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PARTIAL REGULARITY FOR HIGHER ORDER VARIATIONAL PROBLEMS UNDER ANISOTROPIC GROWTH CONDITIONS

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Abstract. We prove a partial regularity result for local minimizers $u\colon \mathbf{R}^n \supset \Omega \to \mathbf{R}^M$ of the variational integral $J(u,\Omega) = \int_\Omega f(\nabla^k u)\,dx$, where k is any integer and f is a strictly convex integrand of anisotropic (p,q)-growth with exponents satisfying the condition $q < p(1+\frac{2}{n})$. This is some extension for the case $n \geq 3$ of the regularity theorem obtained in [BF2].

1. Introduction

In this note we study the regularity properties of local minimizers $u\colon \Omega \to \mathbf{R}^M$ of higher order variational integrals of the form

$$J(w,\Omega) = \int_\Omega f(\nabla^k w)\,dx,$$

where Ω is a domain in $\mathbf{R}^n, n \geq 2$, and $k \geq 2$ denotes a given integer. The symbol $\nabla^k w$ stands for the tensor of all k^{th} order (weak) partial derivatives of the function w, i.e. $\nabla^k w = (D^\alpha w^i)_{|\alpha|=k, 1\leq i\leq M, \alpha\in\mathbf{N}_0^n}$. Our main assumption concerns the energy density f : we consider $f \geq 0$ of class C^2 satisfying with given exponents $1 < p \leq q < \infty$ and with positive constants λ, Λ the anisotropic ellipticity condition

$$\lambda(1+|\sigma|^2)^{\frac{p-2}{2}}|\tau|^2 \leq D^2 f(\sigma)(\tau,\tau) \leq \Lambda(1+|\sigma|^2)^{\frac{q-2}{2}}|\tau|^2 \tag{1.1}$$

being valid for all tensors σ and τ. Note that the left-hand side of (1.1) implies the strict convexity of f, moreover, it is easy to see that

$$a|\tau|^p - b \leq f(\tau) \leq A|\tau|^q + B \tag{1.2}$$

is true with constants $a,\ A > 0,\ b,\ B \geq 0$.

According to (1.2) the appropriate space for local minimizers is the energy class consisting of all Sobolev functions $u \in W^k_{p,\mathrm{loc}}(\Omega;\mathbf{R}^M)$ such that $J(u,\Omega') < \infty$ for

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any subdomain $\Omega' \subset\subset \Omega$, and we say that a function u with these properties is a local J-minimizer if and only if

$$J(u, \Omega') \leq J(v, \Omega')$$

for any $v \in W^k_{p,\text{loc}}(\Omega; \mathbf{R}^M)$ such that $\text{spt}(u - v) \subset\subset \Omega'$, where as above Ω' is an arbitrary subdomain of Ω with compact closure in Ω. For a definition of the Sobolev classes W^k_p, $W^k_{p,\text{loc}}$, etc., we refer the reader to the book of Adams [Ad]. Now we can state our main result:

Theorem 1.1. *Let u denote a local J-minimizer where f satisfies* (1.1). *Suppose further that $p \geq 2$ together with*

$$q < p(1 + \frac{2}{n}). \tag{1.3}$$

Then there is an open subset Ω_0 of Ω such that $\Omega - \Omega_0$ is of Lebesgue measure zero and $u \in C^{k,\nu}_{\text{loc}}(\Omega_0; \mathbf{R}^M)$ for any exponent $0 < \nu < 1$.

Remark 1.1. i) In the two-dimensional case, i.e. $n = 2$, the partial regularity result of Theorem 1.1 can be improved to everywhere regularity which means that actually we have $\Omega_0 = \Omega$. This is outlined in the recent paper [BF2].

ii) The anisotropic first order case, i.e. we have $k = 1$ and f satisfies conditions similar to (1.1), is well investigated: without being complete we mention the papers of Acerbi and Fusco [AF], of Esposito, Leonetti and Mingione [ELM1,2,3] and the results obtained by the second author in collaboration with Bildhauer, see e.g. [BF1]. Further references are contained in the monograph [Bi]. Clearly the above-mentioned results concern the case of vectorvalued functions. The anisotropic scalar situation for first order problems has been discussed before mainly by Marcellini, compare e.g. [Ma1,2,3], with the major result that conditions of the form (1.3) are in fact sufficient for excluding the occurrence of singular points.

iii) If $n \geq 3$ together with $k \geq 2$, then partial $C^{k,\nu}$-regularity of minimizers of the variational integral $\int_\Omega f(\nabla^k u)\, dx$ has been studied in the paper [Kr1] of Kronz. Here the main feature however is the quasiconvexity assumption imposed on f, i.e. the right-hand side of (1.1) is required to hold with $q = p$ and the first inequality in (1.1) is replaced by the hypothesis of uniform strict quasiconvexity with exponent $p \geq 2$. A related result concerning quasimonotone nonlinear systems of higher order with p-growth ($p \geq 2$) is established in [Kr2]. Of course the theorems of Kronz imply our regularity result if we consider (1.1) in the isotropic case $p = q$ together with $p \geq 2$.

For completeness we also like to mention the work of Duzaar, Gastel and Grotowski [DGG] dealing with partial regularity of certain higher order nonlinear elliptic systems and improving earlier results of Giaquinta and Modica established in [GiaMo2].

iv) If the non-autonomous case $I(w, \Omega) := \int_\Omega F(x, \nabla^k w)\, dx$ is considered with integrand $F(x, \sigma)$ satisfying (1.1) uniformly w.r.t. σ, and if in addition we require

$$|D_x D_\sigma F(x, \sigma)| \leq c_1(1 + |\sigma|^2)^{\frac{q-1}{2}}$$

then Theorem 1.1 remains valid, provided (1.3) is replaced by the stronger condition $q < p\,(1 + 1/n)$ and if for example we assume that $F(x, \sigma)$ is given by $F(x, \sigma) = g(x, |\sigma|)$ for a suitable function g. The details are left to the reader, we refer to [ELM3] and [BF3].

Remark 1.2. In Theorem 1.1 we have restricted ourselves to the case $p \geq 2$. In the subquadratic case an application of the techniques used by Carozza, Fusco and Mingione in [CFM] will imply a comparable partial regularity result.

The proof of Theorem 1.1 is organized in two steps. First we introduce a suitable regularization of our variational problem following the lines of [BF2] which leads us to uniform higher integrability and higher weak differentiability results for the solutions of the approximate problems which then extend to our local minimizer. Note that these results are valid for all $p \in (1, \infty)$. In a second step we combine this initial regularity with a blow–up procedure which will give partial regularity as stated in Theorem 1.1. From now on and just for notational simplicity we will assume that $k = 2$ together with $M = 1$. Moreover, we let $n \geq 3$ for obvious reasons. If necessary, we pass to subsequences without explicit indications, and we use the same symbol to denote various constants with different numerical values.

2. Approximation and initial regularity

Let the assumptions of Theorem 1.1 hold but with arbitrary exponent $p \in (1, \infty)$. Consider a local J-minimizer u. We proceed as in [BF2] by fixing two open domains $\Omega_1 \subset\subset \Omega_2 \subset\subset \Omega$. Then we consider the mollification $\overline{u}_m$ of u with radius $1/m, m \in \mathbf{N}$, and let $u_m \in \overline{u}_m + \overset{\circ}{W}{}^2_q(\Omega_2)$ denote the unique solution of the problem

$$J_m(w, \Omega_2) := J(w, \Omega_2) + \rho_m \int_{\Omega_2} (1 + |\nabla^2 w|^2)^{q/2}\, dx \to \min \text{ in } \overline{u}_m + \overset{\circ}{W}{}^2_q(\Omega_2),$$

where we have set

$$\rho_m := \|\overline{u}_m - u\|_{W^2_p(\Omega_2)} \Big[\int_{\Omega_2} (1 + |\nabla^2 \overline{u}_m|^2)^{q/2}\, dx \Big]^{-1}.$$

It is easy to see that (compare [BF2])

$$u_m \rightharpoondown u \text{ in } W^2_p(\Omega_2),\ J(u_m, \Omega_2) \to J(u, \Omega_2),\ J_m(u_m, \Omega_2) \to J(u, \Omega_2)$$

as $m \to \infty$. Next we use the Euler equation

$$\int_\Omega Df_m(\nabla^2 u_m) : \nabla^2 \varphi\, dx = 0,\ \varphi \in \overset{\circ}{W}{}^2_q(\Omega_2), \tag{2.1}$$

$f_m := \rho_m(1 + |\cdot|^2)^{q/2} + f$, with the choice $\varphi := \partial_i(\eta^6 \partial_i u_m)$, $i = 1, \dots, n$, $\eta \in C^\infty_0(\Omega_2)$, $0 \leq \eta \leq 1$, $\eta = 1$ on Ω_1, and get (from now on summation w.r.t. i) with the help of the Cauchy–Schwarz inequality for the bilinear form $D^2 f_m(\nabla^2 u_m)$

$$
\begin{aligned}
&\int_{\Omega_2} \eta^6 D^2 f_m(\nabla^2 u_m)(\partial_i \nabla^2 u_m, \partial_i \nabla^2 u_m)\, dx \\
&\le c\Big\{(\|\nabla^2\eta\|_\infty^2 + \|\nabla\eta\|_\infty^4)\int_{\operatorname{spt}\nabla\eta} |D^2 f_m(\nabla^2 u_m)|\, |\nabla u_m|^2\, dx \\
&\qquad + \|\nabla\eta\|_\infty^2 \int_{\operatorname{spt}\nabla\eta} |D^2 f_m(\nabla^2 u_m)|\, |\nabla^2 u_m|^2\, dx\Big\}
\end{aligned}
\tag{2.2}
$$

where c denotes a finite constant independent of m. Of course this calculation has to be justified with the help of the difference quotient technique using $\varphi = \Delta_{-h}(\eta^6 \Delta_h u_m)$ in (2.1), $\Delta_h u_m(x) := \frac{1}{h}[u_m(x+he_i) - u_m(x)]$. In case that $q \ge 2$, the reader can follow the steps in [BF2] leading from (2.6) to (2.13) where (2.12) has to be adjusted for dimensions $n \ge 3$. If $q < 2$, then we refer to [BF1] or [Bi], pp. 55–57.

Inequality (2.2) implies local uniform higher integrability of the sequence $\{\nabla^2 u_m\}$: let $\chi := \frac{n}{n-2}$ and $s := \frac{p}{2}\chi$. For concentric balls $B_r \subset\subset B_R \subset\subset \Omega_2$ and $\eta \in C_0^\infty(B_R)$, $0 \le \eta \le 1$, $\eta = 1$ on B_r, $|\nabla^\ell \eta| \le c/(R-r)^\ell$, $\ell = 1, 2$, we have by Sobolev's inequality

$$
\begin{aligned}
\int_{B_r} (1+|\nabla^2 u_m|^2)^s\, dx \;&\le\; \int_{B_R} \Big(\eta^3 [1+|\nabla^2 u_m|^2]^{s\frac{n-2}{2n}}\Big)^{2\chi} dx \\
&= \int_{B_R} (\eta^3 h_m)^{2\chi}\, dx \\
&\le\; c\Big(\int_{B_R} |\nabla(\eta^3 h_m)|^2\, dx\Big)^{\frac{n}{n-2}}.
\end{aligned}
$$

Here $h_m := (1+|\nabla^2 u_m|^2)^{p/4}$ is known to be of class $W^1_{2,\mathrm{loc}}(\Omega_2)$ on account of (2.2), and with Young's inequality we deduce

$$
\begin{aligned}
\int_{B_r} (1+|\nabla^2 u_m|^2)^s\, dx &\le c\Big[\int_{B_R} \eta^6 |\nabla h_m|^2\, dx + \int_{B_R} |\nabla \eta^3|^2 h_m^2\, dx\Big]^\chi \\
&=: c[T_1 + T_2]^\chi.
\end{aligned}
\tag{2.3}
$$

From (1.1) and (2.2) we get ($T_{R,r} := B_R - B_r$)

$$
\begin{aligned}
T_1 &\le c(r,R) \int_{T_{R,r}} (1+|\nabla^2 u_m|^2)^{\frac{q-2}{2}} \Big[|\nabla^2 u_m|^2 + |\nabla u_m|^2\Big]\, dx \\
&\le c(r,R)\Big[\int_{T_{R,r}} (1+|\nabla^2 u_m|^2)^{\frac{q}{2}}\, dx + \int_{T_{R,r}} |\nabla u_m|^q\, dx\Big],
\end{aligned}
$$

moreover

$$
T_2 \le c(r,R) \int_{T_{R,r}} (1+|\nabla^2 u_m|^2)^{p/2}\, dx.
$$

Inserting these estimates into (2.3) we find that

$$(2.4)\quad \int_{B_r} (1+|\nabla^2 u_m|^2)^s\,dx \le c(r,R)\Big[\int_{T_{R,r}} (1+|\nabla^2 u_m|^2)^{q/2}\,dx + \int_{T_{R,r}} |\nabla u_m|^q\,dx\Big]^\chi$$

for a constant $c(r,R) = c(R-r)^{-\beta}$ with suitable exponent $\beta > 0$. Fix $\Theta \in (0,1)$ such that

$$\frac{1}{q} = \frac{\Theta}{p} + \frac{1-\Theta}{2s}$$

(note: $2s = p\chi > q$ on account of $q < p(1+\frac{2}{n})$). Then the interpolation inequality implies

$$\|\nabla^2 u_m\|_q \le \|\nabla^2 u_m\|_p^\Theta\,\|\nabla^2 u_m\|_{2s}^{1-\Theta}$$

where the norms are taken over $T_{R,r}$, and we get:

$$(2.5)\qquad \int_{T_{R,r}} |\nabla^2 u_m|^q\,dx \le \Big(\int_{B_R} |\nabla^2 u_m|^p\,dx\Big)^{\Theta q/p}\Big(\int_{T_{R,r}} |\nabla^2 u_m|^{2s}\,dx\Big)^{(1-\Theta)\frac{q}{2s}}.$$

Before applying (2.5) to the first integral on the r.h.s. of (2.4) we discuss the second one: we have (for any $0 < \varepsilon < 1$)

$$(2.6)\qquad \int_{T_{R,r}} |\nabla u_m|^q\,dx \le \varepsilon \int_{T_{R,r}} |\nabla^2 u_m|^q\,dx + c(\varepsilon,R,r)\int_{T_{R,r}} |u_m|^q\,dx,$$

which follows for example from [Mo], Theorem 3.6.9. For the ε-term on the r.h.s. of (2.6) we may use (2.5). By construction we know that $\sup_m \|u_m\|_{W^2_p(\Omega_2)} < \infty$. If $p \ge n$, then the sequence $\{u_m\}$ is uniformly bounded in any space $W^1_t(\Omega_2), t < \infty$, thus we clearly have the boundedness of $\int_{\Omega_2} |u_m|^q\,dx$. So let us assume that $p < n$. Then

$$\sup_m \|u_m\|_{W^1_t(\Omega_2)} < \infty$$

for $t \le \frac{np}{n-p} =: \overline{p}$. In case $\overline{p} \ge n$ we are done. If $\overline{p} < n$, then we obtain

$$\sup_m \|u_m\|_{L^t(\Omega_2)} < \infty$$

for $t \le \frac{n\overline{p}}{n-\overline{p}} = \frac{np}{n-2p}$. Obviously $q \le \frac{np}{n-2p}$ which is a consequence of (1.3) since $p(1+\frac{2}{n}) \le \frac{np}{n-2p}$. Altogether we have shown that

$$(2.7)\qquad \int_{T_{R,r}} |u_m|^q\,dx \le \overline{c}$$

for a constant $\overline{c}$ depending also on Ω_2 and $\sup_m \|u_m\|_{W^2_p(\Omega_2)}$. Returning to (2.4), inserting (2.6) combined with (2.7) and applying (2.5) we have shown that

$$(2.8)\qquad \begin{aligned} \int_{B_r} (1+|\nabla^2 u_m|^2)^s\,dx \le c(R-r)^{-\beta}\Big[&\Big(\int_{\Omega_2} (1+|\nabla^2 u_m|^2)^{\frac{p}{2}}\,dx\Big)^{\Theta q\chi/p} \\ &\cdot \Big(\int_{T_{R,r}} (1+|\nabla^2 u_m|^2)^s\,dx\Big)^{(1-\Theta)\frac{q\chi}{2s}} + \overline{c}\Big]. \end{aligned}$$

Now, from (1.3) it follows that $(1-\Theta)\frac{q\chi}{2s} < 1$, and we may therefore apply Young's inequality on the r.h.s. of (2.8) with the result

$$\begin{aligned}(2.9)\qquad &\int_{B_r} (1+|\nabla^2 u_m|^2)^s\,dx \\ &\leq \int_{T_{R,r}} (1+|\nabla^2 u_m|^2)^s\,dx + c(R-r)^{-\beta_1}\Big[\Big(\int_{\Omega_2} (1+|\nabla^2 u_m|^2)^{\frac{p}{2}}\,dx\Big)^{\beta_2} + \overline{c}\Big],\end{aligned}$$

β_1, β_2 denoting positive exponents. Adding $\int_{B_r} (1+|\nabla^2 u_m|^2)^s\,dx$ on both sides of (2.9) this inequality turns into

$$(2.10)\qquad \int_{B_r} (1+|\nabla^2 u_m|^2)^s\,dx \leq \frac{1}{2}\int_{B_R} (1+|\nabla^2 u_m|^2)^s\,dx + K(R-r)^{-\beta_1},$$

where the constant K on the r.h.s. of (2.9) also depends on $\sup_m \int_{\Omega_2} |\nabla^2 u_m|^p\,dx$. If we use [Gi], Lemma 5.1, p. 81, inequality (2.10) implies the following

Lemma 2.1. *Under the hypothesis of Theorem 1.1 and with the notation introduced before we have that $\{u_m\}$ is uniformly bounded in the space $W^2_{2s,\mathrm{loc}}(\Omega_2)$, $s := \frac{p}{2}\frac{n}{n-2}$. In particular we have that u belongs to $W^2_{q,\mathrm{loc}}(\Omega_2)$. Moreover, the functions $h_m = (1+|\nabla^2 u_m|^2)^{p/4}$ are uniformly bounded in $W^1_{2,\mathrm{loc}}(\Omega_2)$.*

Note that the last statement follows from (2.2) together with $\sup_m \|u_m\|_{W^2_{q,\mathrm{loc}}(\Omega_2)} < \infty$. We return to (2.1) and choose $\varphi = \partial_i(\eta^6\partial_i[u_m - P_m])$ where $\eta \in C_0^\infty(\Omega_2)$, $0 \leq \eta \leq 1$, and P_m denotes a polynomial function of degree ≤ 2. Similar to (2.2) we get (using difference quotients)

$$\begin{aligned}&\int_{\Omega_2} \eta^6 D^2 f_m(\nabla^2 u_m)(\partial_i\nabla^2 u_m, \partial_i\nabla^2 u_m)\,dx \\ &\leq -\int_{\operatorname{spt}\nabla\eta} D^2 f_m(\nabla^2 u_m)\Big(\partial_i\nabla^2 u_m, \nabla^2\eta^6\partial_i[u_m - P_m] \\ &\qquad + 2\nabla\eta^6 \odot \nabla\partial_i(u_m - P_m)\Big)\,dx,\end{aligned}$$

where the sum is taken w.r.t. $i = 1,\dots,n$. We apply the Cauchy–Schwarz inequality to the bilinear form $D^2 f_m(\nabla^2 u_m)$ with the result

$$\begin{aligned}&\int_{\Omega_2} \eta^6 D^2 f_m(\nabla^2 u_m)(\partial_i\nabla^2 u_m, \partial_i\nabla^2 u_m)\,dx \\ (2.11)\qquad &\leq c\Big\{\Big(\|\nabla^2\eta\|^2_\infty + \|\nabla\eta\|^4_\infty\Big)\int_{\operatorname{spt}\nabla\eta} |D^2 f_m(\nabla^2 u_m)|\,|\nabla(u_m - P_m)|^2\,dx \\ &\qquad + \|\nabla\eta\|^2_\infty \int_{\operatorname{spt}\nabla\eta} |D^2 f_m(\nabla^2 u_m)|\,|\nabla^2(u_m - P_m)|^2\,dx\Big\}\end{aligned}$$

in particular $\int_{\Omega_2} \eta^6|\nabla h_m|^2\,dx$ is bounded by the right–hand side of (2.11). We claim

Lemma 2.2. *Let $h := (1+|\nabla^2 u|^2)^{p/4}$ Then the following statements hold:*

i) $h \in W^1_{2,\mathrm{loc}}(\Omega_2)$;
ii) $h_m \rightharpoondown h$ *in* $W^1_{2,\mathrm{loc}}(\Omega_2)$;
iii) $\nabla^\ell u_m \longrightarrow \nabla^\ell u$ *a.e. on* Ω_2, $\ell \le 2$.

If P is a polynomial function of degree ≤ 2, then

$$\begin{aligned}
&\int_{\Omega_2} \eta^6 |\nabla h|^2\, dx \\
&\le c\Big\{ \Big(\|\nabla^2\eta\|^2_\infty + \|\nabla\eta\|^4_\infty \Big) \int_{\operatorname{spt}\nabla\eta} |D^2 f(\nabla^2 u)|\; |\nabla(u-P)|^2\, dx \\
&\quad + \|\nabla\eta\|^2_\infty \int_{\operatorname{spt}\nabla\eta} |D^2 f(\nabla^2 u)|\; |\nabla^2(u-P)|^2\, dx \Big\}
\end{aligned} \tag{2.12}$$

is true for any $\eta \in C^\infty_0(\Omega_2)$, $0 \le \eta \le 1$.

Proof. From Lemma 2.1 we deduce that there exists a function $\hat h \in W^1_{2,\mathrm{loc}}(\Omega_2)$ such that $h_m \rightharpoondown \hat h$ in $W^1_{2,\mathrm{loc}}(\Omega_2)$ and almost everywhere. Suppose that we already have iii). Then i), ii) are trivial. Moreover, if we choose $P_m \equiv P$ in (2.11), Fatou's lemma implies that

$$\int_{\Omega_2} \eta^6 |\nabla h|^2\, dx \le \liminf_{m\to\infty} \int_{\Omega_2} \eta^6 |\nabla h_m|^2\, dx,$$

and we may control the quantities $\int_{\Omega_2} \eta^6 |\nabla h_m|^2\, dx$ with the help of (2.11) in terms of the integrals $\int_{\operatorname{spt}\nabla\eta} |D^2 f_m(\nabla^2 u_m)|\; |\nabla^2 u_m - \nabla^2 P|^2\, dx =: \int_{\operatorname{spt}\nabla\eta} \Phi_m\, dx$ and $\int_{\operatorname{spt}\nabla\eta} |D^2 f_m(\nabla^2 u_m)|\; |\nabla u_m - \nabla P|^2\, dx =: \int_{\operatorname{spt}\nabla\eta} \Psi_m\, dx$. By Lemma 2.1 the integrand Φ_m is uniformly bounded in $L^{1+\varepsilon}(\operatorname{spt}\nabla\eta)$ for some $\varepsilon > 0$, thus $\Phi_m \rightharpoondown: \Phi$ in $L^{1+\varepsilon}(\operatorname{spt}\nabla\eta)$ and therefore $\int_{\operatorname{spt}\nabla\eta} \Phi_m\, dx \to \int_{\operatorname{spt}} \Phi\, dx$. But with the pointwise convergence iii) we see that $\Phi = |D^2 f(\nabla^2 u)|\; |\nabla^2 u - \nabla^2 P|$. Obviously a similar argument applies to $\int_{\operatorname{spt}\nabla\eta} \Psi_m\, dx$ which proves (2.12), and it remains to show iii) just for $\ell = 2$, the other cases are obvious. To this purpose we recall that in fact we have shown that u is in the space $W^2_{q,\mathrm{loc}}(\Omega)$ (due to the arbitrariness of Ω_2) and that by definition u_m is of class $\overline{u}_m + \overset{\circ}{W}{}^2_q(\Omega_2)$. Therefore the following calculations are justified: we have

$$\begin{aligned}
&\int_{\Omega_2} \Big(f(\nabla^2 u_m) - f(\nabla^2 u) \Big)\, dx \\
&= \int_{\Omega_2} Df(\nabla^2 u) : (\nabla^2 u_m - \nabla^2 u)\, dx \\
&\quad + \int_{\Omega_2} \int_0^1 D^2 f\Big(\nabla^2 u + t[\nabla^2 u_m - \nabla^2 u] \Big) \\
&\qquad \cdot (\nabla^2 u_m - \nabla^2 u, \nabla^2 u_m - \nabla^2 u)(1-t)\, dt\, dx.
\end{aligned} \tag{2.13}$$

Note that $\|u-\overline{u}_m\|_{W^2_q(\tilde{\Omega})} \to 0$ for all $\tilde{\Omega} \subset\subset \Omega$, moreover the Euler equation for u implies

$$\int_{\Omega_2} Df(\nabla^2 u) : (\nabla^2 u_m - \nabla^2 u)\, dx = \int_{\Omega_2} Df(\nabla^2 u) : (\nabla^2 \overline{u}_m - \nabla^2 u)\, dx,$$

thus the first term on the r.h.s. of (2.13) vanishes as $m \to \infty$. The same is true for the l.h.s. of (2.13) as it was remarked at the beginning of this section. This implies

$$\lim_{m\to\infty} \int_{\Omega_2} \int_0^1 D^2 f(\nabla^2 u + t[\nabla^2 u_m - \nabla^2 u])(\nabla^2 u_m - \nabla^2 u, \nabla^2 u_m - \nabla^2 u)\, dt\, dx = 0$$

and in the case $p \geq 2$ the claim follows from (1.1). Suppose now that $p < 2$. Then again by (1.1)

$$\int_0^1 \ldots dt \geq \lambda \int_0^1 (1 + |\nabla^2 u + t(\nabla^2 u_m - \nabla^2 u)|^2\Big)^{\frac{p-2}{2}} |\nabla^2 u_m - \nabla^2 u|^2 (1-t)\, dt$$

$$\geq c\Big(1 + [|\nabla^2 u| + |\nabla^2 u_m|]^2\Big)^{\frac{p-2}{2}} |\nabla^2 u_m - \nabla^2 u|^2.$$

For almost all $x \in \Omega_2$ we have

$$h_m(x) \to \hat{h}(x) < \infty,$$

therefore $\lim\limits_{m\to\infty} |\nabla^2 u_m(x)|$ exists and is finite for almost all $x \in \Omega_2$ (by the definition of h_m). If we consider such points $x \in \Omega_2$ and observe that by the above estimate

$$\Big(1 + [|\nabla^2 u| + |\nabla^2 u_m|]^2\Big)^{\frac{p-2}{2}} |\nabla^2 u_m - \nabla^2 u|^2 \to 0 \quad \text{a.e.},$$

then it is immendiate that $|\nabla^2 u_m - \nabla^2 u|^2 \to 0$ a.e., and the claim follows. □

3. Blow-up and partial regularity

In this section we give a proof of Theorem 1.1 where we restrict ourselves to the case that $p \geq 2$. As already remarked the subquadratic case can be handled with techniques introduced in [CFM]. So let the hypothesis of Theorem 1.1 hold. Then we have the following excess–decay lemma which is the key to partial regularity.

Lemma 3.1. *Given a positive number L, define the constant $C^*(L)$ according to (3.11) below and let $C_* := C_*(L) := 2C^*(L)$. Then, for any $\tau \in (0, 1/2)$ there exists $\varepsilon = \varepsilon(\tau, L)$ such that the validity of*

$$|(\nabla^2 u)_{x,r}| \leq L \text{ and } E(x,r) \leq \varepsilon(L,\tau) \tag{3.1}$$

for some ball $B_r(x) \subset\subset \Omega$ implies the estimate

$$E(x, \tau r) \leq \tau^2 C_*(L) E(x,r). \tag{3.2}$$

Here we have set

$$E(x,\rho) := \fint_{B_\rho(x)} |\nabla^2 u - (\nabla^2 u)_{x,\rho}|^2\, dy + \fint_{B_\rho(x)} |\nabla^2 u - (\nabla^2 u)_{x,\rho}|^q\, dy$$

for balls $B_\rho(x)$ compactly contained in Ω, and $\fint_{B_\rho(x)} g\,dy$ or $(g)_{x,\rho}$ denote the mean value of a function g w.r.t. $B_\rho(x)$. Let us recall that we consider the case $p \geq 2$, thus $q > 2$. If $p < 2$ is allowed, then $q < 2$ is possible but the statement of Lemma 3.1 (and thereby partial regularity) remains true if the excess function E then is defined according to [CFM].

Remark 3.1. i) It is well known how to iterate the result of Lemma 3.1 leading to the result that the set of points $x_0 \in \Omega$ such that

$$\limsup_{r \searrow 0} |(\nabla^2 u)_{x_0,r}| < \infty$$

together with $\liminf_{r\searrow 0} E(x_0, r) = 0$ is an open set (of full Lebesgue measure) on which the local minimizer u is of class $C^{2,\nu}$ for any $0 < \nu < 1$. We refer the reader to Giaquinta's textbook [Gia] and mention the papers [GiuMi] of Giusti and Miranda, [Ev] of Evans or the contribution [FH] of Fusco and Hutchinson.

ii) We will give an indirect proof of Lemma 3.1 using the blow-up technique following more or less the ideas of Evans and Gariepy outlined in [Ev] and [EG].

Proof of Lemma 3.1. To argue by contradiction we assume that for $L > 0$ fixed and for some $\tau \in (0, 1/2)$ there exists a sequence of balls $B_{r_m}(x_m) \subset\subset \Omega$ such that

$$|(\nabla^2 u)_{x_m,r_m}| \leq L,\; E(x_m, r_m) =: \lambda_m^2 \underset{m\to\infty}{\longrightarrow} 0, \tag{3.3}$$

$$E(x_m, \tau\, r_m) > C_* \tau^2 \lambda_m^2. \tag{3.4}$$

Now a sequence of rescaled functions is introduced by letting

$$\begin{aligned} a_m &:= (u)_{x_m,r_m},\; A_m := (\nabla u)_{x_m,r_m},\; \Theta_m := (\nabla^2 u)_{x_m,r_m}, \\ \hat{u}_m(z) &:= \frac{1}{\lambda_m r_m^2}\Big[u_m(x_m + r_m z) - a_m - r_m A_m z \\ &\qquad - \frac{1}{2} r_m^2 \Theta_m(z,z) + \frac{1}{2} r_m^2 \fint_{B_1} \Theta_m(\tilde{z},\tilde{z}) d\tilde{z}\Big],\; |z| < 1. \end{aligned}$$

Direct calculations show that

$$\begin{aligned} \nabla \hat{u}_m(z) &= \frac{1}{\lambda_m r_m}\Big[\nabla u(x_m + r_m z) - A_m - \frac{1}{2} r_m \nabla(\Theta_m^{\alpha\beta} z_\alpha z_\beta)\Big], \\ \nabla^2 \hat{u}_m(z) &= \frac{1}{\lambda_m}\Big[\nabla^2 u(x_m + r_m z) - \Theta_m\Big], \end{aligned}$$

moreover, the quantities $(\hat{u}_m)_{0,1}$, $(\nabla \hat{u}_m)_{0,1}$, $(\nabla^2 \hat{u}_m)_{0,1}$ vanish for all m. From our assumptions (3.3) we get

$$\fint_{B_1} |\nabla^2 \hat{u}_m|^2 dz + \lambda_m^{q-2} \fint_{B_1} |\nabla^2 \hat{u}_m|^q dz = \lambda_m^{-2} E(x_m, r_m) = 1, \tag{3.5}$$

and after passing to subsequences which are not relabeled we find (using Poincaré's inequality for deriving (3.7) from (3.5))

$$\Theta_m \to: \Theta, \tag{3.6}$$

$$\hat{u}_m \rightharpoondown: \hat{u} \quad \text{in } W_2^2(B_1), \tag{3.7}$$

$$\lambda_m \nabla^2 \hat{u}_m \to 0 \quad \text{in } L^2(B_1) \text{ and a.e.}, \tag{3.8}$$

$$\lambda_m^{1-2/q} \nabla^2 \hat{u}_m \rightharpoondown 0 \quad \text{in } L^q(B_1). \tag{3.9}$$

After these preparations we claim that the limit function $\hat{u}$ satisfies

$$\int_{B_1} D^2 f(\Theta)(\nabla^2 \hat{u}, \nabla^2 \varphi)\, dz = 0 \quad \forall \varphi \in C_0^\infty(B_1). \tag{3.10}$$

To prove (3.10) we proceed exactly as in [Ev] (see also [BF1] and [Bi], Proposition 3.33) taking into account (3.6), (3.7) and (3.9).

Moreover, the application of Poincaré's inequality in combination with estimate (3.2) from [GiaMo1] and Lemma 7 of [Kr1] (see also [Ca1,2]) give the existence of a constant C^*, only depending on n, L, p, q, λ and Λ, such that

$$⨍_{B_\tau} |\nabla^2 \hat{u} - (\nabla^2 \hat{u})_{0,\tau}|^2\, dz \le C^* \tau^2. \tag{3.11}$$

To be precise, we have

$$⨍_{B_\tau} |\nabla^2 \hat{u} - (\nabla^2 \hat{u})_{0,\tau}|^2\, dz \le c\tau^2 ⨍_{B_\tau} |\nabla^3 \hat{u}|^2\, dz \le c\tau^2 ⨍_{B_{1/2}} |\nabla^3 \hat{u}|^2\, dz,$$

which follows from [GiaMo1], (3.2), applied to the function $v := \partial_\gamma \hat{u}$, $\gamma = 1, \dots, n$. Moreover

$$⨍_{B_{1/2}} |\nabla^3 \hat{u}|^2\, dz \le c \sup_{B_{1/2}} |\nabla^3 \hat{u}|^2 \le c ⨍_{B_1} |\nabla^2 \hat{u}|^2\, dz \le \liminf_{m\to\infty} c ⨍_{B_1} |\nabla^2 \hat{u}_m|^2\, dz \le c,$$

where we used (3.5), (3.7) and [Kr1], Lemma 7. This proves (3.11) for a suitable constant C^*. Clearly (3.11) is in contradiction to (3.4), if we can improve the convergences stated in (3.8) and (3.9) to the strong convergences

$$\nabla^2 \hat{u}_m \to \nabla^2 \hat{u} \quad \text{in } L^2_{\mathrm{loc}}(B_1), \tag{3.12}$$

$$\lambda_m^{1-2/q} \nabla^2 \hat{u}_m \to 0 \quad \text{in } L^q_{\mathrm{loc}}(B_1). \tag{3.13}$$

To verify (3.12) and (3.13) we want to show first for any $0 < \rho < 1$ the identity

$$\lim_{m\to\infty} \int_{B_\rho} \left(1 + |\Theta_m + \lambda_m \nabla^2 \hat{u} + \lambda_m \nabla^2 w_m|^2\right)^{\frac{p-2}{2}} |\nabla^2 w_m|^2\, dz = 0, \tag{3.14}$$

where $w_m := \hat{u}_m - \hat{u}$. Following the basic ideas given in [EG] (see also [BF1] or [Bi], Proposition 3.34) we observe that for all $\varphi \in C_0^\infty(B_1)$, $0 \le \varphi \le 1$,

$$
\begin{aligned}
&\lambda_m^{-2} \int_{B_1} \varphi \Big[f(\Theta_m + \lambda_m \nabla^2 \hat{u}_m) - f(\Theta_m + \lambda_m \nabla^2 \hat{u}) \Big] \, dz \\
&- \lambda_m^{-1} \int_{B_1} \varphi Df\Big(\Theta_m + \lambda_m \nabla^2 \hat{u}\Big) : \nabla^2 w_m \, dz \\
&= \int_{B_1} \int_0^1 \varphi D^2 f\Big(\Theta_m + \lambda_m \nabla^2 \hat{u} + s\lambda_m \nabla^2 w_m\Big)\Big(\nabla^2 w_m, \nabla^2 w_m\Big)(1-s)\, ds\, dz.
\end{aligned}
\tag{3.15}
$$

Obviously (3.14) will follow from the ellipticity of $D^2 f$, if we can show that the left-hand side of (3.15) tends to zero as $m \to \infty$. Using the minimality of u as well as the convexity of f we can estimate

$$
\begin{aligned}
\text{l.h.s. of (3.15)} \le\ & \lambda_m^{-2} \int_{B_1} f\Big(\Theta_m + \lambda_m \nabla^2 [\hat{u}_m + \varphi(\hat{u} - \hat{u}_m)]\Big)\, dz \\
& - \lambda_m^{-2} \int_{B_1} f\Big(\Theta_m + \lambda_m \Big[(1-\varphi)\nabla^2 \hat{u}_m + \varphi \nabla^2 \hat{u}\Big]\Big)\, dz \\
& - \lambda_m^{-1} \int_{B_1} \varphi Df(\Theta_m + \lambda_m \nabla^2 \hat{u}) : \nabla^2 w_m \, dz \\
=:\ & I_1 - I_2 - I_3.
\end{aligned}
$$

Setting

$$
X_m := \Theta_m + \lambda_m \Big[(1-\varphi)\nabla^2 \hat{u}_m + \varphi \nabla^2 \hat{u}\Big],\ Z_m := 2\nabla\varphi \otimes \nabla(\hat{u} - \hat{u}_m) + \nabla^2 \varphi (\hat{u} - \hat{u}_m)
$$

we obtain

$$
\begin{aligned}
I_1 - I_2 =\ & \lambda_m^{-1} \int_{B_1} Df(X_m) : Z_m \, dz \\
& + \int_{B_1} \int_0^1 D^2 f\Big(X_m + s\lambda_m Z_m\Big)(Z_m, Z_m)(1-s)\, ds\, dz \\
\le\ & \lambda_m^{-1} \int_{B_1} Df(X_m) : Z_m \, dz \\
& + c \int_{B_1} \Big(1 + \Big\{|\Theta_m| + \lambda_m |\nabla^2 \hat{u}_m| + \lambda_m |\nabla^2 \hat{u}| + \lambda_m |Z_m|\Big\}^2\Big)^{\frac{q-2}{2}} |Z_m|^2 \, dz.
\end{aligned}
$$

With the notation $\epsilon(m) \to 0$ as $m \to \infty$ we get on account of (3.7) that the last integral can be estimated from above by

$$
c \int_{B_1} \lambda_m^{q-2} |\nabla \hat{u}_m|^{q-2} |Z_m|^2 \, dz + c \int_{B_1} \lambda_m^{q-2} |Z_m|^q \, dz + \varepsilon(m).
$$

Furthermore,

$$
\begin{aligned}
J_1 &:= c\int_{B_1} \lambda_m^{q-2}|\nabla\hat u_m|^{q-2}|Z_m|^2\,dz\\
&\le c\int_{\operatorname{spt}\varphi} \lambda_m^{q-2}|\nabla^2\hat u_m|^{q-2}\Big\{|\nabla\hat u-\nabla\hat u_m|+|\hat u-\hat u_m|\Big\}^2\,dz\\
&\le c\left\{\int_{\operatorname{spt}\varphi}\lambda_m^{q-2}|\nabla^2\hat u_m|^q\,dz\right\}^{1-2/q}\Big\{\lambda_m^{q-2}\int_{\operatorname{spt}\varphi}|\nabla\hat u-\nabla\hat u_m|^q\,dz\\
&\quad+\lambda_m^{q-2}\int_{\operatorname{spt}\varphi}|\hat u-\hat u_m|^q\,dz\Big\}^{2/q}\\
&\le c\left\{\lambda_m^{q-2}\int_{\operatorname{spt}\varphi}|\nabla\hat u-\nabla\hat u_m|^q\,dz+\lambda_m^{q-2}\int_{\operatorname{spt}\varphi}|\hat u-\hat u_m|^q\,dz\right\}^{2/q},
\end{aligned}
$$

where the last inequality follows from (3.9). We also note that due to (3.9) $\lambda_m^{1-2/q}\nabla^k\hat u_m \underset{m\to\infty}{\longrightarrow} 0$ in $L^q(B_1)$ for $k=0,1$. This immediately implies

$$J_1 \le \varepsilon(m)\to 0 \quad \text{as} \quad m\to\infty.$$

Analogous arguments applied to

$$J_2 := c\int_{B_1}\lambda_m^{q-2}|Z_m|^q\,dz$$

guarantee that

$$J_2 \le \varepsilon(m)\to 0 \quad \text{as} \quad m\to\infty.$$

Thus, we arrive at

$$
\begin{aligned}
\text{l.h.s. of (3.15)} \le \varepsilon(m)+\lambda_m^{-1}\Big[&\int_{B_1} Df(X_m):Z_m\,dz\\
&-\int_{B_1} Df(\Theta_m+\lambda_m\nabla^2\hat u):\nabla^2 w_m\varphi\,dz\Big].
\end{aligned}
\tag{3.16}
$$

Next we are going to discuss the last two integrals in (3.16). Since

$$\nabla^2(\varphi w_m) = \nabla^2 w_m\varphi - Z_m,$$

we have that

$$
\begin{aligned}
[\ldots] = &\int_{B_1}\Big(Df(X_m)-Df(\Theta_m+\lambda_m\nabla^2\hat u)\Big):Z_m\,dz\\
&-\int_{B_1} Df\Big(\Theta_m+\lambda_m\nabla^2\hat u\Big):\nabla^2(\varphi w_m)\,dz =: I_4-I_5.
\end{aligned}
$$

From (1.1) and from the requirement that $0 \le \varphi \le 1$ we obtain by recalling the definition of Z_m

$$
\begin{aligned}
I_4 &= \int_{B_1} \Big(Df\Big(\Theta_m + \lambda_m[(1-\varphi)\nabla^2 \hat{u}_m + \varphi \nabla^2 \hat{u}]\Big) - Df(\Theta_m + \lambda_m \nabla^2 \hat{u})\Big) : Z_m \, dz \\
&= \int_{B_1} \int_0^1 \frac{d}{ds} Df\Big(\Theta_m + \lambda_m \nabla^2 \hat{u} + s\lambda_m (1-\varphi)\nabla^2(\hat{u}_m - \hat{u})\Big) ds : Z_m \, dz \\
&= \lambda_m \int_{B_1} \int_0^1 D^2 f\Big(\Theta_m + \lambda_m \nabla^2 \hat{u} + s\lambda_m(1-\varphi)\nabla^2 w_m)(\nabla^2 w_m, Z_m)(1-\varphi)\, ds \, dz \\
&\le \lambda_m c \int_{B_1} \Big(1 + (|\Theta_m| + \lambda_m |\nabla^2 \hat{u}| + \lambda_m |\nabla^2 w_m|)^2\Big)^{\frac{q-2}{2}} \\
&\quad \cdot |\nabla^2 w_m| \Big[|\nabla \varphi| |\nabla w_m| + |\nabla^2 \varphi| |w_m| \Big] \, dz,
\end{aligned}
$$

and similar to the previous discussion of J_1 we get

$$
\lambda_m^{-1} I_4 \to 0 \quad \text{as} \quad m \to \infty.
$$

Finally, we observe that

$$
\begin{aligned}
\lambda_m^{-1} I_5 &= \lambda_m^{-1} \int_{B_1} \Big(Df(\Theta_m + \lambda_m \nabla^2 \hat{u}) - Df(\Theta_m) \Big) : \nabla^2(\varphi w_m) \, dz \\
&= \lambda_m^{-1} \int_{B_1} \int_0^1 D^2 f(\Theta_m + s\lambda_m \nabla^2 \hat{u}) \Big(\lambda_m \nabla^2 \hat{u}, \nabla^2(\varphi w_m) \Big) \, ds \, dz,
\end{aligned}
$$

and, consequently, $\lambda_m^{-1} I_5$ vanishes after passing to the limit $m \to \infty$ on account of the weak convergence (3.7). Summarizing these results we have shown that $\lim_{m\to\infty}$ (l.h.s. of (3.15)) $= 0$.

Therefore, identity (3.14) is proved, and (3.12) immediately follows from (3.14) since we assume that $p \ge 2$. To proceed further, i.e. to prove the strong convergence stated in (3.13), we introduce the auxiliary functions

$$
\Psi_m(z) := \lambda_m^{-1} \Big[(1 + |\Theta_m + \lambda_m \nabla^2 \hat{u}_m(z)|^2)^{p/4} - (1 + |\Theta_m|^2)^{p/4} \Big].
$$

For any $\rho < 1$ Lemma 2.2 implies

$$
\begin{aligned}
&\int_{B_\rho} |\nabla \Psi_m|^2 \, dz = \lambda_m^{-2} \, r_m^{2-n} \int_{B_{\rho r_m}(x_m)} |\nabla h|^2 \, dx \\
&\le c\,(\rho) \lambda_m^{-2} r_m^{2-n} \int_{B_{r_m}(x_m)} |D^2 f(\nabla^2 u)| \cdot \Big\{ r_m^{-2} |\nabla^2 (u-P)|^2 + r_m^{-4} |\nabla(u-P)|^2 \Big\} \, dx.
\end{aligned}
$$

For the last estimate we used inequality (2.12), h being defined in Lemma 2.2 and P representing a polynomial function of degree ≤ 2. If we choose

$$
P(x) := A_m x + \frac{1}{2} \Theta_m (x - x_m, x - x_m) \quad \text{for} \quad x \in B_{r_m}(x_m)
$$

we get

$$\nabla(u(x) - P(x)) = \lambda_m r_m \nabla \hat{u}_m\Big(\frac{x - x_m}{r_m}\Big),$$
$$\nabla^2(u(x) - P(x)) = \lambda_m \nabla^2 \hat{u}_m\Big(\frac{x - x_m}{r_m}\Big).$$

So, taking into account (3.7) and (3.9) we obtain for any $\rho < 1$ the inequality

$$\begin{aligned}\int_{B_1} |\nabla\Psi_m|^2\, dz &\le c(\rho)\int_{B_1} |D^2 f(\Theta_m + \lambda_m\nabla^2\hat{u}_m)| \cdot \Big\{|\nabla^2\hat{u}_m|^2 + |\nabla\hat{u}_m|^2\Big\}\, dz \\ &\le c(\rho) < \infty.\end{aligned} \tag{3.17}$$

In addition, one can write

$$\begin{aligned}|\Psi_m| &\le c\int_0^1 |\nabla^2\hat{u}_m|\Big(1 + |\Theta_m + s\lambda_m\nabla^2\hat{u}_m|^2\Big)^{\frac{p-2}{4}}\, ds \\ &\le c\Big\{|\nabla^2\hat{u}_m| + \lambda_m^{\frac{p-2}{2}}|\nabla^2\hat{u}_m|^{p/2} + 1\Big\}.\end{aligned} \tag{3.18}$$

It follows from (3.14) that

$$\int_{B_\rho} \lambda_m^{p-2}|\nabla^2\hat{u}_m|^p\, dx \le c(\rho) < \infty.$$

Combining the last estimate with (3.17) and (3.18) we can conclude that the sequence Ψ_m is bounded in $W^1_{2,\mathrm{loc}}(B_1)$. Now we proceed as follows: consider a number $M >> 1$ and let

$$U_m := \{z \in B_\rho : \lambda_m|\nabla^2\hat{u}_m| \le M\}.$$

Then

$$\begin{aligned}&\int_{U_m} \lambda_m^{q-2}|\nabla^2\hat{u}_m|^q\, dz \le c\Big\{\int_{U_m} \lambda_m^{q-2}|\nabla^2 w_m|^q\, dz + \int_{U_m} \lambda_m^{q-2}|\nabla^2\hat{u}|^q\, dz\Big\} \\ &\le c\Big\{\int_{U_m} \lambda_m^{q-2}\Big(|\nabla^2\hat{u}_m|^{q-2} + |\nabla^2\hat{u}|^{q-2}\Big)|\nabla^2 w_m|^2\, dz + \int_{U_m} \lambda_m^{q-2}|\nabla^2\hat{u}|^q\, dz\Big\} \\ &\le c\Big\{\int_{B_\rho} (M^{q-2} + |\nabla^2\hat{u}|^{q-2})|\nabla^2 w_m|^2\, dz + \int_{B_\rho} \lambda_m^{q-2}|\nabla^2\hat{u}|^q\, dz\Big\} \\ &\to 0 \quad \text{as} \quad m \to \infty\end{aligned} \tag{3.19}$$

on account of $\nabla^2 w_m \to 0$ in $L^2(B_\rho)$ and $\nabla^2\hat{u} \in L^\infty(B_\rho)$. On the other hand, if we choose M sufficiently large, then on $B_\rho - U_m$ we get

$$\Psi_m(z) \ge c\lambda_m^{-1+p/2}|\nabla^2\hat{u}_m|^{p/2}$$

and, consequently

$$|\nabla^2\hat{u}_m|^q\lambda_m^{q-2} \le c\lambda_m^{2\frac{q}{p}-2}\Psi_m^{\frac{2q}{p}}.$$

Since (1.3) guarantees $\frac{2q}{p} < \frac{2n}{n-2}$ and since Ψ_m is uniformly bounded in $W^1_{2,\mathrm{loc}}(B_1)$, we can conclude

$$\int_{B_\rho - U_m} \lambda_m^{q-2} |\nabla^2 \hat{u}_m|^q \, dz \to 0 \text{ as } m \to \infty \text{ for any } \rho < 1. \tag{3.20}$$

It only remains to note that obviously the results (3.19) and (3.20) provide (3.13), which completes the proof. □

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