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On additive properties of general sequences. (In English)
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The authors give a survey of their papers on additive properties of general sequences and they prove several further results on the range of additive representation functions and on difference sets. Many related unsolved problems are discussed.
Let $\mathbb{N} = \{1, 2, \dots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For $A \subseteq \mathbb{N}_0$ and $n \in \mathbb{N}$ the number of solutions of $n = a + a'$; $a, a' \in A$ is denoted by $r_1(A, n)$. For the additional conditions $a \leq a'$ or $a < a'$ this number is denoted by $r_2(A, n)$ or $r_3(A, n)$ resp. For given ranges of the representation function r_1 two theorems are proved:
Theorem 11. Let $B \subseteq \mathbb{N}_0$. There exists a set $A \subseteq \mathbb{N}_0$ such that B equals $\{r_1(A, n) \mid n \in \mathbb{N}\}$ if and only if either $B = \{0, 1\}$ or $\{0, 1, 2\} \subseteq B$.
Theorem 12. Let $B \subseteq \mathbb{N}_0$. There exists a set $A \subseteq \mathbb{N}_0$ such that B equals $\{m \in \mathbb{N} \mid m = r_1(A, n)$ for infinitely many $n \in \mathbb{N}\}$. Corresponding results are also stated for $i = 2$ and 3.
For $A \subseteq \mathbb{N}_0$ let $D(A) = \{a - a' \mid a \in A, a' \in A, a > a'\}$. Generalizing a theorem by *O. Grošek* and *R. Jajcay* [J. Comb. Math. Comb. Comput. 13, 167-174 (1993; Zbl 777.05025)], the authors show that if a set $B \subseteq \mathbb{N}$ contains arbitrary long sequences of consecutive integers then there exists a set $A \subseteq \mathbb{N}_0$ such that $D(A) = B$. In contrast to sum sets it is possible that a difference set $D(A)$ is 'dense' while A is extremely 'thin' because small differences $d = a - a'$ can be formed using large elements $a, a' \in A$. Two related results are given. The question if for a given infinite set $B \subseteq \mathbb{N}$ the equation $D(A) = B$ with $0 \in A$ can have a unique solution is answered positively in the case that $B = D(A_1)$ for an infinite B_3 sequence A_1 with $0 \in A_1$.
Apart from using Lemma 1 (to prove theorem 12) all proofs use elementary combinatorial methods only.

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Classification:
11B13 Additive bases
11B34 Representation functions
05B10 Difference sets
11B75 Combinatorial number theory
11B83 Special sequences of integers and polynomials

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additive bases; difference bases; B_k sequences; survey; additive representation functions; difference sets; unsolved problems