ON THE EXISTENCE OF A SYMBOLIC AND BIQUADRATIC EQUATION, WHICH IS SATISFIED BY THE SYMBOL OF LINEAR OPERATION IN QUATERNIONS

By

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On the Existence of a Symbolic and Biquadratic Equation, which is satisfied by the Symbol of Linear Operation in Quaternions.

Sir William Rowan Hamilton.

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1. In a recent communication (of June 9, 1862), I showed how the general Linear and Quaternion Function of a Quaternion could be expressed, under a standard quadrinomial form; and how that function, when so expressed, could be inverted.

2. I have since perceived, that whatever *form* be adopted, to represent the *Linear Symbol of Quaternion Operation* thus referred to, that *symbol* always satisfies a certain *Biquadratic Equation*, with *Scalar Coefficients*, of which the *values* depend upon the particular *constants* of the *Function* above referred to.

3. This result, with the properties of the *Auxiliary Linear* and *Quaternion Functions* which which it is connected, appears to me to constitute the most remarkable accession to the *Theory of Quaternions proper*, as distinguished from their *separation* into *scalar* and *vector parts*, and from their *application to Geometry* and *Physics*, which has been made since I had first the honour of addressing the Royal Irish Academy on the subject, in the year 1843.

4. The following is an outline of one of the proofs of the existence of the biquadratic equation, above referred to. Let

$$fq = r \tag{1}$$

be a given linear equation in quaternions; r being a given quaternion, q a sought one, and f the symbol of a linear or distributive operation: so that

$$f(q + q') = fq + fq', \tag{2}$$

whatever two quaternions may be denoted by q and q'.

5. I have found that the *formula of solution* of this equation (1), or the formula of *inversion* of the *function*, f, may be thus stated:

$$nq = nf^{-1}r = Fr; \tag{3}$$

where n is a *scalar constant* depending for its *value*, and F is an auxiliary and *linear symbol* of operation depending for its *form* (or rather for the *constants* which it involves), on the *particular form* of f; or on the special values of the *constants*, which enter into the composition of the *particular function*, fq.

6. We have thus, independently of the particular quaternions, q and r, the equations,

$$Ffq = nq, \quad Ffr = nr; \tag{4}$$

or, briefly and symbolically,

$$Ff = fF = n. \tag{5}$$

7. Changing next f to $f_c = f + c$, that is to say, proposing next to resolve the *new linear equation*,

$$f_c q = fq + cq = r, \tag{6}$$

where c is an *arbitrary scalar*, I find that the *new* formula of solution, or of inversion, may be thus written:

$$f_c F_c = n_c; \tag{7}$$

where

$$F_c = F + cG + c^2 H + c^3, \tag{8}$$

and

$$n_c = n + n'c + n''c^2 + n'''c^3 + c^4; \tag{9}$$

G and H being the symbols (or characteristics) of *two new linear operations*, and n', n'', n''' denoting *three new scalar constants.*

8. Expanding then the symbolical product $f_c F_c$, and comparing powers of c, we arrive at *three new symbolical equations*, namely, the following:

$$fG + F = n'; \quad fH + G = n''; \quad f + H = n'''; \tag{10}$$

by elimination of the symbols, F, G, H, between which and the equation (5), the *symbolical biquadratic*,

$$0 = n - n'f + n''f^2 - n'''f^3 + f^4, \tag{A}$$

is obtained.