Zbl 792.05103
Erdős, Paul; Faudree, Ralph J.; Rousseau, C.C.; Schelp, R.H.
Multipartite graph-tree Ramsey numbers. (In English)
Capobianco, Michael F. (ed.) et al., Graph theory and its applications: East and West. Proceedings of the first China-USA international conference, held in Jinan, China, June 9-20, 1986. New York: New York Academy of Sciences, Ann. N. Y. Acad. Sci. 576, 146-154 (1989). [ISBN 0-89766-579-1/hbk; ISBN 0- 89766-580-5/pbk]

Let $T = T_n$ be a tree of order n, and $F = K(m_0, m_1, \ldots, m_k)$ be the complete multipartite graph with parts of order $m_0 \leq m_1 \leq \cdots \leq m_k$. For n sufficiently large and $m_1 = m_2 = 1$, it has been shown that the Ramsey number $r(F,T) = k(n-1)+1$. This result is generalized by showing that with just $m_0 = 1$ and n sufficiently large,

$$k(n-1)+1 \leq r(F,T) \leq k(r(K(1,m_1),T)-1)+1.$$

Since it is known that $r(K(1,m_1),T) = n$ for the large class of trees that have no vertices of large degree, the upper and lower bounds are frequently identical. In all cases, these bounds are shown to differ by at most k.

For simple graphs F and G, the Ramsey number $r(F,G)$ is the smallest integer p such that if the edges of the complete graph K_p are colored red and blue, either the red subgraph contains a copy of F or the blue subgraph contains a copy of G. If F is a graph with chromatic number $\chi(F)$, then the chromatic surplus $s(F)$ is the smallest number of vertices in a color class under any $\chi(F)$-coloring of the vertices of F.

For any connected graph G of order $n \geq s(F)$, the Ramsey number $r(F,G)$ satisfies the inequality

$$r(F,G) \geq (\chi(F)-1)(n-1)+s(F). \tag{1}$$

This inequality follows from coloring red or blue the edges of a complete graph on $(\chi(F)-1)(n-1)+s(F)-1$ vertices such that the red subgraph is isomorphic to $(\chi(F)-1)K_{n-1} \cup K_{s(F)-1}$ and the blue subgraph is isomorphic to the complement. When equality occurs in (1) we say that G is F-good. The concept of F-goodness generalizes the classical simple result of Chvátal that $r(K_k,T_n) = (k-1)(n-1)+1$ [J. Graph Theory 1, 93 (1977; Zbl 351.05120)], where K_k denotes the complete graph on k vertices and T_n denotes a tree on n vertices. The result of Chvátal has been generalized in many special cases by replacing the complete graph K_k by a graph F of chromatic number k, the tree T_n by a "sparse" graph G of order n, and the number 1 by $s(F)$ (i.e., G is F-good). Even in the case when G is "sparse" but not F-good, the lower bound given in inequality (1) is in most cases a good approximation to the Ramsey number $r(F,G)$.

Our purpose is to investigate the Ramsey number $r(F,T_n)$, when T_n is a large-order tree, and in particular to determine which large trees T_n are F-good. Since $\chi(F)$ is important in determining the value of $r(F,T_n)$, it is natural

to carefully consider the case when $F = K(m_0, m_1, \dots, m_k)$ is a complete multipartite graph.
Not all trees are $K(m_0, m_1, \dots, m_k)$-good when each $m_i \geq 2$ or when $m_0 = 1$ and $m_i \geq 2$ for $1 \leq i \leq k$. For example in [Ann. Discrete Math. 41, 79-89 (1989; Zbl 672.05063)] it was shown that

$$r(K(2,2), K(1, n-1)) > n + n^{1/2} - 5n^{3/10}$$

for n large. However, the principal result of [Graphs Comb. 3, 1-6 (1987; Zbl 612.05046)] is that each large-order tree T_N is $K(1, 1, m_2, m_3, \dots, m_k)$-good. We will generalize this last result by proving the following theorem.
Theorem: If n is sufficiently large, then

$$r(K(1, m_1, \dots, m_k), T_n) \geq \max\{k(n-1), k(r(K(1, m_1), T_n) - 2)\} + 1,$$

and

$$r(K(1, m_1, \dots, m_k), T_n) \leq k\{r(K(1, m_1), T_N) - 1\} + 1.$$

Classification:
05C55 Generalized Ramsey theory
05C05 Trees

Keywords:
tree; multipartite graph; Ramsey number; chromatic number; chromatic surplus