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THE INTERIOR OF DISCRETE PROJECTIVE STRUCTURES IN THE BERS FIBER

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Abstract. The space of all projective structures on a closed surface is a holomorphic vector bundle over the Teichmüller space. In this paper, we restrict the space to the Bers fiber over any fixed underlying complex structure and prove that the interior of the set of discrete projective structures in the Bers fiber consists of those having quasifuchsian holonomy.

1. Statement of the main theorem

The purpose of this paper is to complete a proof of the following theorem due to Shiga and Tanigawa [25]. Notations and terminology are given in the next section.

Theorem. *Let $P(S)$ be the space of all projective structures on an oriented closed surface S of genus $g \geq 2$ and $D(S)$ a subset of those having discrete holonomy representations in* $\mathrm{PSL}(2,\mathbf{C})$. *Then the interior* $\mathrm{Int}(D(S) \cap B(t))$ *of $D(S)$ in each Bers fiber $B(t) \subset P(S)$ consists of projective structures having quasifuchsian holonomy.*

Recent developments in the theory of hyperbolic 3-manifolds, especially an affirmative solution to the Bers density conjecture due to Bromberg [3], enable us to deal with a certain problem that was not covered in the previous arguments.

2. Preliminaries on projective structures

Let S be an oriented closed surface of genus $g \geq 2$. A projective structure on S is a maximal system of local coordinates modeled on the Riemann sphere $\hat{\mathbf{C}} = \mathbf{C}P^1$ whose transition functions are Möbius transformations. A projective structure defines an underlying complex structure on S. For a projective structure φ on S and the universal cover $p\colon \tilde{S} \to S$, we have a developing map $f_\varphi\colon \tilde{S} \to \hat{\mathbf{C}}$, which is a local homeomorphism such that $f_\varphi \circ p^{-1}$ is compatible with the local coordinate system of the projective structure φ. If we provide a complex structure for $\tilde{S}$ from the underlying complex structure of φ, then the developing map $f_\varphi\colon \tilde{S} \to \hat{\mathbf{C}}$ is a holomorphic local homeomorphism.

We consider the deformation space of the projective structures on S. To this end, we assume hereafter that every φ is endowed with marking, namely φ represents

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a pair (ϕ, ω) where ϕ is a projective structure on S and ω is an isotopy class of orientation preserving homeomorphic automorphisms of S. Two marked projective structures $\varphi_1 = (\phi_1, \omega_1)$ and $\varphi_2 = (\phi_2, \omega_2)$ on S are defined to be equivalent if $\phi_1 = \phi_2$ and $\omega_1 = \omega_2$. We denote all the equivalence classes of projective structures on S by $P(S)$. Next consider the correspondence π of a projective structure $\varphi \in P(S)$ to the underlying complex structure $t = \pi(\varphi)$. Then t inherits the marking ω from φ and hence it is regarded as an element of the Teichmüller space $T(S)$, the set of all the equivalence classes of marked complex structures on S.

The Teichmüller space $T(S)$ is a $(3g-3)$-dimensional contractible complex manifold. Consult [9] for basic facts on Teichmüller spaces. Fix a complex structure $t \in T(S)$ and consider the projective structures over the Riemann surface S_t. Then they constitute a $(3g-3)$-dimensional complex vector space. This can be seen, for example, by taking the Schwarzian derivative of the developing map $f_\varphi \colon \tilde{S} \to \hat{\mathbf{C}}$ where S is given a fixed projective structure $\phi(t)$ over t, and by regarding it as an element of the vector space of the holomorphic quadratic differentials on S_t. Thus $P(S)$ is the fiber space $\bigsqcup_{t \in T(S)} \pi^{-1}(t)$ over $T(S)$ with the projection π. Moreover, giving a local trivialization to $P(S)$ canonically, we can regard $\pi \colon P(S) \to T(S)$ as a holomorphic vector bundle, which is isomorphic to the cotangent bundle over $T(S)$. For the zero section $\phi \colon T(S) \to P(S)$, the universal cover $\tilde{S}$ having the projective structure induced from $\phi(t)$ can be embedded in $\hat{\mathbf{C}}$ as a quasidisk $\Delta_{\phi(t)}$. The total space $P(S)$ is a $(6g-6)$-dimensional complex manifold. We call a fiber $B(t) = \pi^{-1}(t)$ over $t \in T(S)$ the *Bers fiber*.

Let V denote the complex manifold of all non-elementary $\mathrm{PSL}(2, \mathbf{C})$-representations of the surface group $\pi_1(S)$ modulo conjugacy: an element of V is represented by $[\theta]$, where $\theta \colon \pi_1(S) \to \mathrm{PSL}(2, \mathbf{C})$ is a homomorphism with a non-elementary image and $[\,\cdot\,]$ means the conjugacy class in $\mathrm{PSL}(2, \mathbf{C})$. The complex dimension of V is $6g-6$ (cf. [20, §4.3]). We define two subsets of V: DK is the set of all elements $[\theta] \in V$ such that $\theta(\pi_1(S))$ is discrete in $\mathrm{PSL}(2, \mathbf{C})$ and $QF \subset DK$ is the set of those having the faithful (injective) θ and the quasifuchsian image $\theta(\pi_1(S))$. Then DK is closed in V because the limit of non-elementary discrete representations is also discrete by Jørgensen [12]. The quasifuchsian space QF is a subdomain of $DK \subset V$, which is biholomorphically equivalent to the product of the Teichmüller spaces $T(S) \times T(S)$ (the Bers simultaneous uniformization).

For a projective structure $\varphi \in P(S)$, take a developing map $f \colon \tilde{S} \to \hat{\mathbf{C}}$. Then the composition $f \circ \gamma$, where $\gamma \in \pi_1(S)$ is identified with an element of the covering transformation group of $p \colon \tilde{S} \to S$, differs from f by the post-composition of a Möbius transformation, which we denote by $\theta_\varphi(\gamma)$. Then the homomorphism $\theta_\varphi \colon \pi_1(S) \to \mathrm{PSL}(2, \mathbf{C})$ is called a *holonomy representation* of $\pi_1(S)$. Moreover, it is known that the holonomy image $G_\varphi = \theta_\varphi(\pi_1(S))$ is non-elementary (cf. [13]). Since the choice of the developing map f has ambiguity, $\theta_\varphi(\gamma)$ depends on f rather than φ, however the conjugacy class is well-defined by φ, and thus we have an element $[\theta_\varphi]$ in V. Hence the correspondence $\mathrm{hol} \colon P(S) \to V$ is defined by $\varphi \mapsto [\theta_\varphi]$, which is called the *holonomy map*. The holonomy map $\mathrm{hol} \colon P(S) \to V$ is a holomorphic immersion (Hejhal, Earle, Gunning and Hubbard. cf. [20, §7.4]). Moreover, hol restricted to each Bers fiber $B(t)$ is injective by the Poincaré theorem due to Kra [15], and is proper by Kapovich [13] and Tanigawa [28]. Hence the image $\mathrm{hol}(B(t))$ is a regular submanifold of V.

We consider the inverse images of DK and QF by hol. Set $D(S) = \mathrm{hol}^{-1}(DK) \subset P(S)$ and call an element of $D(S)$ a *discrete* projective structure. Set $Q(S) = \mathrm{hol}^{-1}(QF) \subset P(S)$ and call an element of $Q(S)$ a *quasifuchsian* projective structure. Each connected component of $Q(S)$ is biholomorphically equivalent to the quasifuchsian space $QF \subset V$ by the holonomy map hol (cf. Ito [10]).

By the Goldman theorem [8], the set of the connected components $\{Q_\lambda\}$ of $Q(S)$ can be labeled by the indices λ in the set $ML_{\mathbf{Z}}(S)$ of the integral points of measured laminations: an element of $ML_{\mathbf{Z}}(S)$ is a free homotopy class of a disjoint union of non-trivial simple closed curves on S, where the null class $\emptyset$ is regarded as $0 \in ML_{\mathbf{Z}}(S)$. To see the correspondence of $\varphi \in Q(S)$ to the index $\lambda \in ML_{\mathbf{Z}}(S)$ for the component Q_λ where φ is, we take the projection of the inverse image of the limit set $\Lambda(G_\varphi)$ of the holonomy image under the developing map

$$\Lambda_\varphi := p \circ f_\varphi^{-1}(\Lambda(G_\varphi)) \subset S.$$

Then Λ_φ is the double of λ in the sense of free homotopy. The complement $S - \Lambda_\varphi$ is denoted by Ω_φ. The Goldman theorem actually asserts that every element $\varphi_\lambda \in Q_\lambda$ is made from some element $\varphi \in Q_0$ by the grafting construction with respect to λ. The correspondence $\varphi \mapsto \varphi_\lambda$ is uniquely determined by the requirement $\mathrm{hol}(\varphi) = \mathrm{hol}(\varphi_\lambda)$, and thus a biholomorphic map $\mathrm{Gr}_\lambda \colon Q_0 \to Q_\lambda$ is defined for each $\lambda \in ML_{\mathbf{Z}}(S)$.

3. Function theory on a fiber

We restrict the holonomy map hol to the Bers fiber $B(t)$ over a fixed complex structure $t \in T(S)$ and investigate the structure of the set $D(S)$ of discrete projective structures in it. In other words, we look at how the image $\mathrm{hol}(B(t))$ passes through the set $DK \subset V$.

The intersection $Q_0 \cap B(t)$ is called the *Bers embedding* of the Teichmüller space, which is biholomorphically equivalent to $T(S)$. In other words, hol maps $Q_0 \cap B(t)$ biholomorphically onto $\{t\} \times T(S) \subset QF$. Extending this observation, we consider the intersection of $B(t)$ with other components Q_λ of $Q(S)$ and the interior of the intersection of $B(t)$ with $D(S)$, where the interior is taken in $B(t)$ with respect to the relative topology. In this direction, there are several works by Shiga [23], [24] and Kra [16] among others. We note the following theorem by Shiga, which is based on the λ-lemma due to Mañé, Sad and Sullivan and a theorem due to Zuravlev.

Proposition 1. *The interior* $\mathrm{Int}(D(S) \cap B(t))$ *contains no projective structure whose developing map is injective except in the Bers embedding* $Q_0 \cap B(t)$.

Actually the Bers fiber $B(t)$ intersects the other components of $Q(S)$: this phenomenon was first discovered by Maskit [19] and later formulated as a problem by Hubbard. An answer to this problem was given independently and in different methods by Tanigawa [27], Gallo [6] and Markaryan [18] as follows.

Proposition 2. *The intersection $Q_\lambda \cap B(t)$ is a non-empty open set in $B(t)$ for every $\lambda \in ML_{\mathbf{Z}}(S)$ and every $t \in T(S)$.*

In this paper, we prove that the interior of $D(S) \cap B(t)$ consists of quasifuchsian projective structures:

$$\mathrm{Int}(D(S) \cap B(t)) = \bigsqcup_{\lambda \in ML_{\mathbf{Z}}(S)} (Q_\lambda \cap B(t)).$$

Shiga and Tanigawa [25] studied this problem and gave a proof for it. However, as we will see later, their arguments did not cover all the possibilities and there still remains certain difficulty to complete the proof. Hereafter in this section, we review basic techniques to solve this problem according to their work. To clarify the arguments, we add some arrangements to the proofs in their original paper.

Take a connected component U of $\mathrm{Int}(D(S) \cap B(t))$. Then, for every $\varphi \in U$, the holonomy representation $\theta_\varphi\colon \pi_1(S) \to \mathrm{PSL}(2, \mathbf{C})$ is faithful (injective) and the holonomy image G_φ has no parabolic elements. This fact is in Kra [14], which is based on *the open mapping property* of holomorphic functions.

Moreover, we consider a holomorphic family of isomorphisms $\{[\theta_\varphi]\}$ over $\varphi \in U$. Then, by a version of the λ-lemma (cf. [1], [4], [26]), the family $\{[\theta_\varphi]\}$ is induced by quasiconformal deformation. Namely, for an arbitrary $G = \theta_{\varphi_0}(\pi_1(S))$ with $\varphi_0 \in U$, the isomorphism $\theta_\varphi \circ \theta_{\varphi_0}^{-1}\colon G \to G_\varphi$ is a conjugation by a quasiconformal automorphism h of the Riemann sphere $\hat{\mathbf{C}}$.

Let $QH(G)$ be the quasiconformal deformation space of the Kleinian group G: an element of $QH(G)$ is a conjugacy class $[\rho]$ of an isomorphism $\rho\colon G \to \mathrm{PSL}(2, \mathbf{C})$ that is induced by a quasiconformal automorphism h of $\hat{\mathbf{C}}$. Here the complex dilatation μ_h of h vanishes almost everywhere on the limit set $\Lambda(G)$ by the Sullivan rigidity theorem (cf. [20, §5.2]). Using the identification $\theta_{\varphi_0}\colon \pi_1(S) \to G$, we can embed $QH(G)$ into V by $[\rho] \mapsto [\rho \circ \theta_{\varphi_0}]$. Moreover, we can parameterize $QH(G)$ via the Teichmüller space $T(\Omega(G)/G)$. That is, there exists a holomorphic covering $w\colon T(\Omega(G)/G) \to QH(G) (\subset V)$, which sends the Teichmüller class $[\mu]$ of a Beltrami coefficient μ for G on $\Omega(G)$ to the conjugacy class of the quasiconformal deformation induced by h_μ (Bers, Kra, Maskit. cf. [20, §5.3 & §7.4]). Here $\Omega(G)$ stands for the region of discontinuity of G and h_μ the quasiconformal automorphism of $\hat{\mathbf{C}}$ whose complex dilatation is μ.

A *singly degenerate group* is a finitely generated Kleinian group whose region of discontinuity is non-empty, connected and simply connected. Since $\mathrm{hol}\colon U \to QH(G)$ is injective and $w\colon T(\Omega(G)/G) \to QH(G)$ is surjective, $T(\Omega(G)/G)$ is not a singleton, and in particular $\Omega(G)$ is not empty. We have already seen that G is purely loxodromic and isomorphic to $\pi_1(S)$. Hence, by the Maskit classification theorem [19], we have the following.

Proposition 3. *Let U be a connected component of* $\mathrm{Int}(D(S) \cap B(t))$ *and G the holonomy image $\theta_{\varphi_0}(\pi_1(S))$ for any $\varphi_0 \in U$. Then G is either quasifuchsian or singly degenerate without parabolic elements.*

Next, we make a holomorphic map $W\colon T(\Omega(G)/G) \to P(S)$ satisfying $\mathrm{hol} \circ W = w$ as follows. For $[\mu] \in T(\Omega(G)/G)$, consider the pull-back $f_{\varphi_0}^* \mu$ of the Beltrami coefficient μ under the developing map $f_{\varphi_0}\colon \Delta_{\phi(t)} (\subset \hat{\mathbf{C}}) \to \hat{\mathbf{C}}$, and take a quasiconformal map $H_\mu\colon \hat{\mathbf{C}} \to \hat{\mathbf{C}}$ that has the complex dilatation $f_{\varphi_0}^* \mu$ on $\Delta_{\phi(t)}$ and 0 on $\hat{\mathbf{C}} - \Delta_{\phi(t)}$. Then

$$h_\mu \circ f_{\varphi_0} \circ H_\mu^{-1}\colon H_\mu(\Delta_{\phi(t)}) \to \hat{\mathbf{C}}$$

is a holomorphic local homeomorphism compatible with $\pi_1(S)$, and thus it defines a projective structure φ_μ on $H_\mu(\Delta_{\phi(t)})/\pi_1(S)$. The map W is defined by the correspondence $[\mu] \mapsto \varphi_\mu$. We can see that $\mathrm{hol}(\varphi_\mu)$ is the quasiconformal deformation of $\mathrm{hol}(\varphi_0)$ by h_μ and thus $\mathrm{hol} \circ W = w$. This in particular implies that $W(T(\Omega(G)/G))$ contains the connected component U of $\mathrm{Int}(D(S) \cap B(t))$.

Remark. To see that the definition of W is independent of the choice of the representative μ in $[\mu]$, we may use quasiconformal isotopy. Let μ_0 and μ_1 be Beltrami coefficients for G on $\Omega(G)$ such that $[\mu_0] = [\mu_1]$, and assume that $W([\mu_0]) \neq W([\mu_1])$. Due to Earle and McMullen [5] (see also [21]), there exists a continuous path μ_t $(0 \leq t \leq 1)$ in the space of the Beltrami coefficients for G on $\Omega(G)$ such that μ_t are in the same Teichmüller class for all t. Then $w([\mu_t]) \in QH(G) \subset V$ is constant independently of t. On the other hand, since $W([\mu_0]) \neq W([\mu_1])$, we can find two distinct points $W([\mu_t])$ and $W([\mu_{t'}])$ arbitrarily close to each other in $P(S)$. However, this contradicts the facts that $\mathrm{hol} \circ W = w$ and that hol is locally injective.

Finally, we consider the case where the holonomy image $G = \theta_{\varphi_0}(\pi_1(S))$ is singly degenerate for $\varphi_0 \in U$. Since $T(\Omega(G)/G)$ is isomorphic to $T(S)$ in this case, its dimension is the same as U. The composition $\pi \circ W$ is a constant map on $W^{-1}(U)$ which is an open set in $T(\Omega(G)/G)$. Then it is constant on the entire $T(\Omega(G)/G)$ by *the theorem of identity.* Thus the image of $T(\Omega(G)/G)$ by W is contained in $B(t)$ as an open set. On the other hand, it is also contained in $D(S)$, for $w(T(\Omega(G)/G)) \subset DK$. Hence $W(T(\Omega(G)/G))$ is contained in U. We conclude the following.

Proposition 4. *Let U be a connected component of* $\mathrm{Int}(D(S) \cap B(t))$. *Assume that the holonomy image $G = \theta_{\varphi_0}(\pi_1(S))$ is singly degenerate for $\varphi_0 \in U$. Then the image of the holomorphic map $W\colon T(\Omega(G)/G) \to P(S)$ coincides with U.*

4. Elimination of singly degenerate projective structures

We say that $\varphi \in P(S)$ is a singly degenerate projective structure if the holonomy representation $\theta_\varphi\colon \pi_1(S) \to \mathrm{PSL}(2, \mathbf{C})$ is faithful and its holonomy image G_φ is a singly degenerate group. Moreover, there are two kinds of singly degenerate projective structures according to their degenerating part. By the Nielsen isomorphism theorem (cf. [17]), any algebraic isomorphism between surface groups is induced geometrically. We apply this to the isomorphism $\theta_\varphi\colon \pi_1(S) \to G_\varphi$ and obtain a homeomorphism $u\colon \tilde{S} \to \Omega(G_\varphi)$ between their universal covers inducing θ_φ. Then φ is a singly degenerate projective structure of *horizontal type* if u is orientation-preserving and of *vertical type* if u is orientation-reversing.

Shiga and Tanigawa [25] proved the following theorem. Actually they did not claim exactly as it stands below, and it seems that they did not consider singly degenerate projective structures of vertical type.

Theorem 1. *There is no singly degenerate projective structure of horizontal type in* $\mathrm{Int}(D(S) \cap B(t))$.

Proof. Let U be a connected component of $\mathrm{Int}(D(S) \cap B(t))$, and assume that $\varphi_0 \in U$ is a singly degenerate projective structure of horizontal type. Set $G = \theta_{\varphi_0}(\pi_1(S))$ and take an orientation-preserving homeomorphism $u\colon \tilde{S} \to \Omega(G)$ inducing θ_{φ_0}. Giving the complex structure $t \in T(S)$, we may assume that u is a quasiconformal map of the universal cover $\tilde{S}$. Then consider the complex dilatation $\mu = \mu_{u^{-1}}$ of u^{-1} on $\Omega(G)$, which is a Beltrami coefficient for G, and take the Teichmüller class $[\mu] \in T(\Omega(G)/G)$. Since $U = W(T(\Omega(G)/G))$ by Proposition 4, we have $\varphi \in U$ such that $\varphi = W([\mu])$ and hence $\mathrm{hol}(\varphi) = w([\mu])$.

On the other hand, consider the composition $h_\mu \circ u\colon \tilde{S} \to h_\mu(\Omega(G))$ which is a conformal homeomorphism. There exists a projective structure $\psi \in B(t)$ whose developing map is $h_\mu \circ u$. Then $\mathrm{hol}(\psi) = w([\mu])$ and thus $\mathrm{hol}(\psi) = \mathrm{hol}(\varphi)$. The Poincaré theorem (Kra [15]) asserts that hol is injective on each fiber $B(t)$, from which $\psi = \varphi$ follows. However, by Proposition 1, if a projective structure $\varphi \in \mathrm{Int}(D(S) \cap B(t))$ has an injective developing map, then it must be a quasifuchsian projective structure. This is a contradiction. □

In this paper, we eliminate the possibility of the existence of singly degenerate projective structures of vertical type in $\mathrm{Int}(D(S) \cap B(t))$. The existence of such projective structures in $P(S)$ is guaranteed by Gallo, Kapovich and Marden [7]. We rely on a partial solution to the Bers density conjecture due to Bromberg [3]. Actually, our arguments work not only to singly degenerate projective structures of vertical type but also to those of horizontal type. Hence we can prove Theorem 3 below simultaneously in both these cases without the reference to Theorem 1.

Bromberg has proved the Bers density conjecture for singly degenerate groups without parabolic elements as in the following theorem. He uses grafting at the infinite end of the hyperbolic 3-manifold for a singly degenerate group and makes a convergent sequence of quasifuchsian cone manifolds. His proof also shows the existence of singly degenerate projective structures of vertical type concretely.

Theorem 2. *Let* $\theta\colon \pi_1(S) \to \mathrm{PSL}(2, \mathbf{C})$ *be a faithful representation whose image is a singly degenerate group without parabolic elements. Then* $[\theta]$ *belongs to the closure of* QF. *Equivalently, a projective structure* $\varphi \in P(S)$ *satisfying* $\mathrm{hol}(\varphi) = [\theta]$ *belongs to the closure of* $Q(S)$.

Furthermore, we can show that this projective structure $\varphi \in P(S)$ is actually in the closure of some connected component of $Q(S)$. Although this fact looks slightly different from the consequence of the above Theorem 2, the following lemma, which is included in a recent work by Ito [11, Cor. 5.6], neatly fills this difference.

Lemma 1. *Let $\varphi \in P(S)$ be a projective structure such that $\mathrm{hol}(\varphi) = [\theta]$ is either on the vertical boundary $\partial T(S) \times T(S)$ or on the horizontal boundary $T(S) \times \partial T(S)$ of the quasifuchsian space $QF = T(S) \times T(S)$. (The Bers embedding $\{t\} \times T(S)$ lies vertically in QF.) Then φ is in the closure of a connected component Q_λ of $Q(S)$ for some $\lambda \in ML_{\mathbf{Z}}(S)$. When $[\theta]$ is on the vertical boundary of QF, the λ must be non-zero.*

Outline of Proof. We sketch the argument in [11] for readers' convenience. It is based on the following length-modulus estimate. In general, for an essential annulus A in a closed hyperbolic surface S of genus g and for a simple closed geodesic α that is freely homotopic to the core curve of A, the conformal modulus $M(A)$ of A and the hyperbolic length $\ell(\alpha)$ of α are related as

$$M(A)\ell(\alpha)^2 \leq 4\pi(g-1).$$

We only consider the case where $[\theta]$ lies on the vertical boundary $\partial T(S) \times T(S)$. The horizontal case is similar. We choose a sequence $\{[\theta_n]\}_{n=1}^{\infty}$ in $QF = T(S) \times T(S)$ having the same second coordinate $\tau \in T(S)$ for all n and converging to $[\theta]$ as $n \to \infty$. Remark that this condition in particular implies the standard convergence $[\theta_n] \to [\theta]$ in the sense of [11]. Take a local inverse η of the holonomy map hol at $[\theta]$ such that $\eta([\theta]) = \varphi$ and consider $\varphi_n = \eta([\theta_n])$ converging to φ. Each φ_n is contained in a connected component Q_{λ_n} of $Q(S)$ for some $\lambda_n \in ML_{\mathbf{Z}}(S) - \{0\}$. To prove the claim, we suppose to the contrary that there are infinitely many distinct such λ_n.

For the sake of simplicity, we explain the proof in the case where each λ_n is a simple closed curve of weight 1. An appropriate modification is possible for the general case. Let $\ell(\lambda_n)$ be the geodesic length of λ_n measured on S_τ. In these circumstances, a Riemann surface S_{t_n} of the complex structure $t_n = \pi(\varphi_n)$ contains an annular domain A_n with the modulus $M(A_n) = 2\pi^2/\ell(\lambda_n)$ whose core curve is freely homotopic to λ_n. Since $\ell(\lambda_n) \to \infty$, we have $M(A_n) \to 0$. Then the above inequality implies $M(A_n)\ell_{t_n}(\lambda_n) \to 0$, where the geodesic length is measured on S_{t_n}. Thus we have $\ell_{t_n}(\lambda_n)/\ell(\lambda_n) \to 0$, from which we see that $t_n = \pi(\varphi_n)$ diverge and so do φ_n. However this contradicts that φ_n converge to φ. □

We also utilize the following claim which characterizes the projective structure φ in the closure of Q_λ. A similar argument can be found in Ito [10].

Lemma 2. *Let $\varphi \in P(S)$ be a singly degenerate projective structure such that φ is in the closure of some connected component Q_λ of $Q(S)$ for $\lambda \in ML_{\mathbf{Z}}(S) - \{0\}$. Then $\Omega_\varphi := p \circ f_\varphi^{-1}(\Omega(G_\varphi)) \subset S$ contains annular domains whose core curves are freely homotopic to λ.*

Proof. We first prove the statement for the case where φ is of vertical type. Let $\{\varphi_n\}_{n=1}^{\infty}$ be a sequence of quasifuchsian projective structures in Q_λ that converges to

φ. We may choose the sequence so that all $\mathrm{hol}(\varphi_n)$ have the same second coordinate $\tau \in T(S)$ in $QF = T(S) \times T(S)$. Since G_φ is a singly degenerate group, the limit sets $\Lambda(G_{\varphi_n})$ of the quasifuchsian groups G_{φ_n} converge to $\Lambda(G_\varphi)$ in the Hausdorff topology (cf. [20, Prop. 7.40]). Since the underlying complex structures $t_n = \pi(\varphi_n)$ converge to $t = \pi(\varphi)$ in $T(S)$ and since the developing maps $f_{\varphi_n} \colon \Delta_{\phi(t_n)} \to \hat{\mathbf{C}}$ converge to $f_\varphi \colon \Delta_{\phi(t)} \to \hat{\mathbf{C}}$ locally uniformly under certain normalization, we see that $\Lambda_{\varphi_n} = p \circ f_{\varphi_n}^{-1}(\Lambda(G_{\varphi_n}))$ converge to $\Lambda_\varphi = p \circ f_\varphi^{-1}(\Lambda(G_\varphi))$ on S in the Hausdorff topology.

Each annular component of $\Omega_{\varphi_n} = S - \Lambda_{\varphi_n}$ has the same conformal modulus for all n because they are the annular covers of the same Riemann surface S_τ with respect to the same simple closed curve in λ. This condition forces the distances between two boundary components of the annuli to be uniformly bounded away from zero ([10, Lemma 4.4]). Then, by the fact that Λ_{φ_n} is the double of λ for all n, we see that $\Omega_\varphi = S - \Lambda_\varphi$ consists of the annular domains whose core curves are freely homotopic to λ.

Similar arguments work for the case where φ is of horizontal type. In this case, the annular components in Ω_{φ_n} degenerate totally but the complement Ω'_{φ_n} of the annuli in Ω_{φ_n} remains as $n \to \infty$ because Ω'_{φ_n} covers the same Riemann surface S_τ for all n. Then Ω_φ is conformally equivalent to Ω'_{φ_n} and hence contains annular subdomains whose core curves are freely homotopic to λ. $\square$

Based on the above Theorem 2 and Lemmas 1 and 2, we obtain the desired assertion as in the following Theorem 3. Then by Proposition 3, this concludes that $\mathrm{Int}(D(S) \cap B(t))$ consists of quasifuchsian projective structures, which is our main theorem stated in Section 1.

Theorem 3. *No singly degenerate projective structure exists in* $\mathrm{Int}(D(S) \cap B(t))$.

Proof. Let U be a connected component of $\mathrm{Int}(D(S) \cap B(t))$ such that $\varphi_0 \in U$ is a singly degenerate projective structure. Since the holonomy representation θ_{φ_0} is faithful and the holonomy image $G = \theta_{\varphi_0}(\pi_1(S))$ has no parabolic elements, Theorem 2 and Lemma 1 conclude that φ_0 is in the closure of some connected component Q_λ of $Q(S)$ for some $\lambda \in ML_{\mathbf{Z}}(S)$. Here λ should be non-zero because the Bers embedding $Q_0 \cap B(t)$ coincides with U when $\lambda = 0$, which cannot contain such φ_0. Then by Lemma 2, $\Omega_{\varphi_0} = p \circ f_{\varphi_0}^{-1}(\Omega(G))$ contains annular domains whose core curves are freely homotopic to $\lambda \in ML_{\mathbf{Z}}(S) - \{0\}$.

Assume that φ_0 is of vertical type. Consider one of the annular domains in $\Omega_{\varphi_0} \subset S_t$, say A. Let $\gamma \in \pi_1(S)$ correspond to the core curve of A, and set $g = \theta_{\varphi_0}(\gamma) \in G$. Then A is conformally equivalent to the annular cover $\Omega(G)/\langle g \rangle$ of the Riemann surface $\Omega(G)/G$. Take a sequence $\{[\mu_n]\}_{n=1}^\infty$ in $T(\Omega(G)/G)$ such that the conformal moduli of $\Omega(\rho_n(G))/\langle \rho_n(g) \rangle$ tend to ∞ for $[\rho_n] := w([\mu_n]) \in QH(G)$. Since $U = W(T(\Omega(G)/G))$ by Proposition 4, $\varphi_n := W([\mu_n])$ belongs to U. Since $\mathrm{hol} \circ W = w$, the domain $\Omega_{\varphi_n} \subset S_t$ contains an annular domain A_n that is conformally equivalent to $\Omega(\rho_n(G))/\langle \rho_n(g) \rangle$. However, since the underlying complex structure t is fixed, it is impossible for S_t to contain the annulus of arbitrarily large modulus.

We can similarly deal with the case where φ_0 is of horizontal type. In this case, Ω_{φ_0} is a non-universal covering surface of $\Omega(G)/G$. Then we can take an appropriate sequence $\{[\mu_n]\}_{n=1}^{\infty}$ in $T(\Omega(G)/G)$ as before so that the corresponding Ω_{φ_n} cannot be contained in S_t. $\square$

5. A remark on projective structures on punctured surfaces

The statements in this paper are restricted to the results concerning projective structures on closed surfaces. However, we can extend all of them to bounded projective structures on finite-area hyperbolic surfaces S, which may have finitely many punctures. Here a projective structure on S is called *bounded* if the corresponding quadratic differential has at most a simple pole at each puncture of S. Actually, standard methods exist in complex-analytic arguments for this generalization and some of the results we refer to in this paper are stated in this way. In this situation, every $\mathrm{PSL}(2,\mathbf{C})$-representation θ of $\pi_1(S)$ should be restricted so that θ sends every cuspidal element of $\pi_1(S)$ to a parabolic element of $\mathrm{PSL}(2,\mathbf{C})$.

A problem for this generalization can exist only in proving Theorem 2, which involves deep geometric analysis of hyperbolic 3-manifolds. In fact, a complete solution of the Bers density conjecture even in the case where a cusp exists settles this problem. Recently, Minsky [22] collaborating with Brock and Canary [2] has completed a proof of the ending lamination conjecture, from which the density conjecture follows. Moreover, several proofs of the density conjecture not relying on the ending lamination conjecture have been also announced. We can see that Theorem 2 is valid even in the setting for finite-area hyperbolic surfaces, and thus obtain the following extension of our main theorem.

Theorem b. *Let $P(S)$ be the space of all bounded projective structures on a finite-area hyperbolic surface S and $D(S)$ a subset of those having discrete holonomy representations in* $\mathrm{PSL}(2,\mathbf{C})$. *Then the interior* $\mathrm{Int}(D(S)\cap B(t))$ *of $D(S)$ in each Bers fiber $B(t)\subset P(S)$ consists of projective structures having quasifuchsian holonomy.*

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