Introduction

It is perhaps evident from several places of this volume that Ramsey theorem played a decisive role in Erdős' combinatorial activity. And perhaps no other part of combinatorial mathematics is so dear to him as Ramsey theory and extremal problems. He was not creating or even aiming for a theory. However, a complex web of results and conjectures did, in fact, give rise to several theories. They all started with modest short papers by Erdős, Szekeres and Turán in the thirties. How striking it is to compare these initial papers with the richness of the later development, described, e.g., by the survey articles of Miki Simonovits (extremal graph theory) and Jeff Kahn (extremal set theory). In addition, the editors of these volumes tried to cover in greater detail the development of Ramsey theory mirrored and motivated by Erdős' papers. In a way (and this certainly is one of the leitmotivs of Erdős' work), there is little difference between, say, density Ramsey type results and extremal problems.

One can only speculate on the origins of density questions. It is clear that in the late 30's and 40's, the time was ripe in ideas which later developed into extremal and density questions: we have not only the Erdős-Turán 1941 paper but also, Erdős and Tomsk 1938 paper on number theory which anticipated extremal theory by determining $n^{3/2}$ as upper bound for C_4-free graphs, the Sperner paper and also the Erdős-Ko-Rado work (which took several decades to get into print). All these ideas, together with Turán's extremal results provided a fruitful cross-interaction of ideas from various fields which, some 30 years later, developed into density Ramsey theorems and extremal theory. We are happy to include in this chapter papers by Gyula Katona (which gives a rare account of Erdős method of encouraging and educating young talented students), and by Vojtěch Rödl and Robin Thomas related to an aspect of Erdős' work. And we include an important paper by Alexander Kostochka which gives a major breakthrough to the Erdős-Rado Δ-system problem (for triples; meanwhile Kostochka succeeded in generalizing the result to k-tuples). Finally, we have the paper of Saharon Shelah, solving a model theoretic Ramsey question of Väänänen, illustrating (once again) that Ramsey theory is alive and well. Let us complement these lines by Erdős' own words:

Hajnal, Rado and I proved

$$2^{c\,k^2} < r_3(k,k) < 2^{2^k} ,$$

we believe that the upper bound is correct or at least is closer to the truth, but Hajnal and I have a curious result: If one colors the triples of a set of n elements by two colors, there always is a set of size $(\log n)^{1/2}$ where the distribution is not just, i.e., one of the colors has more than $0.6\binom{t}{3}$ of the triples for $t = (\log n)^{1/2}$. Nevertheless we believe that the upper bound is correct. It would perhaps change our minds if we could replace 0.6 by $1-\epsilon$ for some $t, t > (\log n)^{\epsilon}$. We never had any method of doing this.

Let H be a fixed graph and let n be large. Hajnal and I conjectured that if $G(n)$ does not contain an induced copy of H then $G(n)$ must contain a trivial subgraph of size $> n^{\epsilon}$, $\epsilon = \epsilon(H)$. We proved this for many special cases but many problems remain. We could only prove $\exp(c\sqrt{\log n})$.

Extremal graph theory

Denote by $T(n;G)$ the smallest integer for which every graph of n vertices and $T(n;G)$ edges contains G as a subgraph. Turán determined $T(n;G)$ if G is $K(r)$ for all r, $T(n,C_4) = (\frac{1}{2}+o(1))n^{3/2}$ was proved by V. T. Sós, Rényi and myself. The exact formula for $T(n,c_4)$ is known if $n = p^2+p+1$ (Füredi).

I have many asymptotic and exact results for $T(n,C_4)$ and many results and conjectures with Simonovits. I have to refer to the excellent book of Bollobás and the excellent survey of Simonovits and a very good recent survey article of Füredi. Here I only state two results of Simonovits and myself: $T(n,G) < cn^{8/5}$ where G is the edge graph of the three-dimensional cube. Also we have fairly exact results for $T(n,K(2,2,2))$.

Ramsey-Turán theorems

The first papers are joint with V. T. Sós and then there are comprehensive papers with Hajnal, Simonovits, V. T. Sós and Szemerédi. Here I only state a result with Bollobás, Szemerédi and myself: For every $\epsilon > 0$ and $n > n_o(\epsilon)$ there is a graph of n vertices $\frac{n^2}{8}(1-\epsilon)$ edges with no $K(4)$ and having largest independent set is $o(n)$, but such a graph does not exist if the number of edges is $\frac{n^2}{8}(1+\epsilon)$. We have no idea what happens if the number of edges is $\frac{n^2}{8}$.

One last Ramsey type problem: Let n_k be the smallest integer (if it exists) for which if we color the proper divisors of n_k by k colors then n_k will be a monochromatic sum of distinct divisors, namely a sum of distinct divisors in a color class. I am sure that n_k exists for every k but I think it is not even known if n_2 exists. It would be of some interest to determine at least n_2. An old problem of R. L. Graham and myself states: Is it true that if m_k is sufficiently large and we color the integers $2 \le t \le m_k$ by k colors then

$$1 = \sum \frac{1}{t_i}$$

is always solvable monochromatically? I would like to see a proof that m_2 exists. (Clearly $m_k \geq n_k$.) Perhaps this is really a Turán type problem and not a Ramsey problem. In other words, if m is sufficiently large and $1 < a_1 < a_2 < \cdots < a_\ell \leq m$ is a sequence of integers for which $\sum_\ell 1/a_\ell > \delta \log m$ then

$$1 = \sum \frac{\epsilon_i}{a_i} \quad (\epsilon_i = 0 \text{ or } 1)$$

is always solvable. I offer 100 dollars for a proof or disproof. Perhaps it suffices to assume that

$$\sum_{a_i < m} \frac{1}{a_i} > C(\log \log m)^2$$

for some large enough C. For further problems of this kind as well as for related results see my book with R. L. Graham. I hope before the year 2000 a second edition will appear.