

One byproduct of hyperbolic fibrations is the 2^{q-1} (not necessarily inequivalent) spreads of $\mathrm{PG}(3,q)$ they spawn via the selection of one ruling family of lines for each of the hyperbolic quadrics. We show how the hyperbolic fibration context can be used to unify the study of these spreads, especially those associated with j-planes. The question of whether a spread spawned from such a fibration could contain any reguli other than the ones it inherits from the fibration plays a significant role in the determination of its automorphism group, as well as being an interesting geometric question in its own right. This information is then used to address the problem of sorting out projective equivalences among the spreads spawned from a given hyperbolic fibration. Plücker coordinates are an important tool in most of these investigations.

1 Introduction

A general setting for hyperbolic fibrations was outlined in [2], the highlights of which follow. The terminology and notation used in this paper will be consistent with that used in [2].

Let $\mathrm{GF}(q)$ denote the finite field of odd order q, and let $\mathrm{GF}(q)^*$ denote the nonzero elements of this field. We let $\square_q$ denote the nonzero squares in $\mathrm{GF}(q)$, while $\not\square_q$ denotes the nonsquares in that field. Throughout the paper, $\mathrm{PG}(n,q)$ will denote n-dimensional projective space over $\mathrm{GF}(q)$ and q will always be an odd prime power. A partition of the points of $\mathrm{PG}(3,q)$ into $q-1$ (mutually disjoint) hyperbolic quadrics and two (skew) lines is called a *hyperbolic fibration*. As usual, we model $\mathrm{PG}(3,q)$ as a 4-dimensional vector space over $\mathrm{GF}(q)$ using homogeneous coordinates. The classical example of a hyperbolic fibration is actually a pencil of quadrics. For a quaternary quadratic form F over $\mathrm{GF}(q)$, let $V(F)$ denote the set of

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zeroes of F in $\mathrm{PG}(3,q)$. When F and G are two such forms, with $V(F) \neq V(G)$, the set $\{V(F+tG) : t \in \mathrm{GF}(q) \cup \{\infty\}\}$ is a *pencil* of quadrics. A pencil consisting of two lines and $q-1$ hyperbolic quadrics of $\mathrm{PG}(3,q)$, whose members are necessarily mutually disjoint, is thus a hyperbolic fibration, which we call a *hyperbolic pencil* or *H-pencil* for short. A set of hyperbolic quadrics will be said to be *linear* if it is contained in an H-pencil.

The examples of hyperbolic fibrations which we will present in the next section are based on the following coordinatization ideas. Suppose ℓ_0 and ℓ_∞ are a pair of skew lines in $\mathrm{PG}(3,q)$. If $\{e_0,e_1\}$ is a basis for ℓ_0 and $\{e_2,e_3\}$ is a basis for ℓ_∞, then $\{e_0,e_1,e_2,e_3\}$ is a basis for $\mathrm{PG}(3,q)$. We let (x_0,x_1,x_2,x_3) denote homogeneous coordinates for $\mathrm{PG}(3,q)$ with respect to this ordered basis. Note that $\ell_0 = V(dx_2^2 + ex_2x_3 + fx_3^2)$ for any d,e,f such that e^2-4df is a nonsquare in $\mathrm{GF}(q)$. Similarly, $\ell_\infty = V(ax_0^2 + bx_0x_1 + cx_1^2)$ for any a,b,c such that b^2-4ac is a nonsquare in $\mathrm{GF}(q)$. Let Q be any quadric which has ℓ_0 and ℓ_∞ as conjugate lines with respect to its associated polarity. Using the basis $\{e_0,e_1,e_2,e_3\}$ as above, Q will have the form $V(ax_0^2 + bx_0x_1 + cx_1^2 + dx_2^2 + ex_2x_3 + fx_3^2)$ for some choice of a,b,c,d,e,f in $\mathrm{GF}(q)$. We abbreviate such a variety by

$$V[a,b,c,d,e,f] = V(ax_0^2 + bx_0x_1 + cx_1^2 + dx_2^2 + ex_2x_3 + fx_3^2).$$

We will sometimes refer to (a,b,c) as the "front half" and (d,e,f) as the "back half" of the variety $V[a,b,c,d,e,f]$.

In all known hyperbolic fibrations, the two (skew) lines of the fibration are conjugate with respect to each of the hyperbolic quadrics. A hyperbolic fibration with this property will be called *regular*. Note that the lines and quadrics of a regular hyperbolic fibration may be represented by six-tuples as above. In fact, typically either the first three or last three coordinates of the six-tuple may be fixed in a description of a hyperbolic fibration. The following result (see [2]) illustrates the appeal of this coordinatization with a fixed "back half".

Proposition 1.1. *Let $V[a,b,c,d,e,f]$ and $V[a',b',c',d,e,f]$ be as above with e^2-4df a nonsquare in* $\mathrm{GF}(q)$.

(a) *$V[a,b,c,d,e,f]$ is a hyperbolic quadric or an elliptic quadric accordingly as b^2-4ac is a nonsquare or nonzero square in* $\mathrm{GF}(q)$.

(b) *$V[a,b,c,d,e,f]$ and $V[a',b',c',d,e,f]$ are disjoint if and only if $(b-b')^2 - 4(a-a')(c-c')$ is a nonsquare in* $\mathrm{GF}(q)$.

The next result tells us something about how much information is required to determine a quadric $V[a,b,c,d,e,f]$ once the "front half" or "back half" are fixed.

Proposition 1.2. *Suppose that Q is a hyperbolic quadric which has ℓ_0 and ℓ_∞ as a pair of conjugate skew lines and has ℓ as a ruling line. Then there exists a unique representation of Q as $V[a,b,c,d,e,f]$ for a given triple (d,e,f) with e^2-4df a nonsquare. Likewise there exists a unique representation of Q as $V[a,b,c,d,e,f]$ for a given triple (a,b,c) with b^2-4ac a nonsquare.*

Proof. Let $\ell = \langle (x_0, x_1, x_2, x_3), (y_0, y_1, y_2, y_3) \rangle$, and write $Q = V[a, b, c, d, e, f]$ for a given triple (d, e, f) with $e^2 - 4df$ a nonsquare. Then we have the following system of equations in the unknowns a, b and c:

$$\begin{aligned} ax_0^2 + {} & bx_0x_1 + cx_1^2 + dx_2^2 + ex_2x_3 + fx_3^2 = 0 \\ ax_0y_0 + {} & \frac{b}{2}(x_0y_1 + x_1y_0) + cx_1y_1 + dx_2y_2 + \frac{e}{2}(x_2y_3 + x_3y_2) + fx_3y_3 = 0 \\ ay_0^2 + {} & by_0y_1 + cy_1^2 + dy_2^2 + ey_2y_3 + fy_3^2 = 0. \end{aligned}$$

Direct computations show that the coefficient matrix of this system has determinant $\frac{1}{2}(x_0y_1 - x_1y_0)^3$. This expression is nonzero unless (x_0, x_1) and (y_0, y_1) are multiples of each other, which is equivalent to ℓ intersecting ℓ_∞. Since ℓ is skew to ℓ_∞, (a, b, c) is uniquely determined by (d, e, f) and the given line ℓ as claimed. Alternately, we could fix (a, b, c) and treat this as a system in the unknowns d, e and f. A similar computation finishes the proof.

One reason for studying hyperbolic fibrations is their use in constructing two-dimensional translation planes. Any hyperbolic fibration gives rise to 2^{q-1} spreads by choosing one of the two ruling families of lines for each hyperbolic quadric in the fibration. We say that these spreads are *spawned* from the fibration. Such a spread will necessarily be partitioned into two lines and $q - 1$ reguli, thus admitting what is often called a *regular elliptic cover*. The two lines in such a partitioning are called the *carriers* of the regular elliptic cover. These 2^{q-1} spreads in turn give rise to 2^{q-1} translation planes of order q^2 whose kernels contain $\mathrm{GF}(q)$, some of which will be isomorphic to one another. We often say that these planes are also spawned from the fibration. It is well known that the translation planes spawned from an H-pencil are the Desarguesian planes and the two-dimensional Andre planes, which include the Hall planes (see [8]).

In this paper we also address the projective equivalence of hyperbolic fibrations. The following result follows immediately from the well known criterion for the equivalence of quadrics over finite projective spaces (see [7], Sections 5.1 and 5.2).

Proposition 1.3. *The regular hyperbolic fibrations*

$$\{V[a_i, b_i, c_i, d, e, f] : i = 1, 2, 3, \dots, q - 1\} \cup \{\ell_0, \ell_\infty\}$$

and $\{V[a_i, b_i, c_i, d', e', f'] : i = 1, 2, 3, \dots, q - 1\} \cup \{\ell_0, \ell_\infty\}$ *with constant back halves are projectively equivalent if and only if* $e^2 - 4df$ *and* $(e')^2 - 4d'f'$ *have the same quadratic character.*

2 Known families of hyperbolic fibrations

The first family of hyperbolic fibrations we discuss is induced by the spreads arising from $(q + 1)$-nests [6], which are known to admit regular elliptic covers. To describe these fibrations in the language of Section 1 we need the following result, whose proof may be found in [2].

Proposition 2.1. *The set* $C = \{z \in \mathrm{GF}(q^2) : z^{q+1} = -1\}$ *of* $(q+1)^{st}$ *roots of* -1 *in* $\mathrm{GF}(q^2)$ *is the union* $C_1 \cup C_2$ *of two equicardinal subsets with the property that the difference of any two distinct elements of* C *is a nonsquare or square in* $\mathrm{GF}(q^2)$ *accordingly as the two elements come from the same or different subsets.*

Let β be a primitive element of $\mathrm{GF}(q^2)$, and let $\varepsilon = \beta^{(1/2)(q+1)}$. Thus $\varepsilon^q = -\varepsilon$, and $\varepsilon^2 = \omega$ is a primitive element of the subfield $\mathrm{GF}(q)$. Using $\{1, \varepsilon\}$ as an ordered basis for $\mathrm{GF}(q^2)$ as a vector space of $\mathrm{GF}(q)$, we express each element $z \in \mathrm{GF}(q^2)$ as $z = z_0 + z_1\varepsilon$ for $z_0, z_1 \in \mathrm{GF}(q)$. Choose $\mu \in \not\square_q$ so that $1 - 4\mu \in \not\square_q$. Let r be a square root of $\dfrac{\omega}{1-4\mu}$ in $\mathrm{GF}(q)$, and let $t_0 \in \mathrm{GF}(q)$ be chosen so that $t_0^2(1-4\mu) - 1 \in \not\square_q$. Define

$$\mathscr{T}_0 = \{V[t, t, \mu t, 1, 1, \mu] : t \in \mathrm{GF}(q)^*, (t - t_0)^2(1 - 4\mu) - 1 \in \not\square_q\} \cup \{\ell_0, \ell_\infty\}.$$

Simple cyclotomy shows that $\mathscr{T}_0$ has $\frac{1}{2}(q+1)$ quadrics, including the two degenerate ones (lines). Another application of Proposition 1.1 shows the other $\frac{1}{2}(q-3)$ quadrics are hyperbolic. In fact, $\mathscr{T}_0$ is a subset of an H-pencil. For any $z \in C_1$, where C_1 is defined as in Proposition 2.1, we define

$$Q_z = Q_{z_0 + z_1\varepsilon} = V[a, b, c, 1, 1, \mu],$$

where $(a, b, c) = t_0(1, 1, \mu) + z_0(0, 1, \frac{1}{2}) + z_1(r, r, \frac{1}{2}r(1 - 2\mu))$. Note that t_0, r, and μ are fixed constants. Defining $\mathscr{N}_1 = \{Q_z : z \in C_1\}$, it is shown in [2] that

$$\mathscr{Q} = \mathscr{T}_0 \cup \mathscr{N}_1 \tag{1}$$

is a hyperbolic fibration. In fact, replacing C_1 by C_2 yields another (projectively equivalent) hyperbolic fibration. It should be noted that $\mathscr{Q}$ contains a linear subset of $\frac{1}{2}(q-3)$ hyperbolic quadrics. After a discussion of automorphism groups, it will become apparent that $\mathscr{Q}$ is indeed induced by a $(q+1)$-nest spread.

Our next family of hyperbolic fibrations was constructed in [2]. The idea is to start with a pencil of quadrics consisting of $\frac{1}{2}(q-1)$ hyperbolic quadrics, $\frac{1}{2}(q+1)$ elliptic quadrics, and one line which partition the points of $\mathrm{PG}(3, q)$ (see [4] for the existence of such pencils). By carefully replacing the $\frac{1}{2}(q+1)$ elliptic quadrics by one line and $\frac{1}{2}(q-1)$ hyperbolic quadrics, mutually disjoint, that cover the same point set as the elliptic quadrics, one obtains a hyperbolic fibration. To describe this fibration, again choose $\mu \in \not\square_q$ such that $1 - 4\mu \in \not\square_q$. Let

$$B = \{b \in \mathrm{GF}(q) : (1 - 4\mu)b^2 + 8\mu b \in \square_q \cup \{0\}, 2b \in \not\square_q\}.$$

For any $b \in B$, the equation $4z^2 - 2bz + \mu b(b-2) = 0$ will have two (possibly equal) roots in $GF(q)$, say c_1 and c_2, since the discriminant of this equation is $4[(1-4\mu)b^2 + 8\mu b]$. As shown in [2],

$$\begin{aligned}\mathscr{B} = &\{V[t, t+1, \mu t, 1, 1, \mu] : t \in GF(q); (t+1)^2 - 4\mu t^2 \in \not\square_q\} \\ &\cup \left\{ V\left[\frac{c_2}{\mu}, b, c_1, 1, 1, \mu\right] : b \in B; c_1, c_2 \in \mathrm{Roots}(4z^2 - 2bz + \mu b(b-2)) \right\} \\ &\cup \{\ell_0, \ell_\infty\} \qquad (2)\end{aligned}$$

is a hyperbolic fibration, obtained by replacing the elliptic quadrics in a pencil of the type described above. Note that one gets two hyperbolic quadrics in $\mathscr{B}$ from each $b \in B$ with $(1-4\mu)b^2 + 8\mu b \in \square_q$.

The only other known hyperbolic fibrations for odd q, to the best of your knowledge, are those induced by the spreads associated with j-planes. For a complete discussion of j-planes, see [10]. Here we give only a brief review of the basic construction. Let $x^2 + gx - f$ be an irreducible polynomial over $GF(q)$, so that $g^2 + 4f \in \not\square_q$, and fix some nonnegative integer j. Consider the cyclic group G of order $q^2 - 1$ acting on $PG(3, q)$ that is induced by all the matrices of the form

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \delta_{s,t}^{-j} & 0 & 0 \\ 0 & 0 & s & t \\ 0 & 0 & ft & s+gt \end{bmatrix},$$

where s and t vary over $GF(q)$, not both 0, and $\delta_{s,t} = s^2 + gst - ft^2$. Let ℓ be the line of $PG(3, q)$ with basis $\{e_0 + e_2, e_1 + e_3\}$, using our previous notation. If $\{\ell_0, \ell_\infty\}$ together with the orbit of ℓ under G is a spread of $PG(3, q)$, then the associated translation plane of order q^2 (defined by f, g and j) is called a *j-plane*.

As pointed out in [10], such spreads (if they exist) admit regular elliptic covers, and hence they must induce hyperbolic fibrations. Moreover, several infinite families of j-planes are shown to exist in [10]. However, from our point of view the induced hyperbolic fibrations seem to constitute a more unifying approach to describing these j-planes as well as many other planes. The point is that only a few of the 2^{q-1} spreads spawned by one of these hyperbolic fibrations admits such a cyclic group of order $q^2 - 1$ and hence generates a j-plane for some j. The other spreads spawned correspond to 2-dimensional translation planes which are not j-planes. For instance, the *pseudo nearfield planes* defined in [10] correspond to certain spreads spawned from these hyperbolic fibrations, yet most often they are not j-planes.

We now discuss three infinite families of hyperbolic fibrations, which we will soon see spawn all the known j-planes of odd order. The general context of a j-fibration will be developed after the following theorem.

Theorem 2.2. *Let $q = p^n$, where p is an odd prime.*

(a) *Fix some* $i \in \{0, 1, 2, \ldots, n\}$, *and choose* $\omega \in \not\square_q$. *Consider the set*

$$\mathcal{J}_0 = \{V[t, 0, -\omega t^{p^i}, 1, 0, -\omega] : t \in \mathrm{GF}(q)^*\} \cup \{\ell_0, \ell_\infty\}.$$

Then $\mathcal{J}_0$ *is a hyperbolic fibration which is a classical H-pencil when* $i = 0$ *or* $i = n$.

(b) *Suppose that* $-3 \in \not\square_q$ *or, equivalently,* $q \equiv 2 \pmod 3$. *Then the set*

$$\mathcal{J}_1 = \{V[t, 3t^2, 3t^3, 1, 3, 3] : t \in \mathrm{GF}(q)^*\} \cup \{\ell_0, \ell_\infty\}$$

is a hyperbolic fibration. The variation $V[t, 0, -\omega t^3, 1, 0, -\omega]$ *may be used when* q *is a power of* 3, *where* ω *is any nonsquare in* $\mathrm{GF}(q)$.

(c) *Suppose that* $5 \in \not\square_q$ *or, equivalently,* $q \equiv \pm 2 \pmod 5$. *Then*

$$\mathcal{J}_2 = \{V[t, 5t^3, 5t^5, 1, 5, 5] : t \in \mathrm{GF}(q)^*\} \cup \{\ell_0, \ell_\infty\}$$

is a hyperbolic fibration. The variation $V[t, 0, -\omega t^5, 1, 0, -\omega]$ *may be used when* q *is a power of* 5, *where again* ω *is any nonsquare in* $\mathrm{GF}(q)$.

Proof. For the first claim, when $i = 0$ or $i = n$, the quadric $V[t, 0, -\omega t, 1, 0, -\omega]$ for $t \neq 0$ is easily seen to be a hyperbolic quadric from Proposition 1.1, and moreover these are the hyperbolic quadrics of an H-pencil with carriers ℓ_0 and ℓ_∞. For other values of i, the quadrics in $\mathcal{J}_0$ other than ℓ_0 and ℓ_∞ are again easily seen to be hyperbolic quadrics. The mutual disjointness follows from Proposition 1.1 and the fact $-4(t - s)(\omega s^{p^i} - \omega t^{p^i}) = 4\omega(t - s)^{p^i+1} \in \not\square_q$.

For the second claim, since $-3 \in \not\square_q$ and $9(t^2 - s^2)^2 - 12(t - s)(t^3 - s^3) = -3(t - s)^4$, a straightforward application of Proposition 1.1 shows that $\mathcal{J}_1$ is a hyperbolic fibration. The variation for $q \equiv 0 \pmod 3$ is similarly seen to yield a hyperbolic fibration.

Finally, for the third claim, since $5 \in \not\square_q$ and $25(t^3 - s^3)^2 - 20(t - s)(t^5 - s^5) = 5(t - s)^2(t^2 + 3ts + s^2)^2$, another application of Proposition 1.1 shows that $\mathcal{J}_2$ is also a hyperbolic fibration. The result similarly holds for the given variation when $q \equiv 0 \pmod 5$.

To discuss any spreads spawned from these fibrations that correspond to j-planes, we first define the general notion of a j-fibration. Using the notation of [10] described above, let $f, g \in \mathrm{GF}(q)$ with $g^2 + 4f \in \not\square_q$. If the hyperbolic quadrics $\{V[t, gt^{j+1}, -ft^{2j+1}, -1, -g, f] : t \in \mathrm{GF}(q)^*\}$ are mutually disjoint and hence form with $\{\ell_0, \ell_\infty\}$ a hyperbolic fibration for some nonnegative integer j, then this fibration is called a *j-fibration*. Straightforward computations show that the cyclic group G of order $q^2 - 1$ defined above fixes ℓ_0 and ℓ_∞ while permuting the hyperbolic quadrics in the above set. Moreover, the line with basis $\{e_0 + e_2, e_1 + e_3\}$ is clearly a ruling line of the hyperbolic quadric $V[1, g, -f, -1, -g, f]$. Hence a spread associated with a j-plane, as previously defined, will induce a j-fibration as above, and conversely a j-fibration will spawn exactly two j-planes.

As shown in [10], the planes of Kantor [11] obtained via ovoids in 8-dimensional hyperbolic space turn out to be j-planes for $j = 1$. Moreover, it is shown for these examples that without loss of generality one may take as parameters $g = 3$ and $f = -3$ for odd q with $q \equiv 2 \pmod 3$, and one may take $g = 0$ and $f = \omega$ for any nonsquare ω when $q \equiv 0 \pmod 3$. Since $V[t, 3t^2, 3t^3, 1, 3, 3]$ and $V[t, 3t^2, 3t^3, -1, -1, -3]$ are projectively equivalent by Proposition 1.3, as are $V[t, 0, -\omega t^3, 1, 0, -\omega]$ and $V[t, 0, -\omega t^3, -1, 0, \omega]$, we see that the hyperbolic fibration $\mathscr{J}_1$ and its alternate form spawn the odd order 1-planes first constructed by Kantor.

The 2-planes constructed in [10] exist for $q \equiv \pm 2 \pmod 5$ and for $q \equiv 0 \pmod 5$. In the former case it is shown that without loss of generality one may take as parameters $g = 5$ and $f = -5$, while in the latter case one may take $g = 0$ and $f = \omega$ for any nonsquare ω. Since $V[t, 5t^3, 5t^5, 1, 5, 5]$ and $V[t, 5t^3, 5t^5, -1, -5, -5]$ are projectively equivalent by Proposition 1.3, as are $V[t, 0, -\omega t^5, 1, 0, -\omega]$ and $V[t, 0, -\omega t^5, -1, 0, \omega]$, we see that the hyperbolic fibration $\mathscr{J}_2$ and its alternate form spawn these 2-planes.

The final infinite family of j-planes constructed in [10] exists for any odd prime power $q = p^n$. In fact, without loss of generality one may choose $j = (p^i - 1)/2$ for any $i = 0, 1, 2, \ldots, n$, and then take $g = 0$ and $f = \omega$ for any nonsquare ω. These j-planes are clearly spawned from the hyperbolic fibration $\mathscr{J}_0$. It should be noted that the 0-planes are Desarguesian, and the $\frac{1}{2}(q-1)$-planes are regular nearfield planes.

It is also shown in [10] that when the parameter $g = 0$ in any odd order j-plane, there is a multiple derivation of the j-plane that yields a $(\frac{1}{2}(q-1) + j)$-plane. This is a generalization of using multiple derivation to obtain a regular nearfield plane from a Desarguesian plane. However, when $g \neq 0$, such multiple derivation will typically not generate a j'-plane for any j'. Thus there appear to be some mistakes in the table listing "sporadic" j-planes in [10], where $g = 1$ in all examples. For instance, when $q = 17$, the table lists j-planes of order 17^2 with parameters $(j, f, g) = (5, 11, 1)$, $(13, 11, 1), (6, 10, 1)$ and $(14, 10, 1)$. Note that $13 = 5 + \frac{1}{2}(17 - 1)$ and $14 = 6 + \frac{1}{2}(17 - 1)$. However, our computations using MAGMA [5] indicate there are no j-planes of order 17^2 for $j = 13$ or $j = 14$. Similar entries occur throughout the table for all orders listed.

Perhaps more interestingly, the remaining planes in this table are actually isomorphic to planes spawned from one of the fibrations listed in Theorem 2.2. This comes about by a simple reparameterization. For instance, consider the 1-planes spawned from fibration $\mathscr{J}_1$ for $q \equiv 2 \pmod 3$. Writing $q = 3k + 2$ in this case and defining $s = t^3$, we see that $(t, t^2, t^3) = (s^{2k+1}, s^{k+1}, s)$ for all $t \in \mathrm{GF}(q)$. Also note that as t varies over $\mathrm{GF}(q)^*$, so does s since $\gcd(3, q-1) = 1$. Hence, in the above example for $q = 17$ (and thus $k = 5$) the spread with parameters $(j, f, g) = (5, 11, 1)$ induces the hyperbolic fibration $\{V[s, s^6, -11s^{11}, -1, -1, 11] : s \in \mathrm{GF}(q)^*\} \cup \{\ell_0, \ell_\infty\} = \{V[t^3, t^2, -11t, -1, -1, 11] : t \in \mathrm{GF}(q)^*\} \cup \{\ell_0, \ell_\infty\}$. Interchanging x_0 and x_1, as well as x_2 and x_3, and then multiplying by $(-11)^{-1} = 3 \in \mathrm{GF}(17)$, we see that $V[t^3, t^2, -11t, -1, -1, 11] \sim V[-11t, t^2, t^3, 11, -1, -1] \sim V[t, 3t^2, 3t^3, -1, -3, -3]$. Using Proposition 1.3 to replace the constant back half $(-1, -1, -3)$ by $(1, 3, 3)$, we obtain precisely the fibration $\mathscr{J}_1$ of Theorem 2.2. From this one easily shows that the given "sporadic" 5-plane of order 17^2 is isomorphic to one of the two 1-planes spawned from $\mathscr{J}_1$. Similar reparameterizations show that every "sporadic" j-plane listed in [10] is isomorphic to a j-plane spawned from one of the hyperbolic fibrations in Theorem 2.2. In fact, we conjecture that any odd order j-plane must be isomorphic to one spawned from one of the fibrations listed in Theorem 2.2.

3 Automorphisms

In this section we discuss the linear stabilizer of a regular hyperbolic fibration with constant back half as well as the stabilizer of any spread spawned from such a fibration. First we make some general statements about the automorphism group of any hyperbolic fibration. Let $\mathscr{F}$ be any hyperbolic fibration, and let $\mathrm{Aut}(\mathscr{F})$ denote the subgroup of $\mathrm{P\Gamma L}(4, q)$ leaving $\mathscr{F}$ invariant. Clearly any element of $\mathrm{Aut}(\mathscr{F})$ leaves invariant the two lines of the fibration (as a set) and permutes the $q-1$ hyperbolic quadrics. If $\mathscr{F}$ has a "large" partial pencil $\mathscr{P}_0$ of some pencil of quadrics $\mathscr{P}$ (not necessarily an H-pencil), then it is conceivable that $\mathrm{Aut}(\mathscr{F})$ will be the stabilizer of $\mathscr{P}_0$ in $\mathrm{Aut}(\mathscr{P})$ (see [2] for the case $\mathscr{F} = \mathscr{B}$). If L is a subgroup of $\mathrm{Aut}(\mathscr{F})$, then L acts on the set of $2(q-1)$ reguli which serve as ruling classes of the hyperbolic quadrics in $\mathscr{F}$. If no L-orbit contains such a regulus and its opposite, then one can always construct a spread $\mathscr{S}$ spawned from $\mathscr{F}$ on which L acts. On the other hand, if a spread $\mathscr{S}$ spawned from $\mathscr{F}$ has no "extra" reguli, then $\mathrm{Aut}(\mathscr{S}) \subseteq \mathrm{Aut}(\mathscr{F})$. An *extra* regulus of a spread $\mathscr{S}$ spawned from $\mathscr{F}$ is any regulus of $\mathscr{S}$ which is not inherited from $\mathscr{F}$. The first nontrivial question we address in this section is determining a lower bound on the number of extra reguli a spawned spread $\mathscr{S}$ must have if $\mathrm{Aut}(\mathscr{S}) \nsubseteq \mathrm{Aut}(\mathscr{F})$.

Theorem 3.1. *Let $\mathscr{F}$ be a hyperbolic fibration and let $\mathscr{S}$ be some spread spawned from $\mathscr{F}$. Suppose that $\mathrm{Aut}(\mathscr{S}) \nsubseteq \mathrm{Aut}(\mathscr{F})$. Then $\mathscr{S}$ has at least $\frac{1}{2}(q+1)$ extra reguli.*

Proof. By assumption there must be an automorphism of the spread $\mathscr{S}$ which maps the inherited regular elliptic cover of $\mathscr{S}$ onto some other regular elliptic cover. This new cover must contain at least one regulus, say $\mathscr{R}$, which is not in the original cover. Thus $\mathscr{R}$ must meet at least $\frac{1}{2}(q-1)$ reguli of the original cover, and this bound is achieved only if $\mathscr{R}$ contains both lines of $\mathscr{F}$. Those $\frac{1}{2}(q-1)$ reguli must not appear in the new cover, and thus the new cover must have at least $\frac{1}{2}(q-1)$ new reguli. If $q > 3$, this implies there are at least 2 extra reguli, which are necessarily disjoint as they lie in a regular elliptic cover. Hence at least one of these reguli does not contain both lines of $\mathscr{F}$, and replacing $\mathscr{R}$ by this regulus in the above argument generates at least $\frac{1}{2}(q+1)$ extra reguli in $\mathscr{S}$. For $q = 3$, the only translation planes of order q^2 are the Desarguesian plane and the Hall plane, whose associated spreads satisfy the theorem.

The bound in the above theorem is sharp. Theorem 10 of [6] is an example of an automorphism of a $(q+1)$-nest spread which is not inherited from the automorphism group of the induced hyperbolic fibration. Such spreads have precisely $\frac{1}{2}(q+1)$ extra reguli, as we shall soon see.

We now restrict our attention to regular hyperbolic fibrations with constant back half. That is, let $\mathscr{F} = \{V[a_i, b_i, c_i, d, e, f] : i = 1, 2, 3, \ldots, q-1\} \cup \{\ell_0, \ell_\infty\}$ as defined in Section 1, where $e^2 - 4df$ and $b_i^2 - 4a_ic_i$ are nonsquares in $\mathrm{GF}(q)$. If $\mathscr{F}$ is not an H-pencil, so that the "discriminant norm" defined on the front half of the quadrics in $\mathscr{F}$ is not the same as that defined on the back half, then the carriers ℓ_0 and ℓ_∞ play different roles. In particular, it is quite easy to see that in this case no automorphism of $\mathscr{F}$ will interchange ℓ_o and ℓ_∞. The following result describes automorphisms of $\mathscr{F}$ that fix every quadric in $\mathscr{F}$. Such automorphisms are said to be in the *kernel* of $\mathscr{F}$, which we denote by $\mathrm{Ker}(\mathscr{F})$. To simplify the notation we normalize the quadrics so that $d = 1$.

Theorem 3.2. *Let $\mathscr{F}$ be a regular hyperbolic fibration with constant back half as above, normalized so that $d = 1$. Let K be the group of collineations of* $\mathrm{PG}(3, q)$ *induced by all the matrices of the form*

$$M_{s,t} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & s & t \\ 0 & 0 & -ft & s + et \end{bmatrix},$$

as s and t vary over $\mathrm{GF}(q)$, *not both* 0, *such that* $s^2 + est + ft^2 = 1$. *Then K is a cyclic group of order* $q + 1$ *contained in* $\mathrm{Ker}(\mathscr{F})$.

Proof. The fact that K is a cyclic group of order $q + 1$ was shown in [10] (also see Section 2). Let

$$A = \begin{bmatrix} a_i & \frac{1}{2}b_i & 0 & 0 \\ \frac{1}{2}b_i & c_i & 0 & 0 \\ 0 & 0 & 1 & \frac{1}{2}e \\ 0 & 0 & \frac{1}{2}e & f \end{bmatrix}$$

be the symmetric matrix representing the quadric $V[a_i, b_i, c_i, 1, e, f]$. A straightforward computation shows that $M_{s,t}AM_{s,t}^{\mathrm{tr}} = A$, where M^{tr} denotes the transpose of M. Hence K fixes each hyperbolic quadric in $\mathscr{F}$. As K clearly fixes ℓ_o and ℓ_∞, the result follows.

It should be noted that K can be interpreted as a cyclic (affine) homology group order $q + 1$ in the translation complement of any translation plane obtained from a spread spawned from $\mathscr{F}$. We now exhibit more collineations in $\mathrm{Ker}(\mathscr{F})$.

Corollary 3.3. $\mathrm{Ker}(\mathscr{F})$ *contains a linear dihedral group of order* $2(q + 1)$.

Proof. Consider the collineation of $\mathrm{PG}(3, q)$ induced by the matrix

$$N = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & e & -1 \end{bmatrix}.$$

Then $N^2 = I$, $NM_{s,t}N = M_{s+et,-t}$, and $NAN^{\mathrm{tr}} = A$ using the above notation. Hence N induces an involution of $\mathrm{PGL}(4, q)$, clearly not in K, which normalizes K and leaves invariant each quadric of $\mathscr{F}$. The result now follows from the previous theorem.

It should be noted that this dihedral group D partitions the lines skew to ℓ_0 and ℓ_∞ into orbits of size $2(q+1)$, which correspond precisely to the hyperbolic quadrics of type $V[a, b, c, 1, e, f]$ as a, b, c vary over $\mathrm{GF}(q)$ with $b^2 - 4ac \in \square_q$. The $q-1$ hyperbolic quadrics of $\mathscr{F}$ cover $q-1$ of these D-orbits. One should also point out that the order of $\mathrm{Ker}(\mathscr{F})$ can be enlarged by another factor of 2 if $b_i = 0$ for all i. Namely, the involution induced by

$$\overline{N} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

is not in D, centralizes D, and leaves invariant each quadric of $\mathscr{F}$ in this case. The collineations in K leave invariant the ruling families (reguli) of each hyperbolic quadric in $\mathscr{F}$ since $\det(M_{s,t}) = 1$, while the involutions N and $\overline{N}$ switch the ruling families of each hyperbolic quadric of $\mathscr{F}$ since $\det(N) = -1 = \det(\overline{N})$ (see [1]).

We now restrict further to the five known families of hyperbolic fibrations, all of which are regular with constant back half and which were described in Section 2. We begin with j-fibrations.

Theorem 3.4. *Consider the j-fibrations $\mathscr{J}_0$, $\mathscr{J}_1$ and $\mathscr{J}_2$ as described in Theorem 2.2. Let $q = p^n$.*

(a) *If $n > 1$ and $i \in \{1, 2, 3, \ldots, n-1\}$, the hyperbolic fibration $\mathscr{J}_0$ admits a linear automorphism group of order $4(q^2-1)$ which is a semidirect product of a cyclic group of order $q^2 - 1$ by a Klein 4-group.*

(b) *The hyperbolic fibration $\mathscr{J}_1$ admits a linear automorphism group of order $2(q^2-1)$ which is a semidirect product of a cyclic group of order q^2-1 by a cyclic group of order 2. If $q \equiv 0 \pmod 3$, then $\mathscr{J}_1$ admits the same group of order $4(q^2-1)$ as did $\mathscr{J}_0$.*

(c) *The hyperbolic fibration $\mathscr{J}_2$ admits a linear automorphism group of order $2(q^2-1)$ which is a semidirect of a cyclic group of order q^2-1 by a cyclic group of order 2. If $q \equiv 0 \pmod 5$, then $\mathscr{J}_2$ admits the same group of order $4(q^2-1)$ as did $\mathscr{J}_0$.*

Proof. The cyclic group G of order $q^2 - 1$ described in Section 2 permutes the hyperbolic quadrics in the associated j-fibration and fixes the lines ℓ_0 and ℓ_∞ (see [10]). Hence, applied to the particular j-fibrations $\mathscr{J}_0$, $\mathscr{J}_1$ and $\mathscr{J}_2$, we get a linear cyclic automorphism group acting on each of these fibrations. In all cases there is a kernel subgroup of order $2(q+1)$ whose intersection with G has order $q+1$. For $\mathscr{J}_0$ and the alternate forms of $\mathscr{J}_1$ and $\mathscr{J}_2$ when $q \equiv 0 \pmod 3$ and $q \equiv 0 \pmod 5$, respectively, there is another factor of 2 in the order of the automorphism group because of the involution induced by $\overline{N}$.

In practice the groups described in Theorem 3.4 are the full linear stabilizers of the given j-fibrations, at least for sufficiently "large" q. For instance, when $q = 11$, MAGMA [5] computations verify that the full automorphism group of $\mathscr{J}_1$ has order $240 = 2(q^2 - 1)$. On the other hand, for $q = 7$, the fibration $\mathscr{J}_2$ has a full automorphism group of order 768, somewhat larger than expected. For $q = 9 = 3^2$ the fibration $\mathscr{J}_1$ (which is identical to $\mathscr{J}_0$ with $i = 1$ in this case) has a full automorphism group of order 640, twice as large as the group described in Theorem 3.4.

One should also discuss the automorphism groups of the spreads spawned by the above fibrations, as these groups are essentially the translation complements of the corresponding translation planes. For "large" q the spawned spreads associated with j-planes have the cyclic group G of order $q^2 - 1$ as the full linear stabilizer. This is to be expected, given the above comments on automorphism groups of j-fibrations, as N and $\overline{N}$ induce involutions which interchange the two reguli of each hyperbolic quadric in these fibrations. In practice, most of the spreads spawned have a full linear stabilizer of order $q+1$, namely the cyclic group K of Theorem 3.2.

We now turn our attention to the H-pencil. While choosing $i = 0$ or $i = n$ in $\mathscr{J}_0$ will yield an H-pencil, this particular representation has $b_i = 0$ for all i, using our previous notation, and hence might be somewhat misleading. We thus prefer to work with the representation

$$\mathscr{H} = \{V[t, t, t\mu, 1, 1, \mu] : t \in \mathrm{GF}(q)^*)\} \cup \{\ell_0, \ell_\infty\},$$

where $\mu \in \not\square_q$ with $1 - 4\mu \in \not\square_q$. This particular H-pencil is the one used in describing the fibration $\mathscr{Q}$, which we will discuss next. Since the "discriminant norms" on the front and back halves of an H-pencil are the same, one would expect automorphisms that interchange ℓ_0 and ℓ_∞, and hence a larger automorphism group.

Theorem 3.5. *The H-pencil $\mathscr{H}$ admits a linear automorphism group of order $8(q^2 - 1)(q + 1)$, which contains a kernel subgroup of order $4(q+1)^2$ isomorphic to the semidirect product of $Z_{q+1} \times Z_{q+1}$ by a Klein 4-group.*

Proof. Let F denote the set of 2×2 matrices of the form

$$A_{v,w} = \begin{bmatrix} v & w \\ -\mu w & v + w \end{bmatrix},$$

as v and w vary over GF(q), not both zero. Since $1 - 4\mu \in \not\square_q$, F is a cyclic group of order $q^2 - 1$ isomorphic to the multiplicative group of GF$(q^2)^*$. Let F_0 be the subgroup of F of order $q + 1$ determined by the matrices with determinant equal to one. Let $R = \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \mu \end{bmatrix}$, so that the hyperbolic quadric $Q_t = V[t, t, t\mu, 1, 1, \mu]$ in $\mathscr{H}$ is represented by the 4×4 matrix $\begin{bmatrix} tR & 0 \\ 0 & R \end{bmatrix}$. We also define the matrix $N_1 = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}$.

Since $ARA^{\mathrm{tr}} = \det(A)R$ for all $A \in F$, the matrices $\begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}$ and $\begin{bmatrix} A & 0 \\ 0 & I \end{bmatrix}$ induce collineations of PG$(3, q)$ that stabilize the H-pencil $\mathscr{H}$. As A varies over F, one obtains a linear collineation group of order $\dfrac{(q^2-1)^2}{q-1} = (q+1)^2(q-1)$. Since $N_1 R N_1^{\mathrm{tr}} = R$, the matrices $\begin{bmatrix} N_1 & 0 \\ 0 & I \end{bmatrix}$ and $\begin{bmatrix} I & 0 \\ 0 & N_1 \end{bmatrix}$ induce involutions in PGL$(4, q)$ that fix each quadric of $\mathscr{H}$. Furthermore, the matrix $\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$ induces an involution that interchanges ℓ_0 and ℓ_∞ while mapping Q_t to $Q_{t^{-1}}$. As all three of these involutions normalize the above collineation group, straightforward computations show that one obtains a linear collineation group of order $8(q+1)^2(q-1)$ leaving $\mathscr{H}$ invariant. Moreover, the collineations induced by $\begin{bmatrix} N_1 & 0 \\ 0 & I \end{bmatrix}$, $\begin{bmatrix} I & 0 \\ 0 & N_1 \end{bmatrix}$, $\begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}$ as A varies over F, and $\begin{bmatrix} B & 0 \\ 0 & I \end{bmatrix}$ as B varies over F_0 induce a normal subgroup of order $4(q+1)^2$ that fix each quadric of $\mathscr{H}$.

As mentioned previously, the spreads spawned from an H-pencil are those associated with the Desarguesian plane and the two-dimensional Andre planes, whose groups are well studied. Here we point out only one connection, which will be very useful when we study the hyperbolic fibration $\mathscr{Q}$ below.

Corollary 3.6. *The two regular spreads spawned from the H-pencil $\mathscr{H}$ have Bruck kernels induced by the matrices $\begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}$, as A varies over F, and $\begin{bmatrix} A^* & 0 \\ 0 & A \end{bmatrix}$, as A varies over F, respectively. Here A^* denotes the classical adjoint of A.*

Proof. The Bruck kernel (see [3]) is a cyclic group of order $q + 1$ acting regularly on the points of each line of a regular spread. As indicated in the above proof, the set of matrices $\begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}$, as A varies over F, induce a cyclic collineation group of order $q + 1$ which leaves invariant each quadric of $\mathscr{H}$. Using the minimal polynomial of a generator for F, one easily sees that the point orbits of this cyclic group are ℓ_o, ℓ_∞, and one ruling family of lines from each of the hyperbolic quadrics in $\mathscr{H}$. These point orbits thus constitute a regular spread on which the given cyclic group acts as a Bruck kernel.

Since $N_1 A N_1 = A^*$ for all $A \in F$, conjugating by the involution $\begin{bmatrix} N_1 & 0 \\ 0 & I \end{bmatrix}$ yields another cyclic subgroup of order $q+1$ contained in $\mathrm{Ker}(\mathscr{H})$, namely the subgroup induced by matrices of the form $\begin{bmatrix} A^* & 0 \\ 0 & A \end{bmatrix}$ as A varies over F. Since A and A^* have the same minimal polynomial, the point orbits are again the lines of a regular spread on which this group acts as a Bruck kernel. The second regular spread is obtained from the first by reversing all the reguli in the first spread which are ruling families of the hyperbolic quadrics in $\mathscr{H}$.

The automorphism group of the hyperbolic fibration $\mathscr{B}$ was well studied in [2] where it was first constructed. In short $\mathscr{B}$ admits a linear automorphism group of order $8(q+1)$ containing a normal dihedral subgroup of order $2(q+1)$ that fixes each quadric in $\mathscr{B}$. Any spread spawned from $\mathscr{B}$ admits a cyclic linear collineation group of order $q+1$, and for "large" q the vast majority of spawned spreads have this cyclic group of order $q+1$ as the full stabilizer of the spread.

Finally, we turn our attention to the fibration $\mathscr{Q}$. We use the same notation as that in Section 2, namely $\mathscr{Q} = \mathscr{T}_0 \cup \mathscr{N}_1$ where $\mathscr{T}_0$ contains $\frac{1}{2}(q-3)$ hyperbolic quadrics from the H-pencil $\mathscr{H}$ and the two lines $\{\ell_0, \ell_\infty\}$, while $\mathscr{N}_1$ contains the $\frac{1}{2}(q+1)$ hyperbolic quadrics Q_z as z varies over C_1. Recall that if $z = z_0 + z_1\varepsilon$, then $Q_z = V[a,b,c,1,1,\mu]$ with $(a,b,c) = t_0(1,1,\mu) + z_0(0,1,\frac{1}{2}) + z_1(r,r,\frac{1}{2}r(1-2\mu))$, where $t_0^2(1-4\mu) - 1 \in \not\square_q$ and $r^2 = \dfrac{\omega}{1-4\mu}$. As was shown in [2], $C_1 = \{\gamma^i : i \equiv 1 \pmod 4\}$ (and $C_2 = \{\gamma^i : i \equiv 3 \pmod 4\}$), where γ is a primitive $2(q+1)^{st}$ root of 1.

Theorem 3.7. *The hyperbolic fibration $\mathscr{Q}$ admits a linear automorphism group of order $2(q+1)^2$, which contains a kernel subgroup isomorphic to a dihedral group of order $2(q+1)$.*

Proof. As A varies over F_0, using the notation developed in the proof of Theorem 3.5, the matrices $\begin{bmatrix} I & 0 \\ 0 & A \end{bmatrix}$ induce a cyclic subgroup of order $q+1$ fixing each quadric of $\mathscr{Q}$. Together with the involution induced by $\begin{bmatrix} I & 0 \\ 0 & N_1 \end{bmatrix}$, this yields a kernel subgroup isomorphic to the semidirect product of Z_{q+1} by Z_2.

We now consider the subgroup of $\mathrm{PGL}(4,q)$ induced by the matrices of the form $\begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}$, as A varies over F_0. This is a cyclic group of order $\frac{1}{2}(q+1)$, as $\begin{bmatrix} -I & 0 \\ 0 & -I \end{bmatrix}$ is one such matrix and it induces the identity collineation. Using the definition of $\mathscr{N}_1$, tedious and messy linear algebraic computations show that $\begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}$, where $A = A_{v,w} = \begin{bmatrix} v & w \\ -\mu w & v+w \end{bmatrix}$ with $\det(A) = 1$, induces a collineation that maps the quadric Q_z to the quadric $Q_{\bar{z}}$, where $\bar{z} = \left(v + \frac{w}{2} + \frac{w}{2r}\varepsilon\right)^{2q} z$. In particular, $\bar{z} \in C_1$ when $z \in C_1$.

Thus the above cyclic subgroup fixes the quadrics of $\mathcal{T}_0$ and permutes the quadrics of $\mathcal{N}_1$. Notice that the collineations induced by $\begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}$ and $\begin{bmatrix} I & 0 \\ 0 & A \end{bmatrix}$, as A varies over F_0, form a group W of order $\frac{1}{2}(q+1)^2$ that cyclically permutes the $q+1$ lines in each of the ruling classes for the hyperbolic quadrics of the linear set $\mathcal{T}_0$.

From Corollary 3.6 we know that the matrices $\begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}$, as A varies over F_0, induce the unique index two subgroup W_0 of the Bruck kernel for one of the two regular spreads spawned from the H-pencil $\mathcal{H}$. We now spawn a spread from $\mathcal{Q}$ using these groups. First we choose a ruling class for each hyperbolic quadric in $\mathcal{T}_0$ so that the resulting reguli all lie in the regular spread whose Bruck kernel contains W_0 as an index two subgroup. Next we let $\mathcal{O}$ be a line orbit under W from the ruling lines of the hyperbolic quadrics in $\mathcal{N}_1$ on which it acts. Thus $\mathcal{O}$ consists of $\frac{1}{2}(q+1)^2$ lines which can be partitioned into $\frac{1}{2}(q+1)$ reguli, one from each hyperbolic quadric of $\mathcal{N}_1$. In particular, we have spawned a spread from $\mathcal{Q}$. On the other hand, $\mathcal{O}$ consists of $q+1$ orbits under W_0 and thus $q+1$ "opposite half-reguli" from the above regular spread. That is, the spread just spawned is a $(q+1)$-nest spread by definition (see [6]). This justifies our earlier claims that $\mathcal{Q}$ is indeed a hyperbolic fibration induced by a $(q+1)$-nest spread.

The computations in Theorems 5 and 9 of [6] now show that there exists an involution in the automorphism group of this $(q+1)$-nest spread, which is not in W, that leaves the fibration $\mathcal{Q}$ invariant by fixing ℓ_0, ℓ_∞, and each hyperbolic quadric in $\mathcal{T}_0$, while permuting the hyperbolic quadrics in $\mathcal{N}_1$. Adding the involution in Ker($\mathcal{Q}$) discussed above, which interchanges the ruling families of each hyperbolic quadric in $\mathcal{Q}$, we obtain a linear collineation group of order $2(q+1)^2$ that stabilizes $\mathcal{Q}$. It should be noted that in [6] it is shown that a $(q+1)$-nest spread also admits a linear stabilizer of order $2(q+1)^2$. These two groups of order $2(q+1)^2$ meet in a subgroup of order $(q+1)^2$.

In practice, at least for "large" enough q, the full linear automorphism group of $\mathcal{Q}$ has order $2(q+1)^2$, except when $t_0 = 0$, in which case the order is $4(q+1)^2$. For large q most of the spreads spawned from $\mathcal{Q}$ have a stabilizer of order $2(q+1)$, which contains as an index two subgroup the cyclic group of order $q+1$ in Ker($\mathcal{Q}$) which does not reverse reguli. However, if one spawns a spread from $\mathcal{Q}$ by choosing a W-orbit $\mathcal{O}$ as in the above proof but not "consistently" choosing ruling classes from $\mathcal{T}_0$ to be in the same regular spread, one can spawn a spread admitting a linear stabilizer of order $(q+1)^2$.

Perhaps most interesting is the fact that $N_1 A N_1 = A^*$ for any $A \in F_0$. Hence, conjugating W_0 by $\begin{bmatrix} I & 0 \\ 0 & N_1 \end{bmatrix}$, we see that the index two subgroup of the "other" Bruck kernel (see Corollary 3.6) is also contained in Aut($\mathcal{Q}$). Thus if we pick reguli from the hyperbolic quadrics of $\mathcal{T}_0$ that are all in the other regular spread, adjoining either one of the two W-orbits on the ruling lines of the quadrics from $\mathcal{N}_1$ will again yield a $(q+1)$-nest spread. Hence there are four $(q+1)$-nest spreads spawned from $\mathcal{Q}$, not necessarily projectively inequivalent. In practice, one obtains in this way at most two inequivalent spreads.

To conclude this section we return to the general case of an arbitrary hyperbolic fibration $\mathcal{F}$, and consider two spreads $\mathcal{S}_1$ and $\mathcal{S}_2$ spawned from $\mathcal{F}$. If neither spread has any extra reguli (reguli other than those inherited from $\mathcal{F}$), then any collineation mapping $\mathcal{S}_1$ to $\mathcal{S}_2$ would necessarily be an automorphism of $\mathcal{F}$. Hence the orders of the stabilizers of $\mathcal{F}$ and the spreads spawned from it would enable one to approximate the number of mutually inequivalent spreads that are spawned. We thus address the issue of extra reguli in the next two sections, leading us naturally to a discussion of Plücker coordinates and the Klein quadric.

4 Plücker correspondence and the Klein quadric

By considering the Plücker correspondence between lines of $\mathrm{PG}(3,q)$ and points of the Klein quadric in $\mathrm{PG}(5,q)$, we will develop some insight on whether a spread spawned by a hyperbolic fibration might contain any reguli other than those which it inherits from the fibration. We know the hyperbolic fibration $\mathcal{Q}$ associated with a $(q+1)$-nest [6] has extra reguli, while we have previously conjectured the hyperbolic fibration $\mathcal{B}$ constructed in [2] does not.

We follow the discussion of Plücker coordinates given in [12]. Recall that we denote points of $\mathrm{PG}(3,q)$ by $P=(x_0,x_1,x_2,x_3)$, using homogeneous coordinates. Consider a line $\ell=\langle(x_0,x_1,x_2,x_3),(y_0,y_1,y_2,y_3)\rangle$, where $P_1=(x_0,x_1,x_2,x_3)$ and $P_2=(y_0,y_1,y_2,y_3)$ are any two distinct points of ℓ. Then the Plücker lift of ℓ, say $\hat{\ell}$, is the point $(p_{01},p_{02},p_{03},p_{12},p_{31},p_{23})$ given by $p_{ij}=x_iy_j-x_jy_i$. Notice that these coordinates are easily seen to be homogeneous, independent of the two points from ℓ used to compute them, and satisfy the equation $p_{01}p_{23}+p_{02}p_{31}+p_{03}p_{12}=0$. The following is well known.

Proposition 4.1. *Each line* $\ell=\langle(x_0,x_1,x_2,x_3),(y_0,y_1,y_2,y_3)\rangle$ *of* $\mathrm{PG}(3,q)$ *corresponds to a point* $\hat{\ell}=(p_{01},p_{02},p_{03},p_{12},p_{31},p_{23})$ *of* $\mathrm{PG}(5,q)$ *given by* $p_{ij}=x_iy_j-x_jy_i$. *This point* $\hat{\ell}$ *lies on the Klein quadric*

$$\mathcal{K}=\{(X_0,X_1,X_2,X_3,X_4,X_5):X_0X_5+X_1X_4+X_2X_3=0\}$$

of $\mathrm{PG}(5,q)$. *Each point of* $\mathcal{K}$ *corresponds to a line of* $\mathrm{PG}(3,q)$, *and two lines of* $\mathrm{PG}(3,q)$ *intersect if and only if the join of their lifts lies on* $\mathcal{K}$. *The points on any line of* $\mathcal{K}$ *correspond to the lines of a plane pencil in* $\mathrm{PG}(3,q)$. *The points on any* (*planar*) *conic lying on* $\mathcal{K}$ *correspond to the lines of a regulus in* $\mathrm{PG}(3,q)$.

Given a regulus $\mathcal{R}$ of $\mathrm{PG}(3,q)$ and the corresponding conic $\mathcal{C}$ on $\mathcal{K}$, we will often refer to the plane π of $\mathrm{PG}(5,q)$ whose section with $\mathcal{K}$ is $\mathcal{C}$ as the *plane associated with* $\mathcal{R}$. It should be noted that a regular spread $\mathcal{S}$ of $\mathrm{PG}(3,q)$ corresponds to a 3-dimensional elliptic quadric lying on $\mathcal{K}$. The solid of $\mathrm{PG}(5,q)$ whose section with $\mathcal{K}$ is this elliptic quadric will be called the *solid associated with* $\mathcal{S}$. Of even more interest to us in this setting is the following observation from [12]. Suppose $[u_0,u_1,u_2,u_3]$ and

$[v_0, v_1, v_2, v_3]$ are two planes of $\mathrm{PG}(3,q)$ which intersect in ℓ, so that $u_0x_0 + u_1x_1 + u_2x_2 + u_3x_3 = 0$, $v_0x_0 + v_1x_1 + v_2x_2 + v_3x_3 = 0$, $u_0y_0 + u_1y_1 + u_2y_2 + u_3y_3 = 0$, and $v_0y_0 + v_1y_1 + v_2y_2 + v_3y_3 = 0$. Then multiplying the first equation by $-v_0$, the second equation by u_0, and adding yields $q_{01}x_1 + q_{02}x_2 + q_{03}x_3 = 0$, where $q_{ij} = u_iv_j - u_jv_i$. Likewise the last two equations yield $q_{01}y_1 + q_{02}y_2 + q_{03}y_3 = 0$. Combining these two equations, one easily checks that $q_{02}p_{12} - q_{03}p_{31} = 0$, or $\dfrac{q_{02}}{q_{03}} = \dfrac{p_{31}}{p_{12}}$. Proceeding in this manner, we conclude that $(q_{23}, q_{31}, q_{12}, q_{03}, q_{02}, q_{01})$ is also a representation of $\hat{\ell}$. In other words, the q coordinates of ℓ are naturally dual, with respect to $\mathscr{K}$, to the p coordinates. We thus have the following result.

Proposition 4.2. *The p (point) and q (plane) coordinates of the line ℓ are connected by the fact that the p coordinates $(p_{01}, p_{02}, p_{03}, p_{12}, p_{31}, p_{23})$ and the q coordinates $(q_{23}, q_{31}, q_{12}, q_{03}, q_{02}, q_{01})$ represent the same point of* $\mathrm{PG}(5,q)$ *on $\mathscr{K}$.*

Now suppose that $\ell = \langle P_1, P_2\rangle$ is a ruling line of the hyperbolic quadric $V[a,b,c,d,e,f]$, where $P_1 = (x_0, x_1, x_2, x_3)$ and $P_2 = (y_0, y_1, y_2, y_3)$. In addition to determining the p coordinates as above we have that $\left[ax_0 + \frac{b}{2}x_1, \frac{b}{2}x_0 + cx_1, dx_2 + \frac{e}{2}x_3, \frac{e}{2}x_2 + fx_3\right]$ and $\left[ay_0 + \frac{b}{2}y_1, \frac{b}{2}y_0 + cy_1, dy_2 + \frac{e}{2}y_3, \frac{e}{2}y_2 + fx_3\right]$ are two planes meeting in ℓ, which allows us to compute the q coordinates as well. This leads us to the following result.

Theorem 4.3. *Let $k \in \mathrm{GF}(q)^*$ be such that $k^2(b^2 - 4ac) = (e^2 - 4df)$ for a hyperbolic quadric $V[a,b,c,d,e,f]$ with ℓ_0 and ℓ_∞ as conjugate skew lines. Let $D = b^2 - 4ac$. Using $(X_0, X_1, X_2, X_3, X_4, X_5)$ as homogeneous coordinates for* $\mathrm{PG}(5,q)$, *the plane π given by*

$$\begin{aligned}
X_0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + kX_5 &= 0\\
(be + kD)X_1 + 2bfX_2 + 2ceX_3 - 4cfX_4 \qquad &= 0\\
2bdX_1 + (be - kD)X_2 + 4cdX_3 - 2ceX_4 \qquad &= 0
\end{aligned}$$

is the plane associated with one of the two ruling classes of the given quadric. Alternately, the last two equations defining π may be replaced by

$$\begin{aligned}
2aeX_1 + 4afX_2 + (be - kD)X_3 - 2bfX_4 \qquad &= 0\\
4adX_1 + 2aeX_2 + 2bdX_3 - (be + kD)X_4 &= 0.
\end{aligned}$$

Proof. Since $b^2 - 4ac$ and $e^2 - 4df$ are both nonsquares in $\mathrm{GF}(q)$, from Proposition 1.1, the definition of k makes sense. Let $\ell = \langle P_1, P_2\rangle$ be a ruling line of $V[a,b,c,d,e,f]$ as above. Computing q_{01} explicitly from the homogeneous coordinates for the two planes meeting in ℓ, we obtain

$$
\begin{aligned}
q_{01} &= \left(ax_0 + \frac{b}{2}x_1\right)\left(\frac{b}{2}y_0 + cy_1\right) - \left(\frac{b}{2}x_0 + cx_1\right)\left(ay_0 + \frac{b}{2}y_1\right) \\
&= \left(\frac{b^2}{4} - ac\right)(x_1 y_0 - x_0 y_1) \\
&= \left(ac - \frac{b^2}{4}\right)p_{01}.
\end{aligned}
$$

One similarly gets $q_{23} = \left(df - \frac{e^2}{4}\right)p_{23}$. With a bit more work one can express each q_{ij} in terms of the p coordinates. Using Proposition 4.2, one then deduces the existence of some nonzero scalar $\lambda \in \mathrm{GF}(q)$ such that the following equations hold:

$$
\begin{aligned}
\lambda p_{01} = q_{23} &= \left(df - \frac{e^2}{4}\right)p_{23} \\
\lambda p_{02} = q_{31} &= -\frac{be}{4}p_{02} - \frac{bf}{2}p_{03} - \frac{ce}{2}p_{12} + cfp_{31} \\
\lambda p_{03} = q_{12} &= \frac{bd}{2}p_{02} + \frac{be}{4}p_{03} + cdp_{12} - \frac{ce}{2}p_{31} \\
\lambda p_{12} = q_{03} &= \frac{ae}{2}p_{02} + afp_{03} + \frac{be}{4}p_{12} - \frac{bf}{2}p_{31} \\
\lambda p_{13} = q_{02} &= adp_{02} + \frac{ae}{2}p_{03} + \frac{bd}{2}p_{12} - \frac{be}{4}p_{31} \\
\lambda p_{23} = q_{01} &= \left(ac - \frac{b^2}{4}\right)p_{01}
\end{aligned}
$$

Solving the first and last of these simultaneously, we find that 4λ must be a square root of $(b^2 - 4ac)(e^2 - 4df)$, which we write as kD using k and D as in the statement of the theorem. The first equation can then be rewritten as $p_{01} + kp_{23} = 0$, as can the last equation.

Now multiply the middle four equations by 4 and replace 4λ to obtain the equations

$$
\begin{aligned}
(be + kD)p_{02} + 2bfp_{03} + 2cep_{12} - 4cfp_{31} &= 0 \\
2bdp_{02} + (be - kD)p_{03} + 4cdp_{12} - 2cep_{31} &= 0 \\
2aep_{02} + 4afp_{03} + (be - kD)p_{12} - 2bfp_{31} &= 0 \\
4adp_{02} + 2aep_{03} + 2bdp_{12} - (be + kD)p_{31} &= 0.
\end{aligned}
$$

The ruling line ℓ has p coordinates which satisfy $p_{01} + kp_{23} = 0$ as well as the above homogeneous linear system of four equations. One easily checks that the first two of these four equations are linearly independent since, for example, the coefficients for p_{12} and p_{31} yield the nonzero 2×2 determinant $-4c(e^2 - 4df)$. Similar computations show that the last two equations are also linearly independent. To see that the rank must be two, note that $X = (0, -4cf, 2ce, 2bf, be - kD, 0)$ and $Y = (0, -2ce, 4cd, be + kD, 2bd, 0)$ are two linearly independent solutions to this homogeneous 4×4 system. The result now follows by observing $Z = (-k, 0, 0, 0, 0, 1)$ satisfies the equation $p_{01} + kp_{23} = 0$ and $\{X, Y, Z\}$ are linearly independent.

We observe that the parameter k given in Theorem 4.3 is critical to the description of the plane π of $\mathrm{PG}(5, q)$ associated with one of the ruling classes of the hyperbolic quadric. Thus we call k^2 the *norm ratio* of the quadric $V[a, b, c, d, e, f]$, where $k^2(b^2 - 4ac) = (e^2 - 4df)$. In particular, the two possible choices for k correspond to the two ruling classes (a regulus and its opposite). One plane lies in the hyperplane $H_k : X_0 + kX_5 = 0$ of $\mathrm{PG}(5, q)$, while the other plane lies in the hyperplane $H_{-k} : X_0 - kX_5 = 0$, for a fixed choice of k.

In fact, a little bit more can be said at this point. The Plücker lift of the line ℓ_0 is the point $\hat{\ell}_0 = (1, 0, 0, 0, 0, 0)$, and the Plücker lift of ℓ_∞ is $\hat{\ell}_\infty = (0, 0, 0, 0, 0, 1)$. Neither of these points lies on H_k or H_{-k} above. If $\mathscr{R}_k$ is the regulus whose associated plane lies in H_k, we let $\mathscr{S}_k$ be the unique regular spread determined by $\mathscr{R}_k$ and ℓ_0. This regular spread necessarily contains ℓ_∞ since ℓ_0 and ℓ_∞ are conjugate skew lines with respect to the polarity associated with the hyperbolic quadric $V[a, b, c, d, ef]$. Similarly, the unique regular spread $\mathscr{S}_{-k}$ determined by $\mathscr{R}_{-k}$ and ℓ_0 necessarily contains ℓ_∞. Since $\hat{\ell}_0$ and $\hat{\ell}_\infty$ satisfy the last two equations in Theorem 4.3 (either form), these two equations (or the alternate pair) determine the solid Γ_k associated with the regular spread $\mathscr{S}_k$. Replacing k by $-k$ in this pair of equations determines the solid Γ_{-k} associated with $\mathscr{S}_{-k}$. Note that $\Gamma_k \cap H_k$ and $\Gamma_{-k} \cap H_{-k}$ are the two planes associated with the ruling classes (reguli) of the given hyperbolic quadric.

We now separate off a technical lemma concerning 2×2 matrices.

Lemma 4.4. *Consider the two symmetric nonzero matrices* $A = \begin{bmatrix} a & \frac{1}{2}b \\ \frac{1}{2}b & c \end{bmatrix}$ *and* $\bar{A} = \begin{bmatrix} \bar{a} & \frac{1}{2}\bar{b} \\ \frac{1}{2}\bar{b} & \bar{c} \end{bmatrix}$ *over* $\mathrm{GF}(q)$, *with* $b^2 - 4ac, \bar{b}^2 - 4\bar{a}\bar{c} \in \not\square_q$. *Then there is a unique* $t \in \mathrm{GF}(q)$ *such that* $A + t\bar{A}$ *is singular if and only if* A *and* $\bar{A}$ *are* $\mathrm{GF}(q)$-*scalar multiples of one another.*

Proof. If A and $\bar{A}$ are $\mathrm{GF}(q)$-scalar multiples of one another, a straightforward computation shows that $det(A + t\bar{A}) = 0$ for a unique value of $t \in \mathrm{GF}(q)$. Conversely, suppose that $det(A + \bar{A}) = 0$ for a unique $t \in \mathrm{GF}(q)$. Consider a projective plane $\pi = \mathrm{PG}(2, q)$ with homogeneous coordinates (x, y, z). Let $\mathscr{C}$ be the conic in π with equation $y^2 - 4xz = 0$. Recalling that q is odd, another straightforward computation shows that (x_0, y_0, z_0) is an interior (exterior) point of $\mathscr{C}$ precisely when $y_0^2 - 4x_0z_0$ is a nonsquare (nonzero square) in $\mathrm{GF}(q)$. Taking the entries of the matrices A and $\bar{A}$, we treat $P = (a, b, c)$ and $\bar{P} = (\bar{a}, \bar{b}, \bar{c})$ as points of π. The hypotheses imply P and $\bar{P}$ are interior points of $\mathscr{C}$.

If P and $\bar{P}$ were distinct points of π, then the line $\ell = \langle P, \bar{P} \rangle$ could not be a tangent line of $\mathscr{C}$. Note that any point of ℓ, other than $\bar{P}$, looks like $\langle P + t\bar{P} \rangle$ for some $t \in \mathrm{GF}(q)$ and thus has homogeneous coordinates $(a + t\bar{a}, b + t\bar{b}, c + t\bar{c})$. Such a point lies on $\mathscr{C}$ if and only if $(b + t\bar{b})^2 - 4(a + \bar{a})(c + \bar{c}) = 0$, which is true if and only if $det(A + t\bar{A}) = 0$. By assumption this occurs for precisely one value of $t \in \mathrm{GF}(q)$, contradicting the fact that ℓ cannot be tangent to $\mathscr{C}$. Hence it must be the case that $P = \bar{P}$, implying A and $\bar{A}$ are $\mathrm{GF}(q)$-scalar multiples of one another.

We close this section with a result which details possible intersection patterns for the planes associated with the reguli which rule hyperbolic quadrics with the same "back half". The notation used is that given prior to Lemma 4.4.

Theorem 4.5. *Consider two distinct hyperbolic quadrics $V[a, b, c, d, e, f]$ and $V[\bar{a}, \bar{b}, \bar{c}, d, e, f]$, pick one ruling family of lines for each quadric, and let π and $\bar{\pi}$ be the planes of* $\mathrm{PG}(5, q)$ *associated with these reguli $\mathscr{R}_k$ and $\mathscr{R}_{\bar{k}}$. There are three possibilities for the intersection of the distinct planes π and $\bar{\pi}$. If π and $\bar{\pi}$ meet in a line, then $\langle \pi, \bar{\pi} \rangle = \Gamma_k = \Gamma_{\bar{k}}$. That is, in this case there exists a regular spread containing both $\mathscr{R}_k$ and $\mathscr{R}_{\bar{k}}$. If π and $\bar{\pi}$ meet in a point, then $k = \bar{k}$ and $\langle \pi, \bar{\pi} \rangle = H_k = H_{\bar{k}}$. If π and $\bar{\pi}$ are disjoint, then $\langle \pi, \bar{\pi} \rangle = \mathrm{PG}(5, q)$.*

Proof. To determine the intersection of π and $\bar{\pi}$ it suffices to consider the 6×6 coefficient matrix for the defining equations of these planes in $\mathrm{PG}(5, q)$, for which we take

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & k \\ 0 & be + kD & 2bf & 2ce & -4cf & 0 \\ 0 & 2bd & be - kD & 4cd & -2ce & 0 \\ 1 & 0 & 0 & 0 & 0 & \bar{k} \\ 0 & 2\bar{a}e & 4\bar{a}f & \bar{b}e - \bar{k}\bar{D} & -2\bar{b}f & 0 \\ 0 & 4\bar{a}d & 2\bar{a}e & 2\bar{b}d & -(\bar{b}e + \bar{k}\bar{D}) & 0 \end{bmatrix}.$$

Note that we choose the first description from Theorem 4.3 for π and the alternate description for $\bar{\pi}$. The nature of $\pi \cap \bar{\pi}$ is determined by the rank of this matrix. We partition this matrix into block diagonal form by considering the first and third rows together with the first and last columns. These two rows are clearly independent if and only if $k \neq \bar{k}$, and thus contribute either 1 or 2 toward the rank of our matrix.

We use Laplace's formula to compute the determinant of the 4×4 block by taking a partition consisting of the first two rows and the last two rows of this block. Computing the six 2×2 determinants of the first two rows yields

$$[2ak, 2d, -(e + bk), e - bk, -2f, -2ck]$$

after the common factor $2ckD$ is factored out. Similarly, the six 2×2 determinants of the last two rows are

$$[2\bar{a}\bar{k}, 2d, -(e + \bar{b}\bar{k}), e - \bar{b}\bar{k}, -2f, -2\bar{c}\bar{k}]$$

with a common factor of $2\bar{a}\bar{k}\bar{D}$. Hence, using Laplace expansion, the determinant of the 4×4 block is $4\bar{a}c\bar{k}k\bar{D}D$ times the usual inner product of the "vectors" $(2ak, 2d, -(e+bk), e-bk, 2f, -2ck)$ and $(-2\bar{c}\bar{k}, 2f, e-\bar{b}\bar{k}, -(e+\bar{b}\bar{k}), 2d, 2\bar{a}\bar{k})$, which simplifies to

$$4\bar{a}c\bar{k}k\bar{D}D(2k\bar{k}(b\bar{b} - 2\bar{a}c - 2a\bar{c}) - 2(e^2 - 4df)).$$

Note that $\bar{a}, c, \bar{k}, k, \bar{D}$ and D are all nonzero by our assumptions on the coordinates of the hyperbolic quadrics.

Thus, if the above determinant is zero, then $k^2\bar{k}^2(b\bar{b} - 2\bar{a}c - 2a\bar{c})^2 = (e^2 - 4df)^2 = k^2 D\bar{k}^2\bar{D}$ and thus $(b\bar{b} - 2\bar{a}c - 2a\bar{c})^2 = (b^2 - 4ac)(\bar{b}^2 - 4\bar{a}\bar{c})$. This implies the discriminant of the polynomial $f(t) = \det(A + t\bar{A})$ is zero, where A and $\bar{A}$ are defined as in Lemma 4.4. Thus A and $\bar{A}$ are $\mathrm{GF}(q)$-scalar multiples of one another by Lemma 4.4, and thus $\bar{a} = sa, \bar{b} = sb$, and $\bar{c} = sc$ for some $s \in \mathrm{GF}(q)^*$. Note that $s \neq 1$ as $A \neq \bar{A}$ by assumption. Now $\bar{D} = \bar{b}^2 - 4\bar{a}\bar{c} = s^2(b^2 - 4ac) = s^2 D$. Since $\bar{k}^2\bar{D} = e^2 - 4df = k^2 D$, we have $(s\bar{k})^2 = k^2$. In fact, we have $0 = k\bar{k}(b\bar{b} - 2\bar{a}c - 2a\bar{c}) - (e^2 - 4df) = k\bar{k}s(b^2 - 4ac) - (e^2 - 4df) = k\bar{k}sD - k^2 D$ and hence $k = s\bar{k}$.

This further implies that $2\bar{a}\bar{k} = 2sa\left(\frac{k}{s}\right) = 2ak$, and then similar computations show that the 6-tuple of 2×2 determinants of the first two rows of the 4×4 block are a scalar multiple of the 6-tuple of 2×2 determinants of the last two rows of this block. That is, the rank of the 4×4 block is two whenever its determinant of zero. Moreover, in this case $k \neq \bar{k}$ as $s \neq 1$. Therefore the 2×2 block in the block diagonal form also has rank two. Geometrically, this is the case when $\pi \cap \bar{\pi}$ is a line and therefore $\langle \pi, \bar{\pi} \rangle$ is a solid. What we have shown is that in this case necessarily $\Gamma_k = \Gamma_{\bar{k}} = \langle \pi, \bar{\pi} \rangle$ since the first two equations of the 4×4 block represent Γ_k and the last two equations represent $\Gamma_{\bar{k}}$. That is, when $\pi \cap \bar{\pi}$ is a line, the regular spread represented by the intersection of $\mathscr{K}$ with the solid $\Gamma_k = \Gamma_{\bar{k}}$ contains both reguli $\mathscr{R}_k$ and $\mathscr{R}_{\bar{k}}$.

Suppose now that the determinant of the 4×4 block is nonzero. Then the rank of the 6×6 system is five when $k = \bar{k}$, and the rank is six when $k \neq \bar{k}$. Geometrically, this implies that if $\pi \cap \bar{\pi}$ is a point, then $k = \bar{k}$ and $H_k = H_{\bar{k}} = \langle \pi, \bar{\pi} \rangle$. Clearly, if $\pi \cap \bar{\pi}$ is empty, then $\langle \pi, \bar{\pi} \rangle = \mathrm{PG}(5, q)$.

5 Extra reguli

Any spread spawned from a hyperbolic fibration necessarily has $q - 1$ (mutually disjoint) reguli. Any additional regulus contained in such a spread is called an extra regulus as previously defined, and the purpose of this section is to discuss the existence of such reguli in the spreads spawned from the known hyperbolic fibrations of Section 2.

Consider a regular hyperbolic fibration of the type discussed above, say with constant back half. Let $Q_i = V[a_i, b_i, c_i, d, e, f]$ for $i = 1, 2, 3, \ldots, q-1$ be the hyperbolic quadrics of such a fibration, where $d^2 - 4ef$ is a nonsquare of $\mathrm{GF}(q)$. Let $\mathscr{R}$ be an extra regulus of a spread $\mathscr{S}$ spawned from this hyperbolic fibration, and let $\mathscr{R}_i$ be the regulus of $\mathcal{S}$ chosen as the ruling family of lines on Q_i, for $i = 1, 2, 3, \ldots, q-1$. Thus $\mathcal{S} = \mathcal{R}_1 \cup \mathcal{R}_2 \cup \cdots \cup \mathcal{R}_{q-1} \cup \{\ell_0, \ell_\infty\}$. Since $\mathcal{R} \neq \mathcal{R}_i$ for $i = 1, 2, 3, \ldots, q-1$, we know that $\mathcal{R}$ has at most two lines in common with each $\mathcal{R}_i$. Let π be the plane in $\mathrm{PG}(5, q)$ associated with $\mathcal{R}$, and let $\hat{\ell}_0$ and $\hat{\ell}_\infty$ be the Plücker lifts of the lines ℓ_0 and ℓ_∞, respectively.

Suppose first that ℓ_0 and ℓ_∞ are lines of $\mathcal{R}$, and thus $\hat{\ell}_0$ and $\hat{\ell}_\infty$ are points of π. Let ℓ be another line of $\mathcal{R}$, and let $\hat{\ell}$ be its Plücker lift. Necessarily, ℓ is a line of $\mathcal{R}_i$ for some i. Let π_i be the plane associated with the regulus $\mathcal{R}_i$. Thus $\Gamma_i = \langle \pi_i, \hat{\ell}_0 \rangle = \langle \pi_i, \hat{\ell}_\infty \rangle$ is the solid associated with the (unique) regular spread containing $\mathcal{R}_i$, ℓ_0, and ℓ_∞, as discussed after the proof of Theorem 4.3. Since $\mathcal{R}$ is determined by any three of its lines, we have $\pi = \langle \hat{\ell}_0, \hat{\ell}_\infty, \hat{\ell} \rangle$ and thus $\pi \subseteq \Gamma_i$ in this case.

Similarly, suppose $\mathcal{R}$ contains exactly one of $\{\ell_0, \ell_\infty\}$, say ℓ_0. Since $\mathcal{R}$ has $q+1$ lines, necessarily $\mathcal{R}$ contains two lines ℓ_1 and ℓ_2 from some $\mathcal{R}_i$. Hence $\pi = \langle \hat{\ell}_1, \hat{\ell}_2, \hat{\ell}_0 \rangle \subseteq \langle \pi_i, \hat{\ell}_0 \rangle = \Gamma_i$ once again.

Finally, suppose that $\mathcal{R}$ contains neither ℓ_0 nor ℓ_∞. In that case $\mathcal{R}$ must contain two lines from $\mathcal{R}_i$ and two lines from $\mathcal{R}_j$ for some $i \neq j$. Letting π_i and π_j be the planes associated with $\mathcal{R}_i$ and $\mathcal{R}_j$, respectively, we see that π meets each of π_i and π_j in a line, and these lines must intersect. Hence $\pi_i \cap \pi_j \neq \emptyset$, and either $\langle \pi_i, \pi_j \rangle = \Gamma_i = \Gamma_j$ or $\langle \pi_i, \pi_j \rangle = H_i = H_j$ from Theorem 4.5, where H_i and H_j are the hyperplanes discussed in that theorem. Therefore, either $\pi \subseteq \Gamma_i = \Gamma_j$ or $\pi \subseteq H_i = H_j$ in this case. In summary, after considering all possible cases for $\mathcal{R}$ and its associated plane π, either $\pi \subseteq \Gamma_i$ or $\pi \subseteq H_i$ for some i, $1 \leqslant i \leqslant q-1$.

These two possibilities are now considered in more detail. Suppose first that $\pi \subseteq \Gamma_i$ for some i, and thus the extra regulus $\mathcal{R}$ is contained in the regular spread $\mathcal{S}_i$ determined by $\mathcal{R}_i$ and ℓ_0 (or ℓ_∞). The lines of $\mathcal{S}_i$ are ruling lines of the quadrics in the H-pencil $\{V[ta_i, tb_i, tc_i, d, e, f] : t \in \mathrm{GF}(q)^*\} \cup \{\ell_0, \ell_\infty\}$. Since at least $\frac{1}{2}(q-1)$ $\mathcal{R}_j$'s share a line with $\mathcal{R}$ in this case, Theorem 1.2 implies that at least $\frac{1}{2}(q-1)$ Q_j's constitute a (linear) subset of $\{V[ta_i, tb_i, tc_i, d, e, f] : t \in \mathrm{GF}(q)^*\}$.

On the other hand, suppose $\pi \subseteq H_i$ for some i, where the equation of the hyperplane H_i is $X_0 + k_i X_5 = 0$. Recall that neither ℓ_0 nor ℓ_∞ is a line of $\mathcal{R}$ in this case. Thus every line ℓ of $\mathcal{R}$ uniquely determines k_i as $-X_0/X_5$, where $\hat{\ell} = [X_0, X_1, X_2, X_3, X_4, X_5]$. Since at least $\frac{1}{2}(q+1)$ $\mathcal{R}_j$'s share a line with $\mathcal{R}$, at least $\frac{1}{2}(q+1)$ Q_j's have the same norm ratio, namely k_i^2, as defined after the proof of Theorem 4.3.

We now discuss the hyperbolic fibrations of Section 2. Suppose $\mathcal{R}$ is an extra regulus in some spread $\mathcal{S}$ spawned from the hyperbolic fibration $\mathcal{B}$ constructed in [2]. As shown in [2], $\mathcal{B}$ has no linear subset of hyperbolic quadrics of size greater than two. Moreover, straightforward computations show that a given norm ratio can occur at most four times. Thus our previous discussion implies we must have $\frac{1}{2}(q-1) \leqslant 2$ or $\frac{1}{2}(q+1) \leqslant 4$. That is, we must have $q \leqslant 7$ for an extra regulus to exist in any spread spawned from $\mathcal{B}$. As pointed out in [2], such a phenomenon does occur when $q = 7$.

Next we turn to the hyperbolic fibrations arising from j-planes. For the hyperbolic fibration $\mathcal{J}_1$ more straightforward computations show that at most four hyperbolic quadrics have the same norm ratio, and no two hyperbolic quadrics form a linear subset. Thus the existence of an extra regulus in any spread spawned from $\mathcal{J}_1$ implies

$q \leqslant 7$ as above. Since $\mathscr{J}_1$ exists only for $q \equiv 0, 2 \pmod 3$, we must in fact have $q \leqslant 5$ for such an extra regulus to exist. Computations using MAGMA [5] show that indeed this does happen for $q = 5$.

For the hyperbolic fibration $\mathscr{J}_2$ the largest size of a linear subset of hyperbolic quadrics is two, while the greatest number of hyperbolic quadrics with the same norm ratio is six. Therefore computations similar to those above show that the existence of an extra regulus in any spread spawned from $\mathscr{J}_2$ implies $q \leqslant 11$. However, for $\mathscr{J}_2$ to exist, we must have $q \equiv 0, \pm 2 \pmod 5$, and thus necessarily $q \leqslant 7$ when such extra reguli occur.

Finally, consider the hyperbolic fibration $\mathscr{J}_0$. If $i = 0$ or $i = n$, then $\mathscr{J}_0$ is an H-pencil and there can be many extra reguli in the Andre spreads spawned from $\mathscr{J}_0$. However, for the other values of i, computations similar to those above show that there are no extra reguli in any spawned spread.

We collect these results in the form of a theorem.

Theorem 5.1. *For $q > 5$ any spread spawned from the hyperbolic fibration $\mathscr{J}_1$ has no extra reguli. For $q > 7$ any spread spawned from $\mathscr{B}$ or $\mathscr{J}_2$ has no extra reguli. When $i \neq 0$ and $i \neq n$, using our previous notation, any spread spawned from $\mathscr{J}_0$ has no extra reguli.*

As mentioned in [6], the spreads arising from $(q+1)$-nests admit a regular elliptic cover and also admit $\frac{1}{2}(q+1)$ extra reguli. These extra reguli are actually mutually disjoint and partition the replacement set for the $(q+1)$-nest. Thus these extra reguli cover the same line set as $\frac{1}{2}(q+1)$ reguli from the given regular elliptic cover. Another way of saying this is that the spread actually admits two different regular elliptic covers that share $\frac{1}{2}(q-3)$ reguli (in a partial linear set). We now show that the above extra reguli are the only ones possible in any spread spawned from the hyperbolic fibration $\mathscr{Q}$ of Section 2, at least for sufficiently large q.

Theorem 5.2. *If $q \geqslant 9$, there are precisely $q+1$ reguli among the $2q^2$ lines lying on the quadrics of $\mathscr{Q} = \mathscr{T}_0 \cup \mathscr{N}_1$, other than the $2(q-1)$ ruling classes of the hyperbolic quadrics of $\mathscr{Q}$. Moreover, these reguli are contained in the ruling lines of the hyperbolic quadrics of $\mathscr{N}_1$. In particular, any spread spawned from $\mathscr{Q}$ will have at most $\frac{1}{2}(q+1)$ extra reguli.*

Proof. As discussed after the proof of Theorem 3.7 and using the notation developed in that theorem, adjoining either one of the two W-orbits on the ruling lines of the quadrics from $\mathscr{N}_1$ to the lines in a linear set of reguli that rule the hyperbolic quadrics of $\mathscr{T}_0$ will generate together with $\{\ell_0, \ell_\infty\}$ a $(q+1)$-nest spread. Such spreads, as shown in [6], each contain $\frac{1}{2}(q+1)$ extra reguli, and these reguli have their lines contained in the appropriate W-orbit. Thus we get at least $q+1$ extra reguli among the lines lying on the quadrics of $\mathscr{Q}$. Each such regulus will contain precisely two (skew) ruling lines from each of the hyperbolic quadrics in $\mathscr{N}_1$. Computer computations using MAGMA [5] show the theorem is valid for $q = 9$, and so we assume $q \geqslant 11$ from now on.

Suppose $\mathscr{R}$ is any extra regulus contained in the ruling lines of the quadrics in the fibration $\mathscr{Q}$. Since $\mathscr{Q}$ has no linear subset of hyperbolic quadrics of size greater than $\frac{1}{2}(q-3)$, our previous discussion on extra reguli implies that we must have at least $\frac{1}{2}(q+1)$ hyperbolic quadrics in $\mathscr{Q}$ with the same norm ratio. Straightforward computations show that the $\frac{1}{2}(q+1)$ quadrics in $\mathscr{N}_1$ do indeed have the same norm ratio k_0^2, where $k_0^2[t_0^2(1-4\mu)-1]=1-4\mu$. Thus any extra regulus $\mathscr{R}$ must have its associated plane π contained in the hyperplane H_{k_0} of $\mathrm{PG}(5,q)$ whose equation is $X_0+k_0X_5=0$, where we have arbitrarily chosen k_0 to be one of the two square roots of k_0^2. It should be noted that it is conceivable that up to two hyperbolic quadrics in $\mathscr{T}_0$ also have norm ratio k_0^2. For each hyperbolic quadric of $\mathscr{Q}$ with this norm ratio, we now choose a ruling family of lines so that the plane associated with this regulus lies in H_{k_0}.

Let $\mathscr{R}_1,\mathscr{R}_2,\dots,\mathscr{R}_{(1/2)(q+1)}$ be the reguli chosen from the quadrics of $\mathscr{N}_1$ as above, and let $\pi_1,\pi_2,\dots,\pi_{(1/2)(q+1)}$ be the associated planes in $\mathrm{PG}(5,q)$. The proof of Theorem 4.5 implies that any two distinct planes from $\pi_1,\pi_2,\dots,\pi_{(1/2)(q+1)}$ meet precisely in the point $P=(-k_0,0,0,0,0,1)$. Recall that neither $\hat{\ell}_0$ nor $\hat{\ell}_\infty$ lies on the hyperplane H_{k_0}, and hence neither ℓ_0 nor ℓ_∞ lies in any extra regulus $\mathscr{R}$.

Let Γ be the solid of $\mathrm{PG}(5,q)$ defined by the equations $X_0=0=X_5$. Then $P\notin\Gamma$ and $\Gamma\subseteq H_{k_0}$. Thus $m_i=\pi_i\cap\Gamma$ is a line for all i. Since $\pi_i\cap\pi_j=\{P\}$ for $i\neq j$, the lines $m_1,m_2,\dots,m_{(1/2)(q+1)}$ in Γ are mutually skew. Letting $\mathscr{K}$ denote the Klein quadric as before, $\Gamma\cap\mathscr{K}$ represents those lines meeting ℓ_0 and ℓ_∞, and therefore each line m_i is a passant of the conic $\pi_i\cap\mathscr{K}$ representing the regulus $\mathscr{R}_i$. If ρ is the polarity associated with $\mathscr{K}$, then $P^\rho=\langle\hat{\ell}_0,\hat{\ell}_\infty\rangle$. Since $P\in\langle\hat{\ell}_0,\hat{\ell}_\infty\rangle$, we have $P^\rho\supseteq\Gamma$ and thus $P^{\sigma_i}=m_i$ for each i, where σ_i is the polarity associated with the conic $\pi_i\cap\mathscr{K}$. That is, P is an interior point of $\pi_i\cap\mathscr{K}$ for each i.

Next let $\overline{\mathscr{R}}_1,\overline{\mathscr{R}}_2,\dots,\overline{\mathscr{R}}_{(1/2)(q+1)}$ be the set of known extra reguli described in the first paragraph of the proof whose associated planes lie in H_{k_0} (the other known set of $\frac{1}{2}(q+1)$ extra reguli have associated planes that lie in H_{-k_0}). Any one of these reguli shares two lines with each of the $\mathscr{R}_i$'s. In particular, each plane $\bar{\pi}_j$ associated with $\overline{\mathscr{R}}_j$ meets each π_i in a line, and thus each $\bar{\pi}_j$ contains the point P. Let $\overline{m}_j=\Gamma\cap\bar{\pi}_j$, and note that the $\overline{m}_j$'s form another set of mutually skew lines in the solid Γ. Since each $\bar{\pi}_j$ meets each π_i in a secant line to $\mathscr{K}$ through P, each $\overline{m}_j$ meets each m_i (in the unique point of Γ on the above $\mathscr{K}$-secant line through P). That is, the m_i's and the $\overline{m}_j$'s are "opposite half-reguli".

We now define a point Q of any m_i to be "good" if the line PQ is secant to $\mathscr{K}\cap\pi_i$ (equivalently, secant to $\mathscr{K}$). Since we are assuming $q\geqslant 11$ and any extra regulus $\mathscr{R}$ has at most four lines lying on the quadrics in $\mathscr{T}_0$, we know that $\mathscr{R}$ must share two lines with at least two of the $\mathscr{R}_i$'s. This implies that the associated plane π also contains the point P. As P is an interior point of $\mathscr{K}\cap\pi_i$, this further implies that $\mathscr{R}$ shares either zero or two lines with each $\mathscr{R}_i$. Moreover, $\mathscr{R}$ shares two lines with (at least) three different $\mathscr{R}_i$'s. Thus the line $m=\pi\cap\Gamma$ must meet at least three of the m_i's, each in a "good" point. However, each m_i has exactly $\frac{1}{2}(q+1)$ "good" points, and the $\overline{m}_j$'s are the only lines of Γ meeting at least three m_i's in "good" points. Thus $m=\overline{m}_j$ for some j, and hence $\mathscr{R}=\overline{\mathscr{R}}_j$. That is, the only extra reguli are the ones described in the first paragraph of the proof.

Corollary 5.3. *If $q \geqslant 7$, any $(q+1)$-nest spread will have precisely $\frac{1}{2}(q+1)$ extra reguli.*

Proof. The result follows from the above theorem if $q \geqslant 9$. For $q = 7$ the result has been verified by computer using the software package MAGMA [5].

It should be noted that every extra regulus from Theorem 5.2 is some ruling family for a hyperbolic quadric of the form $V[1, 1, \mu, d, e, f]$, where $(d, e, f) = k_0^2 t_0(1, 1, \mu) + k_0^2 z_0(0, 1, \frac{1}{2}) + k_0^2 z_1(r, r, \frac{1}{2}r(1-2\mu))$ and $z_0 + z_1\varepsilon$ is a $(q+1)^{st}$ root of -1 in $GF(q^2)$. This can be verified by straightforward, but messy, computations and an application of Proposition 1.2.

For small values of q the hyperbolic fibration $\mathcal{Q}$ does indeed have extra reguli other than the ones described in Theorem 5.2. MAGMA [5] computations show that for $q = 5$ there is a fibration $\mathcal{Q}$ which has 24 extra reguli. One of the $(q+1)$-nest spreads spawned is an irregular nearfield spread, which has 12 extra reguli. This particular fibration spawns three mutually inequivalent spreads (among the 16 distinct spreads spawned). In addition to the irregular nearfield spread, there is another $(q+1)$-nest spread (whose extra reguli are precisely those specified in Corollary 5.3) and a spread which has no extra reguli. The latter spread corresponds to a j-plane with $j = 1$. For $q = 7$ a fibration $\mathcal{Q}$ was constructed which spawns 6 mutually inequivalent spreads (among the 64 possibilities). Only one of these is a $(q+1)$-nest spread, whose only extra reguli are those mentioned in Corollary 5.3. Again one of the spreads corresponds to a j-plane, this time for $j = 2$, and has no extra reguli. Nonetheless, the fibration $\mathcal{Q}$ itself has 24 extra reguli.

We close this section by returning to the question of estimating the number of mutually inequivalent spreads spawned from a general hyperbolic fibration $\mathcal{F}$. If no spread spawned from $\mathcal{F}$ has any extra reguli, then any projective equivalence between two spawned spreads would arise from an automorphism of $\mathcal{F}$. For "large" q the fibrations $\mathcal{B}$, $\mathcal{J}_0$, $\mathcal{J}_1$ and $\mathcal{J}_2$ are of this type. For these fibrations sorting out projective equivalences among the spawned spreads is the same as determining the number of orbits under the action of $\mathrm{Aut}(\mathcal{F})$ on this set of spawned spreads. Our work in Section 3 shows that any j-fibration admits a linear automorphism group of order $2(q^2-1)$. In practice, as long as the j-fibration is not an H-pencil, this group is either the full automorphism group of the fibration or an index two subgroup of the full automorphism group. When q is a large prime, the above group seems to be the full automorphism group. There are only two j-spreads spawned from a given j-fibration, and they are projectively equivalent. Most spreads spawned from any j-fibration, other than an H-pencil, have a full linear stabilizer of order $q+1$. Thus the number of mutually inequivalent spreads spawned from a j-fibration, other than an H-pencil, is approximately $2^{q-2}/(q-1)$, at least for large primes q. Similar computations show that the number of mutually inequivalent spreads spawned from the hyperbolic fibration $\mathcal{B}$, at least for large primes q, is approximately 2^{q-4}. So these hyperbolic fibrations are indeed robust producers of odd order two-dimensional translation planes, and seem to unify a number of previous construction techniques.

6 Concluding remarks

As mentioned after the proof of Theorem 3.2, any spread spawned from a regular fibration with constant back half will generate a translation plane admitting a cyclic (affine) homology group of order $q+1$ in its translation complement. Conversely, in [9] Jha and Johnson show that any translation plane admitting a cyclic homology group of order $q+1$ must arise from a spread admitting a regular elliptic cover and hence inducing a hyperbolic fibration. In fact, the work in [9] shows that the resulting hyperbolic fibration must be regular. It may well be the case that this regular fibration must have constant back half, although this is not completely evident.

However, even if true, this would show only that given the existence of a cyclic homology group of order $q+1$ in the translation complement of some translation plane, the associated spread must be spawned from *some* regular hyperbolic fibration with constant back half. We are most interested in proving that *any* fibration which spawns such a spread must be regular with constant back half. In other words, it is conceivable that such a spread could be spawned from two (or more) hyperbolic fibrations, and these "parent" fibrations might not be of the same type. Of course, this would imply, among other things, that the given spread has extra reguli.

Another interesting question is whether any spread of $PG(3,q)$ admitting a regular elliptic cover (and hence inducing a hyperbolic fibration) must correspond to a translation plane which admits a cyclic (affine) homology group of order $q+1$. This would be the full converse of the main result in [9], and we conjecture this to be true. As stated above, our comments after Theorem 3.2 indicate this conjecture is true when the induced hyperbolic fibration is regular with constant back half.

A major open question is whether all hyperbolic fibrations have their two lines forming a conjugate pair with respect to each of the quadrics in the fibration; that is, are all hyperbolic fibrations regular? While we feel this is probably not the case, it seems to be very difficult to construct such examples. Furthermore, all known examples of regular hyperbolic fibrations have the property that they may be represented by forms that agree on a two dimensional subspace of the underlying four dimensional vector space. More precisely, they may be represented so that they have the same form when restricted to the underlying space of one of their two lines. That is, in the language of this paper, they have "constant front or back half". Is this property necessary for *any* regular hyperbolic fibration? Here we believe the answer is yes, but have so far been unable to prove this.

As mentioned at the end of the previous section, we believe that hyperbolic fibrations constitute a robust and unifying way of thinking about many translation plane constructions. In particular, we conjecture that any *j*-plane, as defined in [10], must correspond to a spread spawned from one of the three hyperbolic fibrations described in Theorem 2.2.

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